Study of Mathematical Models for Dengue,Tuberculosis & Covid-19

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by

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Study of Mathematical Models for Dengue,Tuberculosis & Covid-19" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Mridini Gawas and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Study of Mathematical Models for Dengue,Tuberculosis & Covid-19" is a bonafide work carried out by Mr. Amar Bhagat under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " Study of Mathematical Models for Dengue,Tuberculosis & Covid-19." This survey is divided into four chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of the Mathematical modelling in Epidemiology

SECOND CHAPTER:

This chapter deals with mathematical modelling of dengue.In this topic we have discussed some basic properties ,talked about reproduction number and studied stability analysis of the model.

THIRD CHAPTER:

This chapter focuses on the mathematical representation of tuberculosis dynamics. Within this discussion, we have explored fundamental characteristics, delved into the concept of the reproduction number, and conducted an analysis of the model's stability.

FOURTH CHAPTER:

In this chapter, we've delved into the mathematical portrayal of the dynamics surrounding COVID-19. Throughout our exploration, we've examined the foundational traits, delved into the notion of the reproduction number, and scrutinized the stability of the model.

Keywords: equilibrium points; Reproduction number ; local stability; global stability;

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ABSTRACT

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of the Mathematical modelling in Epidemiology

SECOND CHAPTER:

In this model, once an individual recovers from the infection or disease, they are assumed to be immune and cannot be reinfected. However, this immunity is specific only to the virus strain that caused the initial infection; recovery from one strain does not confer immunity against the other three strains.

The model is simplified into a two-dimensional planar system. It is observed that the endemic state is considered stable if the basic reproductive number (R_0) of the disease exceeds one. This finding is consistent with the results obtained from a transmission model incorporating immunity.

THIRD CHAPTER:

This study delves into the analysis of three distinct tuberculosis models: SIR, SEIR, and BSEIR Mathematical Models. It investigates the progressive enhancement in realism from one model to the next. Our focus extends to examining the stability of these models concerning their equilibrium points. Through our analysis, we identify the specific conditions under which these models exhibit stability.

FOURTH CHAPTER:

The COVID-19 pandemic, emerging in 2019, represents one of the most severe global health crises in recent memory due to its rapid spread and high mortality rates. Its impact surpasses that of previous outbreaks like MERS in South Korea and SARS in

the Middle East, instilling fear and concern worldwide. To understand and predict the dynamics of this deadly disease, we are developing a dynamic model. Through mathematical techniques such as the Routh-Hurwitz criteria and the construction of Lyapunov functions, we aim to assess both local and global stability. This analysis will provide insights into the stability of disease-free and diseased states, offering crucial information for effective control measures.

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Chapter 1

Introduction

A differential equation is a mathematical equation that involves an unknown function and one or more of its derivatives with respect to an independent variable. The equation expresses a relationship between the function and its rates of change, reflecting how the function evolves or behaves over the given variable.

system of differential equation:

A system of differential equations involves multiple equations, each describing the rate of change of one or more dependent variables with respect to an independent variable. These systems are commonly used to model complex relationships where the behavior of one variable is dependent on the behavior of others.

The general form of a system of a n first-order ordinary differential equations (ODEs) is often written as:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, t) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, t) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, t)
\end{aligned}$$

Here, $x_1, x_2, \ldots, x_n$ are the dependent variables, t is the independent variable (often representing time), and $f_1, f_2, \ldots, f_n$ are functions defining the rates of change of the corresponding variables.

Mathematical Modelling:

Mathematical modelling is the process of describing a real world problem in mathematical terms, usually in the form of equations, and then using these equations both to help understand the original problem, and also to discover new features about the problem.

Epidemiology:

Epidemiology is the branch of medical science that deals with the study of the distribution and determinants of health-related events or conditions in populations, and the application of this study to control health problems. It involves the systematic investigation of the factors that influence the occurrence and spread of diseases, injuries, or health-related events within specific populations.

Dengue:

Dengue is a mosquito-borne viral infection caused by the dengue virus, primarily transmitted to humans by the Aedes mosquitoes. Symptoms include high fever, severe headache, pain behind the eyes, joint and muscle pain, fatigue, rash .

Tuberculosis:

Tuberculosis (TB) is an infectious disease that most often affects the lungs and is caused by a type of bacteria. It spreads through the air when infected people cough, sneeze or spit. Tuberculosis is preventable and curable.

Covid-19:

Covid-19 stands for Coronavirus Disease 2019. It's an infectious illness caused by the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), first identified in December 2019 in Wuhan, China. This virus primarily spreads through respiratory droplets and can cause a wide range of symptoms, from mild respiratory issues to severe illness, and can lead to complications or death, particularly in older adults or those with underlying health conditions.

Formulation of a mathematical model:

The formulation of a mathematical model using differential equations involves expressing the relationships between variables in a system in terms of differential equations

Reproduction number:

The reproduction number, often denoted as R_0 , is a crucial epidemiological concept used to measure the transmission potential of an infectious disease within a population.

Specifically, R_0 represents the average number of secondary infections produced by one infected individual in a completely susceptible population

Equilibrium points:

The equilibrium points represent the states where the system is at rest, as the rates of change are zero at those points. Analyzing the stability and behavior of the system around these equilibrium points is crucial for understanding its dynamics.

Endemic Equilibrium :

The endemic equilibrium represents a stable state in the population where the disease persists at a non-zero level. In this equilibrium, there is a balance between the rates of infection and recovery, leading to a constant, non-zero prevalence of the disease.

Disease-Free Equilibrium :

The disease-free equilibrium represents a state in the population where no individuals are infected with the disease. At this equilibrium point, all compartments related to the disease (such as susceptible, infected, and recovered) have constant values, and the spread of the disease is not occurring.

Stability analysis:

Stability analysis helps to understand whether small perturbations from an equilibrium point lead to convergence (stable behavior) or divergence (unstable behavior) over time. There are two main types of stability: local stability and global stability.

Local stability :

Local stability focuses on the behavior of solutions in the immediate vicinity of a specific equilibrium point. It examines how small perturbations from that equilibrium point evolve over time.

Global stability:

Global stability considers the behavior of the entire system over its entire state space. It examines whether all trajectories in the system, regardless of initial conditions, converge to a specific equilibrium point.

Chapter 2

Analysis of Dengue Disease

2.1 Introduction

Mathematical models have found extensive application across diverse fields of infectious disease epidemiology. Since the early 1900s, researchers have engaged in mathematical modeling of dengue disease transmission within both human and vector populations. Notable contributions include works by L. Esteva and C. Vargas [4], Z. Feng and J.X. Velasco-Hernandez[5], Gideon A. Ngwa and William S. Shu [6], and N. Nuraini, E. Socewono, and K.A. Sidarto [7],[8],[9].

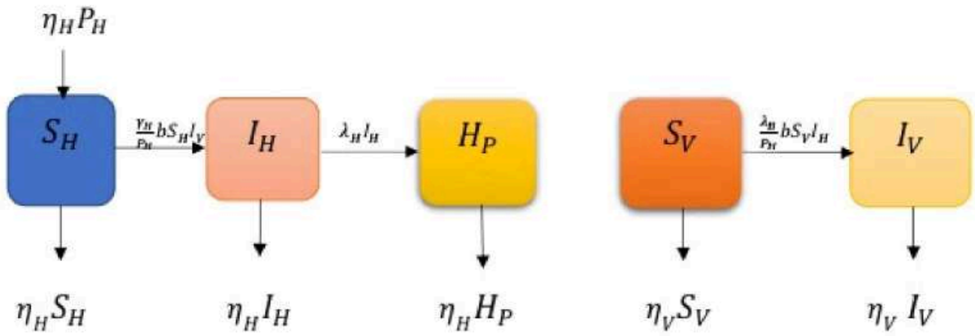
Numerous studies have explored infection models within human populations, conducted by researchers such as H.W. Hethcote and J.W. Van Ark [10],[11]. These models serve a variety of purposes, including comparing, planning, implementing, evaluating, and optimizing detection, prevention, therapy, and control programs.

Furthermore, epidemiological modeling plays a crucial role in designing and analyzing epidemiological surveys. It helps in identifying essential data to collect, detecting trends, making broad predictions, and estimating forecast uncertainties.

2.2 Esteva and Vargas mathematical Model for Dengue Fever Transmission

Formation of model

Let P_H and P_V represent the sizes of the human and vector populations, respectively. We consider that the human population remains constant over time. This implies that the birth rate and death rate of the human population are equal, both denoted by η_H.
The vector population is assumed to be governed by $\boxed{\dfrac{dP_v}{dt} = A - n_v P_v}$
To get equilibirium point we equate $\frac{dP_v}{dt} = 0$
$\therefore A - n_v P_v = 0$
$\therefore P_v = \frac{A}{N_V}$
Therefore vector approaches to $\frac{A}{N_V}$ as t tends to infinity.
The host population is divided into susceptible individuals denoted by S_H, infected individuals denoted by I_H, and recovered individuals denoted by H_P. The vector population, due to its short lifespan, is divided into susceptible vectors represented by S_V and infected vectors represented by I_V.
Schematic diagram of the model is :



Model involving both populations are

$$\begin{aligned}
\frac{dS_H}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H \\
\frac{dI_H}{dt} &= \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \lambda_H) I_H \\
\frac{dH_P}{dt} &= \lambda_H I_H - \eta_H H_P \\
\frac{dS_V}{dt} &= A - \frac{\lambda_B}{P_H} b S_V I_H - \eta_v S_v \\
\frac{dI_v}{dt} &= \frac{\lambda_B}{P_H} b S_V I_H - \eta_v I_v
\end{aligned} \tag{2.1}$$

with two conditions $P_H = S_H + I_H + H_P$ and $P_V = S_V + I_V$

where ,

P_H =Total host population

S_H = number of susceptible in the host population

I_H = number of infected in the host population

H_p = number of immunes in the host population

P_V = Total vector population

S_V = number of susceptible in the vector population

I_V = number of infected in the vector population

η_H= The birth or death rate in the host population

η_v= The death rate in the vector population

γ_H = The transmission probability from vector to host

λ_H = The recovery rate in the host population

λ_B = The transmission probability from host to vector

b = The biting rate of the vector

A = The recruitment rate.

The subset T defined by the equations $S_H + I_H + H_p = P_H$ and $S_V + I_V = \frac{A}{\eta_v}$ is an invariant region for system (2.1), because any solution starting in T satisfies

$$\frac{d}{dt}(S_H + I_H + H_p) = \frac{dP_H}{dt} = 0$$

and

$$\frac{d}{dt}(S_V + I_V) = 0$$

Therefore, all paths approaches T. Therefore, it is enough to study the asymptotic behavior of solutions of (2.1) in invariant set T

2.2.1 Positivity of solutions

To show that $S_H \geq 0, I_H \geq 0, H_P \geq 0, S_V \geq 0, I_v \geq 0$

$$\begin{aligned}
\frac{dS_H}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H \\
\frac{dS_H}{dt} &\geq -\frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H \\
\frac{dS_H}{dt} &\geq -(\frac{\gamma_H}{P_H} b I_v + \eta_H) S_H \\
\frac{dS_H}{S_H} &\geq -(\frac{\gamma_H}{P_H} b I_v + \eta_H) dt
\end{aligned}$$

$$\ln S_H \geq -(\frac{\gamma_H}{P_H} b I_v + \eta_H)t + c^*$$
$$S_H \geq c\exp(-(\frac{\gamma_H}{P_H} b I_v + \eta_H)t)$$
$$\therefore \boxed{S_H \geq 0}$$

$$\frac{dI_H}{dt} = \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \lambda_H) I_H$$
$$\frac{dI_H}{dt} \geq -(\eta_H + \lambda_H) I_H$$
$$\frac{dI_H}{I_H} \geq -(\eta_H + \lambda_H) dt$$
$$\ln I_H \geq -(\eta_H + \lambda_H)t + c^*$$
$$I_H \geq c\exp(-(\eta_H + \lambda_H)t)$$
$$\therefore \boxed{I_H \geq 0}$$

$$\frac{dH_P}{dt} = \lambda_H I_H - \eta_H H_P$$
$$\frac{dH_P}{dt} \geq -\eta_H H_P$$
$$\frac{dH_P}{H_P} \geq -\eta_H dt$$
$$\ln H_P \geq -\eta_H t + c^*$$
$$H_P \geq c\exp(-\eta_H t)$$
$$\therefore \boxed{H_P \geq 0}$$

$$\frac{dS_V}{dt} = A - \frac{\lambda_B}{P_H} b S_V I_H - \eta_v S_v$$

$$\frac{dS_V}{dt} \geq -\frac{\lambda_B}{P_H} b S_V I_H - \eta_v S_v$$
$$\frac{dS_V}{dt} \geq -(\frac{\lambda_B}{P_H} b I_H + \eta_v) S_v$$
$$\frac{dS_V}{S_V} \geq -(\frac{\lambda_B}{P_H} b I_H + \eta_v) dt$$
$$\ln S_V \geq -(\frac{\lambda_B}{P_H} b I_H + \eta_v) t + c^*$$
$$S_V \geq c \exp(-(\frac{\lambda_B}{P_H} b I_H + \eta_v) t)$$

$$\therefore \boxed{S_V \geq 0}$$

$$\frac{dI_v}{dt} = \frac{\lambda_B}{P_H} b S_V I_H - \eta_v I_v$$
$$\frac{dI_v}{dt} \geq -\eta_v I_v$$
$$\frac{dI_v}{I_v} \geq -\eta_v dt$$
$$\ln I_v \geq -\eta_v t + c^*$$
$$I_v \geq c \exp(-\eta_v t)$$

$$\therefore \boxed{I_v \geq 0}$$

$$\therefore S_H \geq 0, I_H \geq 0, H_P \geq 0, S_V \geq 0, I_v \geq 0$$

2.2.2 Boundedness of the solution

To show that $\Omega = \{(S_H, I_H, H_P, S_V, I_V) \in \mathbb{R}^5_+;\ P_H = \text{constant}, P_V \leq \frac{A}{\eta_v}\}$

$$\frac{dP_H}{dt} = \frac{dS_H}{dt} + \frac{dI_H}{dt} + \frac{dH_P}{dt}$$
$$\frac{dP_H}{dt} = \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H$$

$$
\begin{aligned}
&+\frac{\gamma_H}{P_H}bS_H I_v-(\eta_H+\lambda_H)I_H\\
&+\lambda_H I_H-\eta_H H_P\\
\frac{dP_H}{dt}&=\eta_H P_H-\eta_H(S_H+I_H+H_P)\\
\frac{dP_H}{dt}&=\eta_H P_H-\eta_H P_H\\
\frac{dP_H}{dt}&=0\\
\therefore P_H&=\text{constant}
\end{aligned}
$$

also,

$$
\begin{aligned}
\frac{dP_V}{dt}&=\frac{dS_V}{dt}+\frac{dI_V}{dt}\\
\frac{dP_V}{dt}&=A-\frac{\lambda_B}{P_H}bS_V I_H-\eta_v S_v\\
&\quad+\frac{\lambda_B}{P_H}bS_V I_H-\eta_v I_v\\
\frac{dP_V}{dt}&=A-\eta_v(S_V+I_V)\\
\frac{dP_V}{dt}&=A-\eta_v P_V
\end{aligned}
$$

Now,$A-\eta_v P_V\geq 0$ if $A\geq\eta_v P_V$

which is $\frac{A}{\eta_v}\geq P_V$

i.e $\frac{A}{\eta_v}\geq S_V+I_V\geq 0$

Therefore it can be concluded that P_V is bounded as

$\frac{A}{\eta_v}\geq P_V\geq 0$

Therefore $\frac{A}{\eta_v}$ is an upperbound of P_V

Therefore $\Omega=\{(S_H,I_H,H_P,S_V,I_V)\in\mathbb{R}^5_+;\ P_H=\text{constant}, P_V\leq\frac{A}{\eta_v}\}$

2.2.3 Reduction of system of differential equations

Substituting,

$$x = \frac{S_H}{P_H}, \quad y = \frac{I_H}{P_H}, \quad u = \frac{H_P}{P_H}, \quad v = \frac{S_V}{\frac{A}{\eta_v}}, \quad z = \frac{I_V}{\frac{A}{\eta_v}} \tag{2.2}$$

we get,

$$x + y + u = 1, \quad v + z = 1$$

consider the equation ,

$$\frac{dS_H}{dt} = \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H$$

then substituting (2.2) in the above equation:

$$\begin{aligned}
\frac{d(xP_H)}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b(xP_H)(\frac{A}{\eta_v} z) - \eta_H (xP_H) \\
\frac{dx}{dt} &= \eta_H - \frac{\gamma_H bA}{P_H \eta_v} xz - \eta_H x \\
\frac{dx}{dt} &= \eta_H (1 - x) - \frac{\gamma_H bA}{P_H \eta_v} xz \\
\text{Let } \alpha_1 &= \frac{\gamma_H bA}{P_H \eta_V}
\end{aligned}$$

$$\therefore \frac{dx}{dt} = \eta_H (1 - x) - \alpha_1 xz \tag{2.3}$$

Consider the equation,

$$\frac{dI_H}{dt} = \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \lambda_H) I_H$$

then substituting (2.2) in the above equation:

$$\begin{aligned}
\frac{d(yP_H)}{dt} &= \frac{\gamma_H}{P_H} b(xP_H)(\frac{A}{\eta_v} z) - (\eta_H + \lambda_H)(yP_H) \\
\frac{dy}{dt} &= \frac{\gamma_H bA}{P_H \eta_v} xz - (\eta_H + \lambda_H) y \\
\text{Let } \alpha_2 &= \eta_H + \lambda_H
\end{aligned}$$

$$\therefore \frac{dy}{dt} = \alpha_1 xz - \alpha_2 y \tag{2.4}$$

Consider the equation,

$$\frac{dH_P}{dt} = \lambda_H I_H - \eta_H H_P$$

then substituting (2.2) in the above equation:

$$\begin{aligned}
\frac{d(uP_H)}{dt} &= \lambda_H(yP_H) - \eta_H(uP_H) \\
\frac{du}{dt} &= \lambda_H y - \eta_H u \\
\frac{d(1-x-y)}{dt} &= \lambda_H y - \eta_H(1-x-y) \\
-\frac{dx}{dt} - \frac{dy}{dt} &= \lambda_H y - \eta_H(1-x-y) \\
\frac{dx}{dt} + \frac{dy}{dt} &= -\lambda_H y + \eta_H - x\eta_H - y\eta_H \\
\eta_H(1-x) - \alpha_1 xz + \alpha_1 xz - \alpha_2 y &= -\lambda_H y + \eta_H - x\eta_H - y\eta_H \\
\therefore 0 &= 0
\end{aligned}$$

Therefore this equation is eliminated

Consider the equation,

$$\frac{dS_V}{dt} = A - \frac{\lambda_B}{P_H} bS_V I_H - \eta_v S_v$$

then substituting (2.2) in the above equation:

$$\begin{aligned}
\frac{d(\frac{A}{\eta_v}v)}{dt} &= A - \frac{\lambda_B}{P_H} b(\frac{A}{\eta_v})((yP_H)v) - \eta_v(\frac{A}{\eta_v}v) \\
\frac{dv}{dt} &= \eta_v - \frac{\lambda_B}{P_H} b((yP_H)v) - \eta_v v \\
\frac{dv}{dt} &= -\lambda_B bvy + \eta_v(1-v)
\end{aligned}$$

$$\frac{dv}{dt} = -\lambda_B bvy + \eta_v z$$

$$\text{Let}\alpha_3 = b\lambda_B \quad \text{and} \quad \alpha_4 = \eta_V$$

$$\frac{dv}{dt} = -\alpha_3 vy + \alpha_4 z$$

$$\frac{d(1-z)}{dt} = -\alpha_3(1-z)y + \alpha_4 z$$

$$\therefore \frac{dz}{dt} = \alpha_3(1-z)y - \alpha_4 z \tag{2.5}$$

Consider the equation,

$$\frac{dI_v}{dt} = \frac{\lambda_B}{P_H} bS_V I_H - \eta_v I_v$$

then substituting (2.2) in the above equation:

$$\frac{d(\frac{A}{\eta_v}z)}{dt} = \frac{\lambda_B}{P_H} b(\frac{A}{\eta_v})((yP_H)v) - \eta_v(\frac{A}{\eta_v}z)$$

$$\frac{dz}{dt} = b\lambda_B(1-z)y - \eta_v z$$

$$\therefore \frac{dz}{dt} = \alpha_3(1-z)y - \alpha_4 z$$

Therefore, the system is reduced to:

$$\begin{aligned}
\frac{dx}{dt} &= \eta_H(1-x) - \alpha_1 xz \\
\frac{dy}{dt} &= \alpha_1 xz - \alpha_2 y \\
\frac{dz}{dt} &= \alpha_3(1-z)y - \alpha_4 z
\end{aligned} \tag{2.6}$$

Where ,

$$\alpha_1 = \frac{\gamma_H bA}{P_H \eta_V},$$
$$\alpha_2 = \eta_H + \lambda_H,$$
$$\alpha_3 = b\lambda_B,$$
$$\alpha_4 = \eta_V$$

2.2.4 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$$

$$\therefore \eta_H(1-x) - \alpha_1 xz = 0$$
$$\eta_H - \eta_H x - \alpha_1 xz = 0$$
$$\eta_H = x(\eta_H + \alpha_1 z)$$

$$x = \frac{\eta_H}{\eta_H + \alpha_1 z} \tag{2.7}$$

$$\therefore \alpha_1 xz - \alpha_2 y = 0$$
$$\alpha_1 xz = \alpha_2 y$$
$$y = \frac{\alpha_1 xz}{\alpha_2}$$

$$y = \frac{\eta_H \alpha_1 z}{\alpha_2(\eta_H + \alpha_1 z)} \tag{2.8}$$

$$\alpha_3(1-z)y - \alpha_4 z = 0$$

$$\alpha_3(1-z)(\frac{\eta_H \alpha_1 z}{\alpha_2(\eta_H + \alpha_1 z)}) - \alpha_4 z = 0$$

$$(\alpha_3(1-z)\frac{\eta_H \alpha_1}{\alpha_2(\eta_H + \alpha_1 z)} - \alpha_4)z = 0$$

$$\therefore z = 0 \quad \text{or} \quad \frac{\alpha_1 \alpha_3}{\alpha_2}(1-z)\frac{\eta_H}{\eta_H + \alpha_1 z} - \alpha_4 = 0$$

$$\therefore z = 0 \quad \text{or} \quad \frac{\alpha_1 \alpha_3}{\alpha_2}(1-z)\frac{\eta_H}{\eta_H + \alpha_1 z} = \alpha_4$$

$$\therefore z = 0 \quad \text{or} \quad \alpha_1 \alpha_3(1-z)\eta_H = \alpha_2 \alpha_4(\eta_H + \alpha_1 z)$$

$$\therefore z = 0 \quad \text{or} \quad \alpha_1 \alpha_3 \eta_H - \alpha_1 \alpha_3 \eta_H z = \alpha_2 \alpha_4 \eta_H + \alpha_1 \alpha_2 \alpha_4 \eta_H z$$

$$\therefore z = 0 \quad \text{or} \quad z = \frac{(\alpha_1 \alpha_3 - \alpha_2 \alpha_4)\eta_H}{\alpha_1(\eta_H \alpha_3 + \alpha_2 \alpha_4)} \tag{2.9}$$

If $z = 0$ then $x = 1$ and $y = 0$

$\therefore$ disease-free equilibrium point is $D_1 = (1, 0, 0)$

$$\text{If} \quad z = \frac{(\alpha_1 \alpha_3 - \alpha_2 \alpha_4)\eta_H}{\alpha_1(\eta_H \alpha_3 + \alpha_2 \alpha_4)} \quad \text{then,}$$

$$x = \frac{\eta_H}{\eta_H + \alpha_1\left(\dfrac{(\alpha_1\alpha_3 - \alpha_2\alpha_4)\eta_H}{\alpha_1(\eta_H\alpha_3 + \alpha_2\alpha_4)}\right)}$$

$$x = \frac{1}{1 + \dfrac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\eta_H\alpha_3 + \alpha_2\alpha_4}}$$

$$x = \frac{\eta_H\alpha_3 + \alpha_2\alpha_4}{\eta_H\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_3 - \alpha_2\alpha_4}$$

Therefore

$$x = \frac{\eta_H\alpha_3 + \alpha_2\alpha_4}{\eta_H\alpha_3 + +\alpha_1\alpha_3}$$

$$y = \frac{\eta_H\alpha_1(\dfrac{(\alpha_1\alpha_3 - \alpha_2\alpha_4)\eta_H}{\alpha_1(\eta_H\alpha_3 + \alpha_2\alpha_4)})}{\alpha_2(\eta_H + \alpha_1(\dfrac{(\alpha_1\alpha_3 - \alpha_2\alpha_4)\eta_H}{\alpha_1(\eta_H\alpha_3 + \alpha_2\alpha_4)})}$$

$$y = \frac{\eta_H(\dfrac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\eta_H\alpha_3 + \alpha_2\alpha_4})}{\alpha_2(1 + \dfrac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\eta_H\alpha_3 + \alpha_2\alpha_4})}$$

$$y = \frac{\eta_H(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_2(\eta_H\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_3 - \alpha_2\alpha_4)}$$

Therefore

$$y = \frac{\eta_H(\alpha_1\alpha_3 - \alpha_2\alpha_4)}{\alpha_2\alpha_3(\eta_H + \alpha_1)}$$

$\therefore$ endemic equilibrium point is

$$D_2 = (x_0, y_0, z_0) = \left(\frac{\eta_H \alpha_3 + \alpha_2 \alpha_4}{\eta_H \alpha_3 + + \alpha_1 \alpha_3}, \frac{\eta_H(\alpha_1 \alpha_3 - \alpha_2 \alpha_4)}{\alpha_2 \alpha_3 (\eta_H + \alpha_1)}, \frac{(\alpha_1 \alpha_3 - \alpha_2 \alpha_4)\eta_H}{\alpha_1(\eta_H \alpha_3 + \alpha_2 \alpha_4)} \right)$$

Therefore, we get two equilibrium points $D_1 = (1,0,0)$ and $D_2 = (x_0, y_0, z_0)$.

2.2.5 Reproduction number

We use next generation matrix method to find reproduction number. In F we put those terms from equation (2.6) which helps in growing secondary infection and in V we put all other terms with opposite signs.

$$F = \begin{bmatrix} -\alpha_1 xz \\ \alpha_1 xz \\ \alpha_3(1-z)y \end{bmatrix}, V = \begin{bmatrix} -\eta_H(1-x) \\ \alpha_2 y \\ \alpha_4 z \end{bmatrix}$$

Then by taking the jacobian of above we get,

$$J_F = \begin{bmatrix} -\alpha_1 z & 0 & -\alpha_1 x \\ \alpha_1 z & 0 & \alpha_1 x \\ 0 & \alpha_3(1-z) & \alpha_3 y \end{bmatrix}, J_V = \begin{bmatrix} \eta_H & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix} 0 & 0 & -\alpha_1 \\ 0 & 0 & \alpha_1 \\ 0 & \alpha_3 & 0 \end{bmatrix}, J_{V(D_1)} = \begin{bmatrix} \eta_H & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix}$$

$$J_{V(D_1)}^{-1} = \frac{1}{\eta_H \alpha_2 \alpha_4} \begin{bmatrix} \alpha_2\alpha_4 & 0 & 0 \\ 0 & -\eta_H\alpha_4 & 0 \\ 0 & 0 & -\eta_H\alpha_2 \end{bmatrix}$$

$$J_{V(D_1)}^{-1} = \begin{bmatrix} \frac{1}{\eta_H} & 0 & 0 \\ 0 & -\frac{1}{\alpha_2} & 0 \\ 0 & 0 & -\frac{1}{\alpha_4} \end{bmatrix}$$

$$B = J_{F(D_1)} J_{V(D_1)}^{-1} = \begin{bmatrix} 0 & 0 & \frac{\alpha_1}{\alpha_4} \\ 0 & 0 & -\frac{\alpha_1}{\alpha_4} \\ 0 & -\frac{\alpha_3}{\alpha_2} & 0 \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(B - \lambda I) = \left| B - \lambda I \right| = 0$$

Therefore, we have

$$\begin{vmatrix} -\lambda & 0 & \frac{\alpha_1}{\alpha_4} \\ 0 & -\lambda & -\frac{\alpha_1}{\alpha_4} \\ 0 & -\frac{\alpha_3}{\alpha_2} & -\lambda \end{vmatrix} = 0$$

then the characteristic equation of the above matrix we get,

$$\lambda(\lambda^2 - \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}) = 0$$

$$\therefore \lambda = 0 \quad \text{or} \quad \lambda = \pm\sqrt{\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}}$$

So, the reproduction number (R_0) is determined by the greatest eigenvalue:

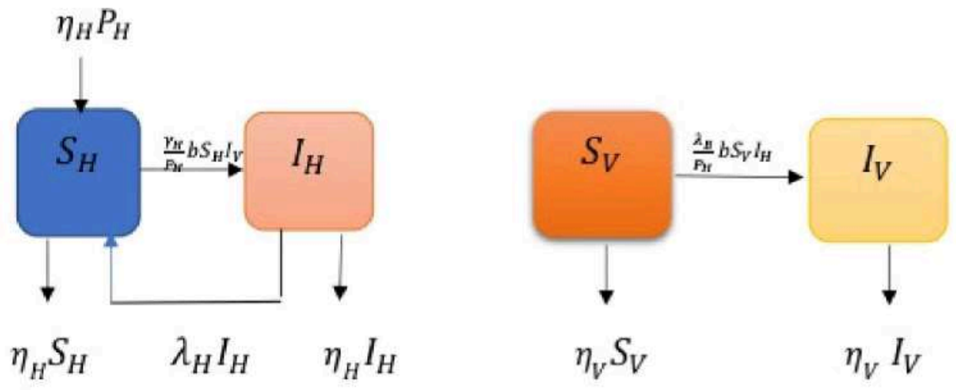
$$R_0 = \sqrt{\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}}$$

2.3 Transmision model excluding immunity

2.3.1 Formation of model

Let P_H and P_V represent the sizes of the human and vector populations, respectively. We consider that the human population remains constant over time. This implies that the birth rate and death rate of the human population are equal, both denoted by η_H.
The vector population is assumed to be governed by $\boxed{\frac{dP_v}{dt} = A - n_v P_v}$
To get equilibirium point we equate $\frac{dP_v}{dt} = 0$
$\therefore A - n_v P_v = 0$
$\therefore P_v = \frac{A}{N_V}$
Therefore vector approaches to $\frac{A}{N_V}$ as t tends to infinity.
The host population is divided into susceptible individuals denoted by S_H, infected individuals denoted by I_H. The vector population, due to its short lifespan, is divided into susceptible vectors represented by S_V and infected vectors represented by I_V.
Here we take into account that there are various type of viruses identified regarding dengue .If a person is recovered from dengue it does not imply that person is immune to other type of dengue .And we assume the immune sub population is negligible. Schematic

diagram of the model is :



Model involving both populations are

$$\begin{aligned}
\frac{dS_H}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H + \lambda_H I_H \\
\frac{dI_H}{dt} &= \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \lambda_H) I_H \\
\frac{dS_V}{dt} &= A - \frac{\lambda_B}{P_H} b S_V I_H - \eta_v S_v \\
\frac{dI_v}{dt} &= \frac{\lambda_B}{P_H} b S_V I_H - \eta_v I_v
\end{aligned} \tag{2.10}$$

with two conditions $P_H = S_H + I_H$ and $P_V = S_V + I_V$

where ,

P_H =Total host population

S_H = number of susceptible in the host population

I_H = number of infected in the host population

P_V = Total vector population

S_V = number of susceptible in the vector population

I_V = number of infected in the vector population

η_H= The birth or death rate in the host population

η_v= The death rate in the vector population

γ_H = The transmission probability from vector to host

λ_H = The rate at which infected become susceptible

λ_B = The transmission probability from host to vector

b = The biting rate of the vector

A = The recruitment rate.

The subset T defined by the equations $S_H + I_H = P_H$ and $S_V + I_V = \frac{A}{\eta_v}$ is an invariant region for system (2.10), because any solution starting in T satisfies

$$\frac{d}{dt}(S_H + I_H) = \frac{dP_H}{dt} = 0$$

and

$$\frac{d}{dt}(S_V + I_V) = 0$$

Therefore, all paths approaches T.

Therefore, it is enough to study the asymptotic behavior of solutions of (2.10) in invariant set T

2.3.2 Positivity of solutions

To show that $S_H \geq 0, I_H \geq 0, S_V \geq 0, I_v \geq 0$

$$\frac{dS_H}{dt} = \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H + \lambda_H I_H$$
$$\frac{dS_H}{dt} \geq -\frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H$$

$$\begin{aligned}
\frac{dS_H}{dt} &\geq -(\frac{\gamma_H}{P_H} bI_v + \eta_H)S_H \\
\frac{dS_H}{S_H} &\geq -(\frac{\gamma_H}{P_H} bI_v + \eta_H)dt \\
\ln S_H &\geq -(\frac{\gamma_H}{P_H} bI_v + \eta_H)t + c^* \\
S_H &\geq c\exp(-(\frac{\gamma_H}{P_H} bI_v + \eta_H)t)
\end{aligned}$$

$$\therefore \boxed{S_H \geq 0}$$

$$\begin{aligned}
\frac{dI_H}{dt} &= \frac{\gamma_H}{P_H} bS_H I_v - (\eta_H + \lambda_H)I_H \\
\frac{dI_H}{dt} &\geq -(\eta_H + \lambda_H)I_H \\
\frac{dI_H}{I_H} &\geq -(\eta_H + \lambda_H)dt \\
\ln I_H &\geq -(\eta_H + \lambda_H)t + c^* \\
I_H &\geq c\exp(-(\eta_H + \lambda_H)t)
\end{aligned}$$

$$\therefore \boxed{I_H \geq 0}$$

$$\begin{aligned}
\frac{dS_V}{dt} &= A - \frac{\lambda_B}{P_H} bS_V I_H - \eta_v S_v \\
\frac{dS_V}{dt} &\geq -\frac{\lambda_B}{P_H} bS_V I_H - \eta_v S_v \\
\frac{dS_V}{dt} &\geq -(\frac{\lambda_B}{P_H} bI_H + \eta_v)S_v \\
\frac{dS_V}{S_V} &\geq -(\frac{\lambda_B}{P_H} bI_H + \eta_v)dt \\
\ln S_V &\geq -(\frac{\lambda_B}{P_H} bI_H + \eta_v)t + c^* \\
S_V &\geq c\exp(-(\frac{\lambda_B}{P_H} bI_H + \eta_v)t)
\end{aligned}$$

$$\therefore \boxed{S_V \geq 0}$$

$$\begin{aligned}
\frac{dI_v}{dt} &= \frac{\lambda_B}{P_H} b S_V I_H - \eta_v I_v \\
\frac{dI_v}{dt} &\geq -\eta_v I_v \\
\frac{dI_v}{I_v} &\geq -\eta_v dt \\
\ln I_v &\geq -\eta_v t + c^* \\
I_v &\geq c \exp(-\eta_v t)
\end{aligned}$$

$$\therefore \boxed{I_v \geq 0}$$

$$\therefore S_H \geq 0, I_H \geq 0, S_V \geq 0, I_v \geq 0$$

2.3.3 Boundedness of the solution

To show that $\Omega = \{(S_H, I_H, S_V, I_V) \in \mathbb{R}^4_+;\ P_H = \text{constant}, P_V \leq \frac{A}{\eta_v}\}$

$$\begin{aligned}
\frac{dP_H}{dt} &= \frac{dS_H}{dt} + \frac{dI_H}{dt} \\
\frac{dP_H}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H + \lambda_H I_H \\
&\quad + \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \lambda_H) I_H \\
\frac{dP_H}{dt} &= \eta_H P_H - \eta_H (S_H + I_H) \\
\frac{dP_H}{dt} &= \eta_H P_H - \eta_H P_H \\
\frac{dP_H}{dt} &= 0
\end{aligned}$$

$$\therefore P_H = \text{constant}$$

also,

$$\begin{aligned}
\frac{dP_V}{dt} &= \frac{dS_V}{dt} + \frac{dI_V}{dt} \\
\frac{dP_V}{dt} &= A - \frac{\lambda_B}{P_H} bS_V I_H - \eta_v S_v \\
&\quad + \frac{\lambda_B}{P_H} bS_V I_H - \eta_v I_v \\
\frac{dP_V}{dt} &= A - \eta_v (S_V + I_V) \\
\frac{dP_V}{dt} &= A - \eta_v P_V
\end{aligned}$$

Now, $A - \eta_v P_V \geq 0$ if $A \geq \eta_v P_V$

which is $\frac{A}{\eta_v} \geq P_V$

i.e $\frac{A}{\eta_v} \geq S_V + I_V \geq 0$

Therefore it can be concluded that P_V is bounded as

$\frac{A}{\eta_v} \geq P_V \geq 0$

Therefore $\frac{A}{\eta_v}$ is an upperbound of P_V

Therefore $\Omega = \{(S_H, I_H, H_P, S_V, I_V) \in \mathbb{R}^5_+;\ P_H = \text{constant}, P_V \leq \frac{A}{\eta_v}\}$

2.3.4 Reduction of system of differential equations

Substituting,

$$x = \frac{S_H}{P_H}, \quad y = \frac{I_H}{P_H}, \quad v = \frac{S_V}{\frac{A}{\eta_v}}, \quad z = \frac{I_V}{\frac{A}{\eta_v}} \tag{2.11}$$

we get,

$$x + y = 1 \quad , \quad v + z = 1$$

consider the equation ,

$$\frac{dS_H}{dt} = \eta_H P_H - \frac{\gamma_H}{P_H} b S_H I_v - \eta_H S_H + \lambda_H I_H$$

then substituting (2.11) in the above equation:

$$\begin{aligned}
\frac{d(xP_H)}{dt} &= \eta_H P_H - \frac{\gamma_H}{P_H} b(xP_H)\left(\frac{A}{\eta_v}z\right) - \eta_H(xP_H) \\
&\quad + \lambda_H(yP_H) \\
\frac{dx}{dt} &= \eta_H - \frac{\gamma_H bA}{P_H\eta_v} xz - \eta_H x + \lambda_H y \\
\frac{dx}{dt} &= \eta_H(1-x) - \frac{\gamma_H bA}{P_H\eta_v} xz + \lambda_H y \\
\frac{d(1-y)}{dt} &= \eta_H y - \frac{\gamma_H bA}{P_H\eta_v}(1-y)z + \lambda_H y \\
\frac{dy}{dt} &= \frac{\gamma_H bA}{P_H\eta_v}(1-y)z - (\lambda_H + \eta_H)y \\
\text{Let } \alpha_1 &= \frac{\gamma_H bA}{P_H\eta_V} \\
\text{and} \quad \alpha_2 &= \lambda_H + \eta_H
\end{aligned}$$

$$\frac{dy}{dt} = \alpha_1(1-y)z - \alpha_2 y \tag{2.12}$$

Consider the equation,

$$\frac{dI_H}{dt} = \frac{\gamma_H}{P_H} b S_H I_v - (\eta_H + \gamma_H) I_H$$

then substituting (2.11) in the above equation:

$$\begin{aligned}
\frac{d(yP_H)}{dt} &= \frac{\gamma_H}{P_H} b(xP_H)(\frac{A}{\eta_v}z) - (\eta_H + \gamma_H)(yP_H) \\
\frac{dy}{dt} &= \frac{\gamma_H bA}{P_H\eta_v} xz - (\eta_H + \gamma_H)y
\end{aligned}$$

$$\frac{dy}{dt} = \alpha_1(1-y)z - \alpha_2 y \tag{2.13}$$

Consider the equation,

$$\frac{dS_V}{dt} = A - \frac{\lambda_B}{P_H} b S_V I_H - \eta_v S_v$$

then substituting (2.11) in the above equation:

$$\begin{aligned}
\frac{d(\frac{A}{\eta_v}v)}{dt} &= A - \frac{\lambda_B}{P_H} b(\frac{A}{\eta_v})((yP_H)v) - \eta_v(\frac{A}{\eta_v}v) \\
\frac{dv}{dt} &= \eta_v - \frac{\lambda_B}{P_H} b((yP_H)v) - \eta_v v \\
\frac{dv}{dt} &= -\lambda_B bvy + \eta_v(1-v) \\
\frac{dv}{dt} &= -\lambda_B bvy + \eta_v z \\
\text{Let} \alpha_3 &= b\lambda_B \quad \text{and} \quad \alpha_4 = \eta_V \\
\frac{dv}{dt} &= -\alpha_3 vy + \alpha_4 z \\
\frac{d(1-z)}{dt} &= -\alpha_3(1-z)y + \alpha_4 z
\end{aligned}$$

$$\frac{dz}{dt} = \alpha_3(1-z)y - \alpha_4 z \tag{2.14}$$

Consider the equation,

$$\frac{dI_v}{dt} = \frac{\lambda_B}{P_H} b S_V I_H - \eta_v I_v$$

then substituting (2.11) in the above equation:

$$\begin{aligned}
\frac{d(\frac{A}{\eta_v}z)}{dt} &= \frac{\lambda_B}{P_H} b(\frac{A}{\eta_v})((yP_H)v) - \eta_v(\frac{A}{\eta_v}z) \\
\frac{dz}{dt} &= b\lambda_B(1-z)y - \eta_v z \\
\frac{dz}{dt} &= \alpha_3(1-z)y - \alpha_4 z
\end{aligned}$$

Therefore, the system is reduced to:

$$\begin{aligned}\frac{dy}{dt} &= \alpha_1(1-y)z - \alpha_2 y \\ \frac{dz}{dt} &= \alpha_3(1-z)y - \alpha_4 z\end{aligned} \tag{2.15}$$

Where ,

$$\alpha_1 = \frac{\gamma_H bA}{P_H \eta_V},$$

$$\alpha_2 = \eta_H + \lambda_H,$$

$$\alpha_3 = b\lambda_B,$$

$$\alpha_4 = \eta_V$$

2.3.5 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\frac{dy}{dt} = \frac{dz}{dt} = 0$$

$$\therefore \alpha_1(1-y)z - \alpha_2 y = 0$$

$$z = \frac{\alpha_2 y}{\alpha_1(1-y)} \tag{2.16}$$

$$\alpha_3(\alpha_1(1-y) - \alpha_2 y)y - \alpha_4 \alpha_2 y = 0$$

$$\alpha_3(1-z)y-\alpha_4 z=0$$

$$\alpha_3(\alpha_1-\alpha_1 y-\alpha_2 y)y-\alpha_4\alpha_2 y=0$$

$$y(\alpha_1\alpha_3-\alpha_1\alpha_3 y-\alpha_2\alpha_3 y-\alpha_2\alpha_4)=0$$

$$\therefore y=0 \quad \text{or} \quad y=\alpha_1\alpha_3-\alpha_1\alpha_3 y-\alpha_2\alpha_3 y-\alpha_2\alpha_4=0$$

$$\therefore y=0 \quad \text{or} \quad y=\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_1\alpha_3+\alpha_2\alpha_4}$$

$$\therefore y=0 \quad \text{or} \quad y=\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}=0 \tag{2.17}$$

If $y=0$ then $z=0$

$\therefore$ disease-free equilibrium point is $D_1=(0,0)$

$$\text{If} \quad y=\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)} \quad \text{then,}$$

$$z=\frac{\alpha_2\left(\dfrac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}\right)}{\alpha_1\left(1-\left(\dfrac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}\right)\right)}$$

$$z=\frac{\alpha_2(\alpha_1\alpha_3-\alpha_2\alpha_4)}{\alpha_1(\alpha_1\alpha_3-\alpha_2\alpha_3-\alpha_1\alpha_3+\alpha_2\alpha_4)}$$

$$z=\frac{\alpha_2(\alpha_1\alpha_3-\alpha_2\alpha_4)}{\alpha_1\alpha_2(\alpha_3+\alpha_4)}$$

Therefore

$$z=\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_1(\alpha_3+\alpha_4)}$$

$\therefore$ endemic equilibrium point is

$$D_2=(y_0,z_0)=\left(\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)},\frac{\alpha_1\alpha_3-\alpha_2\alpha_4}{\alpha_1(\alpha_3+\alpha_4)}\right)$$

Therefore, we get two equilibrium points $D_1 = (0,0)$ and $D_2 = (y_0, z_0)$.

2.3.6 Reproduction number

We use next generation matrix method to find reproduction number. In F we put those terms from equation (2.15) which helps in growing secondary infection and in V we put all other terms with opposite signs.

$$F = \begin{bmatrix} \alpha_1(1-y)z \\ \alpha_3(1-z)y \end{bmatrix}, \; V = \begin{bmatrix} \alpha_2 y \\ \alpha_4 z \end{bmatrix}$$

Then by taking the jacobian of above, we get

$$J_F = \begin{bmatrix} \alpha_1 z & \alpha_1(1-y) \\ \alpha_3(1-z) & \alpha_3 y \end{bmatrix}, \; J_V = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_4 \end{bmatrix}$$

Then evaluating above with respect to disease free equilibrium point, we get

$$J_{F(D_1)} = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_3 & 0 \end{bmatrix}, \; J_{V(D_1)} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_4 \end{bmatrix}$$

$$J_{V(D_1)}^{-1} = \frac{1}{\alpha_2 \alpha_4} \begin{bmatrix} \alpha_4 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$

$$J_{V(D_1)}^{-1} = \begin{bmatrix} \frac{1}{\alpha_2} & 0 \\ 0 & \frac{1}{\alpha_4} \end{bmatrix}$$

$$B = J_{F(D_1)} J_{V(D_1)}^{-1} = \begin{bmatrix} 0 & \dfrac{\alpha_1}{\alpha_4} \\ \dfrac{\alpha_3}{\alpha_2} & 0 \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(B - \lambda I) = \left| B - \lambda I \right| = 0$$

Therefore, we have

$$\begin{vmatrix} -\lambda & \dfrac{\alpha_1}{\alpha_4} \\ \dfrac{\alpha_3}{\alpha_2} & -\lambda \end{vmatrix} = 0$$

then the characteristic equation of the above matrix we get,

$$\lambda^2 - \frac{\alpha_1 \alpha_3}{\alpha_2 \alpha_4} = 0$$

$$\therefore \quad \lambda = \pm \sqrt{\frac{\alpha_1 \alpha_3}{\alpha_2 \alpha_4}}$$

So, the reproduction number (R_0) is determined by the greatest eigenvalue:

$$R_0 = \sqrt{\frac{\alpha_1 \alpha_3}{\alpha_2 \alpha_4}}$$

2.3.7 Linearisation of system with respect to equilibrium points

Jacobian of the linearised system is denoted as J:

$$J = \begin{bmatrix} -\alpha_1 z - \alpha_2 & \alpha_1 - \alpha_1 y \\ \alpha_3 - \alpha_3 z & -\alpha_3 y - \alpha_4 \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_1 = (0,0)$ yields the Jacobian matrix:

$$J_1 = \begin{bmatrix} -\alpha_2 & \alpha_1 \\ \alpha_3 & -\alpha_4 \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_2 = (y_0, z_0)$ yields the Jacobian matrix:

$$J_2 = \begin{bmatrix} -\alpha_1(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_1(\alpha_3+\alpha_4)}) - \alpha_2 & \alpha_1 - \alpha_1(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}) \\ \alpha_3 - \alpha_3(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_1(\alpha_3+\alpha_4)}) & -\alpha_3(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}) - \alpha_4 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4}{\alpha_3+\alpha_4} - \alpha_2 & \alpha_1 - \alpha_1(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_3(\alpha_1+\alpha_2)}) \\ \alpha_3 - \alpha_3(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_1(\alpha_3+\alpha_4)}) & \frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4}{\alpha_1+\alpha_2} - \alpha_4 \end{bmatrix}$$

consider

$$\frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4}{\alpha_3+\alpha_4} - \alpha_2$$

$$= \frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_2\alpha_3 - \alpha_2\alpha_4}{\alpha_3 + \alpha_4}$$

$$= -\alpha_3(\frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4})$$

consider

$$\alpha_1 - \alpha_1(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_3(\alpha_1 + \alpha_2)})$$

$$= \frac{\alpha_3\alpha_1^2 + \alpha_1\alpha_2\alpha_3 - \alpha_3\alpha_1^2 + \alpha_1\alpha_2\alpha_4}{\alpha_3(\alpha_1 + \alpha_2)}$$

$$= \frac{\alpha_1\alpha_2}{\alpha_3}(\frac{\alpha_3 + \alpha_4}{\alpha_1 + \alpha_2})$$

consider

$$\alpha_3 - \alpha_3(\frac{\alpha_1\alpha_3 - \alpha_2\alpha_4}{\alpha_1(\alpha_3 + \alpha_4)})$$

$$= \frac{\alpha_3^2\alpha_1 + \alpha_1\alpha_3\alpha_4 - \alpha_3^2\alpha_1 + \alpha_2\alpha_3\alpha_4}{\alpha_1(\alpha_3 + \alpha_4)}$$

$$= \frac{\alpha_3\alpha_4}{\alpha_1}(\frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4})$$

consider

$$\frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4}{\alpha_1 + \alpha_2} - \alpha_4$$

$$= \frac{-\alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_1\alpha_4 - \alpha_2\alpha_4}{\alpha_1 + \alpha_2}$$

$$= -\alpha_1(\frac{\alpha_1 + \alpha_4}{\alpha_1 + \alpha_2})$$

$$\therefore J_2 = \begin{bmatrix} -\alpha_3\left(\frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4}\right) & \frac{\alpha_1\alpha_2}{\alpha_3}\left(\frac{\alpha_3 + \alpha_4}{\alpha_1 + \alpha_2}\right) \\ \frac{\alpha_3\alpha_4}{\alpha_1}\left(\frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4}\right) & -\alpha_1\left(\frac{\alpha_1 + \alpha_4}{\alpha_1 + \alpha_2}\right) \end{bmatrix}$$

2.3.8 Stability Analysis

Theorem 2.3.8.1. *The Dengue disease-free equilibrium* $D_1 = (0,0)$ *of the system is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. : In order for the equilibrium point of the system to be stable, the eigenvalues must be negative or have a negative real part.

Denote $J_1 = B = \begin{bmatrix} -\alpha_2 & \alpha_1 \\ \alpha_3 & -\alpha_4 \end{bmatrix}$

We can get the characteristic equation by solving

$$\det(B - \lambda I) = \left| B - \lambda I \right| = 0$$

Therefore,

$$\begin{vmatrix} -\alpha_2 - \lambda & \alpha_1 \\ \alpha_3 & -\alpha_4 - \lambda \end{vmatrix} = 0$$

Therefore,

$$\lambda^2 + (\alpha_2 + \alpha_4)\lambda + \alpha_2\alpha_4 - \alpha_1\alpha_3 = 0$$

is the characteristic equation.

Using Rauth Hurwitz criterion ,a second degree polynomial with all coefficients positive will obviously have negative roots

$$\text{clearly} \quad \alpha_2 + \alpha_4 > 0 \quad \text{and}$$

$$\begin{aligned}\alpha_2\alpha_4 - \alpha_1\alpha_3 > 0 &\iff \\ \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4} < 1 &\iff \\ \sqrt{\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}} < 1 &\iff \\ R_0 < 1 &\end{aligned}$$

$\therefore$ Disease free equilibrium point is locally asmptotically stable iff $R_0 < 1$,otherwise it is unstable. □

Theorem 2.3.8.2. *The Dengue disease-present equilibrium (endemic equilibrium)* $D_2 = (y_0, z_0)$ *of the system is asymptotically stable if* $R_0 > 1$ *and unstable if* $R_0 < 1$.

Proof. : Denote

$$J_2 = C = \begin{bmatrix} -\alpha_3\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right) & \frac{\alpha_1\alpha_2}{\alpha_3}\left(\frac{\alpha_3+\alpha_4}{\alpha_1+\alpha_2}\right) \\ \frac{\alpha_3\alpha_4}{\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right) & -\alpha_1\left(\frac{\alpha_1+\alpha_4}{\alpha_1+\alpha_2}\right) \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(C - \lambda I) = \left|C - \lambda I\right| = 0$$

Therefore,

$$\begin{vmatrix} -\alpha_3\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right) - \lambda & \frac{\alpha_1\alpha_2}{\alpha_3}\left(\frac{\alpha_3+\alpha_4}{\alpha_1+\alpha_2}\right) \\ \frac{\alpha_3\alpha_4}{\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right) & -\alpha_1\left(\frac{\alpha_1+\alpha_4}{\alpha_1+\alpha_2}\right) - \lambda \end{vmatrix} = 0$$

$$\therefore (\lambda+\alpha_3\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right))(\lambda+\alpha_1\left(\frac{\alpha_1+\alpha_4}{\alpha_1+\alpha_2}\right))-\frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_3\alpha_1}\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right)\left(\frac{\alpha_1+\alpha_4}{\alpha_1+\alpha_2}\right)$$

$$\therefore \lambda^2+(\alpha_1\left(\frac{\alpha_1+\alpha_4}{\alpha_1+\alpha_2}\right)+\alpha_3\left(\frac{\alpha_1+\alpha_2}{\alpha_3+\alpha_4}\right))\lambda+\alpha_1\alpha_3-\alpha_2\alpha_4=0$$

$$\therefore \lambda^2+(\frac{\alpha_1(\alpha_3+\alpha_4)^2+\alpha_3(\alpha_1+\alpha_2)^2}{(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)})\lambda+\alpha_1\alpha_3-\alpha_2\alpha_4=0$$

then the characteristic equation of above matrix we get

$\lambda^2+a_0\lambda+\alpha_1\alpha_3-\alpha_2\alpha_4=0$

where $a_0=\dfrac{\alpha_1(\alpha_3+\alpha_4)^2+\alpha_3(\alpha_1+\alpha_2)^2}{(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)}$

Using Rauth Hurwitz criterion ,a second degree polynomial with all coefficients positive will obviously have negative roots

Clearly we can see that $a_0>0$ and

$$\begin{aligned}\alpha_1\alpha_3-\alpha_2\alpha_4>0 &\iff\\ \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}>1 &\iff\\ \sqrt{\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}}>1 &\iff\end{aligned}$$

$$R_0 > 1$$

$\therefore$ Endemic equilibrium point is locally asmptotically stable iff $R_0 > 1$,otherwise it is unstable. □

Chapter 3

Analysing of Tuberculosis

3.1 Introduction

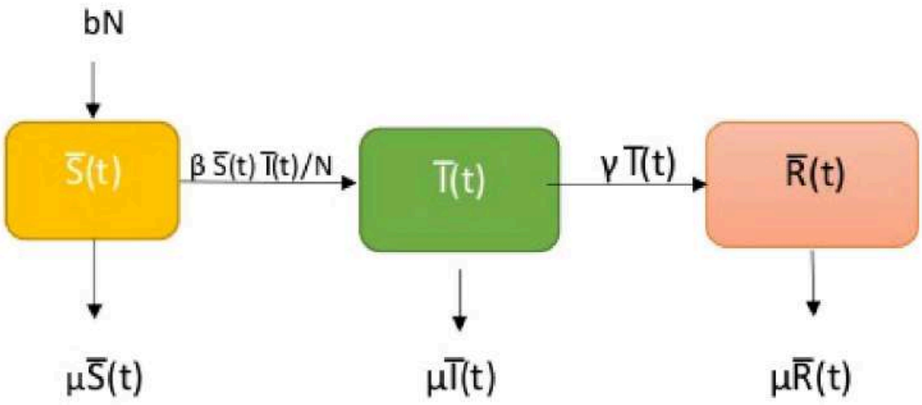
The paper discusses the impact of Tuberculosis (TB) on health, politics, and the economy, emphasizing the importance of mathematical models in understanding and controlling epidemics. It traces the history of epidemic modeling, particularly focusing on TB, and highlights the role of vaccination in preventing the spread of the disease. The study introduces three epidemiological models (modified SIR, SEIR, and BSEIR) to analyze TB dynamics in Turkey, considering factors like birth and death rates, exposed individuals, and prevention. The models are calibrated using real data from the World Health Organization, revealing that the basic reproduction number (R_0) for all models is less than 1, indicating controlled disease spread. The stability analysis demonstrate that the model incorporating vaccination (BSEIR) provides realistic predictions. The first study on the mathematical modelling of the spread of disease was proposed by

Bernoulli in 1766[12].Kermack and McKendrick[13] proposed a deterministic model to describe SIR.Chavez and Feng [14] focused on four models to understand the disease transmission dynamics of TB. Yali Yang et[15] evaluated the cost of control strategies by using an SEIR model. Side[16] proposed a SIR and an SEIR models for TB and analysed these models. Zhang [17]set up a new mathematical model for TB in China using the data from January 2005 to December 2012. Xu [18]proposed a mathematical model to investigate the control and precautions in Guangdong of China. Liu [19] proposed a mixed vaccination strategy that is the combination of constant vaccination and pulse vaccination. Egbetade and Ibrahim [20]set up a new mathematical model incorporating treatment, migration and vaccination. Rangkuti [21]explained the spread of TB in North Sumatera Indonesia using VSEIR, which was created by adding the vaccination parameter to the SEIR model. Egonmwan[22] formulated a mathematical model that incorporates vaccination of newborn children and older susceptible individuals into the transmission dynamics of TB in a population.

3.2 Mathematical Modelling of Tuberculosis through SIR model

3.2.1 Formation of model

Assuming the total population remains constant over time which implies that the birth rate and death rate are same. This model contains three compartments susceptible,infected,recovered. susceptible are those individuals who are not infected but they can get the infection,infected are those individuals who has the tb infection and can transmit the disease,Recovered are those who are recovered from tb and are immune to the disease.



$\therefore$ Nonlinear system of differential equations is

$$\begin{aligned}\frac{d\bar{S}(t)}{dt} &= bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)\\ \frac{d\bar{I}(t)}{dt} &= \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\gamma+\mu)\bar{I}(t)\\ \frac{d\bar{R}(t)}{dt} &= \gamma\bar{I}(t) - \mu\bar{R}(t)\end{aligned} \tag{3.1}$$

with initial conditions

$\bar{S}(0) \geq 0, \bar{I}(0) \geq 0,\ \bar{R}(0) \geq 0$

where,

$$N = \text{Total population}$$

$$\bar{S}(t) = \text{number of susceptible individuals}$$

$$\bar{I}(t) = \text{number of infected individuals}$$

$$\bar{R}(t) = \text{number of recovered individuals}$$

$$b = \text{The birth rate of the population}$$

$$\mu = \text{The death rate of the population}$$

$$\beta = \text{The transmission rate}$$

$$\gamma = \text{The recovery rate}$$

Total population is denoted by $N = \bar{S}(t) + \bar{I}(t) + \bar{R}(t)$

β is the rate at which disease is transmitted from infected individuals to susceptible.γ is the rate at which infected individuals are recovered.vertical transmission are not taken into account in this model.

3.2.2 Positivity of solutions

Theorem 3.2.2.1. *If $\bar{S}(0) \geq 0$, $\bar{I}(0) \geq 0$, $\bar{R}(0) \geq 0$, then the solutions of the system of equations $\bar{S}(t)$, $\bar{I}(t)$, $\bar{R}(t)$ are positive for all $t \geq 0$.*

Proof. :

$$\frac{d\bar{S}(t)}{dt} = bN - \frac{\beta \bar{S}(t)\bar{I}(t)}{N} - \mu \bar{S}(t)$$
$$\frac{d\bar{S}(t)}{dt} \geq -\frac{\beta \bar{S}(t)\bar{I}(t)}{N} - \mu \bar{S}(t)$$
$$\frac{d\bar{S}(t)}{dt} \geq -(\frac{\beta \bar{I}(t)}{N} + \mu)\bar{S}(t)$$
$$\frac{d\bar{S}(t)}{\bar{S}(t)} \geq -(\frac{\beta \bar{I}(t)}{N} + \mu)dt$$
$$\ln \bar{S}(t) \geq -(\frac{\beta \bar{I}(t)}{N} + \mu)t + c^*$$

$$\bar{S}(t) \geq c\exp(-(\frac{\beta\bar{I}(t)}{N}+\mu)t)$$
$$\therefore \boxed{\bar{S}(t) \geq 0}$$

$$\frac{d\bar{I}(t)}{dt} = \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\gamma+\mu)\bar{I}(t)$$
$$\frac{d\bar{I}(t)}{dt} \geq -(\gamma+\mu)\bar{I}(t)$$
$$\frac{d\bar{I}(t)}{\bar{I}(t)} \geq -(\gamma+\mu)dt$$
$$\ln\bar{I}(t) \geq -(\gamma+\mu)t + c^*$$
$$\bar{I}(t) \geq c\exp(-(\gamma+\mu)t)$$
$$\therefore \boxed{\bar{I}(t) \geq 0}$$

$$\frac{d\bar{R}(t)}{dt} = \gamma\bar{I}(t) - \mu\bar{R}(t)$$
$$\frac{d\bar{R}(t)}{dt} \geq -\mu\bar{R}(t)$$
$$\frac{d\bar{R}(t)}{\bar{R}(t)} \geq -\mu dt$$
$$\ln\bar{R}(t) \geq -\mu t + c^*$$
$$\bar{R}(t) \geq c\exp(-\mu t)$$
$$\therefore \boxed{\bar{R}(t) \geq 0}$$

$$\therefore \bar{S}(t) \geq 0, \bar{I}(t) \geq 0, \bar{R}(t) \geq 0$$

□

3.2.3 Boundedness of the solution

Theorem 3.2.3.1. *All feasible solutions $\bar{S}(t)$, $\bar{I}(t)$, $\bar{R}(t)$ of the system of equations are bounded by the region* $\Omega = \{(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \in \mathbb{R}^3_+;\ N = constant\}$.

Proof. :

$$\frac{dN}{dt} = \frac{d\bar{S}(t)}{dt} + \frac{d\bar{I}(t)}{dt} + \frac{d\bar{R}(t)}{dt}$$
$$\frac{dN}{dt} = bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t) + \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\gamma+\mu)\bar{I}(t) + \gamma\bar{I}(t) - \mu\bar{R}(t)$$
$$\frac{dN}{dt} = bN - \mu(\bar{S}(t) + \bar{I}(t) + \bar{R}(t))$$
$$\frac{dN}{dt} = bN - \mu N$$
$$\frac{dN}{dt} = (b-\mu)N$$
$$\frac{dN}{dt} = 0$$
$$\therefore N = constant$$
$$\therefore \Omega = \{(\bar{S}(t), \bar{I}(t), \bar{R}(t)) \in \mathbb{R}^3_+; N = constant\} \text{is feasible region.}$$

□

3.2.4 Reduction of system of differential equations

Substituting,

$$S(t) = \frac{\bar{S}(t)}{N}, \quad I(t) = \frac{\bar{I}(t)}{N}, \quad R(t) = \frac{\bar{R}(t)}{N} \tag{3.2}$$

we get,

$$1 = S(t) + I(t) + R(t)$$

Consider the equation,

$$\frac{d\bar{S}(t)}{dt} = bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)$$

then substituting (3.2) in the above equation:

$$\begin{aligned}\frac{d(NS(t))}{dt} &= bN - \frac{\beta(NS(t))(NI(t))}{N} - \mu NS(t)\\ \frac{S(t)}{dt} &= b - \beta S(t)I(t) - \mu S(t)\\ \frac{S(t)}{dt} &= \mu - \beta S(t)I(t) - \mu S(t)\\ \frac{S(t)}{dt} &= -\beta S(t)I(t) + \mu(1 - S(t))\end{aligned}$$

Consider the equation,

$$\frac{d\bar{I}(t)}{dt} = \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\gamma+\mu)\bar{I}(t)$$

then substituting (3.2) in the above equation:

$$\begin{aligned}\frac{d(NI(t))}{dt} &= \frac{\beta(NS(t))(NI(t))}{N} - (\gamma+\mu)NI(t)\\ \frac{I(t)}{dt} &= \beta S(t)I(t) - (\gamma+\mu)I(t)\end{aligned}$$

Consider the equation,

$$\frac{d\bar{R}(t)}{dt} = \gamma\bar{I}(t) - \mu\bar{R}(t)$$

then substituting (3.2) in the above equation:

$$\begin{aligned}\frac{d(NR(t))}{dt} &= \gamma(NI(t)) - \mu(NR(t))\\ \frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t)\end{aligned}$$

$$\begin{aligned}
\frac{d(1-S(t)-I(t))}{dt} &= \gamma I(t) - \mu(1-S(t)-I(t)) \\
-\frac{dS(t)}{dt} - \frac{dI(t)}{dt} &= \gamma I(t) - \mu + \mu S(t) + \mu I(t) \\
\beta S(t)I(t) - \mu(1-S(t)) - \beta S(t)I(t) + (\gamma+\mu)I(t) &= \gamma I(t) - \mu + \mu S(t) + \mu I(t) \\
\therefore 0 &= 0
\end{aligned}$$

Therefore, the system is reduced to:

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta S(t)I(t) + \mu(1-S(t)) \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - (\gamma+\mu)I(t)
\end{aligned} \tag{3.3}$$

3.2.5 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \frac{dI(t)}{dt} = 0 \\
-\beta S(t)I(t) + \mu(1-S(t)) &= 0 \\
-\beta S(t)I(t) + \mu - \mu S(t) &= 0 \\
\mu &= S(t)(\beta I(t) + \mu)
\end{aligned}$$

$$\therefore S(t) = \frac{\mu}{\beta I(t) + \mu} \tag{3.4}$$

also,

$$I(t) = \frac{\mu(1-S(t))}{\beta S(t)} \tag{3.5}$$

$$\beta S(t)I(t) - (\gamma+\mu)I(t) = 0$$
$$(\beta S(t) - (\gamma+\mu))I(t) = 0$$
$$\therefore I(t) = 0 \text{ or } \beta S(t) - (\gamma+\mu) = 0$$

$$\therefore I(t) = 0 \text{ or } S(t) = \frac{\gamma+\mu}{\beta} \tag{3.6}$$

$$\text{If } I(t) = 0, \text{ then}$$
$$S(t) = 1$$
$$\therefore \text{disease-free equilibrium point is } D_1 = (1,0)$$

$$\text{If } S(t) = \frac{\gamma+\mu}{\beta}, \text{ then}$$
$$I(t) = \frac{\mu\left(1-\frac{\gamma+\mu}{\beta}\right)}{\beta\left(\frac{\gamma+\mu}{\beta}\right)}$$
$$\therefore I(t) = \frac{\mu(\beta-\gamma-\mu)}{\beta(\mu+\gamma)}$$
$$\therefore \text{endemic equilibrium point is } D_2 = (S_2^*, I_2^*)$$
$$\text{where } S_2^* = \frac{\gamma+\mu}{\beta}, \text{ and } I_2^* = \frac{\mu(\beta-\gamma-\mu)}{\beta(\mu+\gamma)}$$

$\therefore$ we get two equilirium points $D_1 = (S_1^*, I_1^*) = (1,0)$ and $D_2 = (S_2^*, I_2^*)$

3.2.6 Reproduction number

We use the next generation matrix method to find the reproduction number. In F we put those terms from equation (3.3) which help in growing secondary infection, and in V we put all other terms with opposite signs.

$$F=\begin{bmatrix}-\beta S(t)I(t)\\ \beta S(t)I(t)\end{bmatrix},\ V=\begin{bmatrix}-\mu(1-S(t))\\ (\gamma+\mu)I(t)\end{bmatrix}$$

$$J_F=\begin{bmatrix}-\beta I(t) & -\beta S(t)\\ \beta I(t) & \beta S(t)\end{bmatrix},\ J_V=\begin{bmatrix}\mu & 0\\ 0 & \gamma+\mu\end{bmatrix}$$

$$J_{F(D_1)}=\begin{bmatrix}0 & -\beta\\ 0 & \beta\end{bmatrix},\ J_{V(D_1)}=\begin{bmatrix}\mu & 0\\ 0 & \gamma+\mu\end{bmatrix}$$

$$J^{-1}_{V(D_1)}=\begin{bmatrix}\frac{1}{\mu} & 0\\ 0 & \frac{1}{\gamma+\mu}\end{bmatrix}$$

$$A=J_{F(D_1)}J^{-1}_{V(D_1)}=\begin{bmatrix}0 & -\dfrac{\beta}{\mu+\gamma}\\ 0 & \dfrac{\beta}{\mu+\gamma}\end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(A-\lambda I)=\begin{vmatrix}A-\lambda I\end{vmatrix}=0$$

$$\begin{vmatrix} -\lambda & -\dfrac{\beta}{\mu+\gamma} \\ 0 & \dfrac{\beta}{\mu+\gamma}-\lambda \end{vmatrix} = 0$$

then the characteristic equation of the above matrix we get,

$\lambda^2 - \dfrac{\beta}{\mu+\gamma}\lambda = 0$

$\lambda(\lambda - \dfrac{\beta}{\mu+\gamma}) = 0$

$\therefore \lambda = 0$ or $\lambda = \dfrac{\beta}{\mu+\gamma}$

So we choose the greatest eigenvalue to get the reproduction number as $R_0 = \dfrac{\beta}{\mu+\gamma}$

3.2.7 Linearisation of system with respect to equilibrium points

The Jacobian of the linearized system is denoted as $J = \begin{bmatrix} -\beta I(t) - \mu & -\beta S(t) \\ \beta I(t) & \beta - \gamma - \mu \end{bmatrix}$

Linearisation of the system with respect to equilibrium point $D_1 = (S_1^*, I_1^*) = (1,0)$ gives the Jacobian matrix

$J_1 = \begin{bmatrix} -\mu & -\beta \\ 0 & \beta - \gamma - \mu \end{bmatrix}$

Linearisation of the system with respect to equilibrium point $D_2 = (S_2^*, I_2^*)$ gives the Jacobian matrix

$J_2 = \begin{bmatrix} -\dfrac{\mu\beta(\beta-\gamma-\mu)}{\beta(\mu+\gamma)} - \mu & -\frac{\beta(\gamma+\mu)}{\beta} \\ \dfrac{\mu\beta(\beta-\gamma-\mu)}{\beta(\mu+\gamma)} & \frac{\beta(\gamma+\mu)}{\beta} - \gamma - \mu \end{bmatrix}$

$$J_2 = \begin{bmatrix} -\dfrac{\mu(\beta-\gamma-\mu+\mu+\gamma)}{\mu+\gamma} & -\gamma-\mu \\ \dfrac{\mu(\beta-\gamma-\mu)}{\mu+\gamma} & 0 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -\dfrac{\mu\beta}{\mu+\gamma} & -\gamma-\mu \\ \dfrac{\mu(\beta-\gamma-\mu)}{\mu+\gamma} & 0 \end{bmatrix}$$

3.2.8 Stability Analysis

In order to the equilibrium point of the system to be stable ,the eigenvalue must be negative or have negative real part.

Denote $J_1 = B = \begin{bmatrix} -\mu & -\beta \\ 0 & \beta-\gamma-\mu \end{bmatrix}$

We can get the characteristic equation by solving

$$\det(B-\lambda I) = \left|B-\lambda I\right| = 0$$

$$\begin{vmatrix} -\mu-\lambda & -\beta \\ 0 & \beta-\gamma-\mu-\lambda \end{vmatrix} = 0$$

$$\therefore -(\mu+\lambda)(\beta-\gamma-\mu-\lambda) = 0$$

$$\therefore -\mu\beta+\mu\gamma+\mu^2+\mu\lambda-\beta\lambda+\gamma\lambda+\mu\lambda+\lambda^2 = 0$$

$$\therefore \lambda^2+\lambda(2\mu-\beta+\gamma)\lambda+\mu^2+\mu\gamma-\mu\beta = 0 \text{is the characteristic equation.}$$

Using Rauth Hurwitz criterion ,a second degree polynomial with all coefficients positive will obviously have negative roots

$$\mu^2 + \mu\gamma - \mu\beta > 0 \iff$$
$$\mu + \gamma - \beta > 0 \iff$$
$$\frac{\beta}{\mu + \gamma} < 1 \iff$$
$$R_0 < 1$$

Also,

$$2\mu - \beta + \gamma > 0 \iff$$
$$\mu - \beta + \gamma > 0 \iff$$
$$\frac{\beta}{\mu + \gamma} < 1 \iff$$
$$R_0 < 1$$

Therefore, the disease-free equilibrium point is locally asymptotically stable if and only if $R_0 < 1$; otherwise, it is unstable.

Denote $J_2 = C = \begin{bmatrix} -\frac{\mu\beta}{\mu+\gamma} & -\gamma - \mu \\ \frac{\mu(\beta-\gamma-\mu)}{\mu+\gamma} & 0 \end{bmatrix}$

We can get the characteristic equation by solving

$$\det(C - \lambda I) = \left| C - \lambda I \right| = 0$$

Therefore,

$$\begin{vmatrix} -\frac{\mu\beta}{\mu+\gamma} - \lambda & -\gamma - \mu \\ \frac{\mu(\beta-\gamma-\mu)}{\mu+\gamma} & -\lambda \end{vmatrix} = 0$$

The characteristic equation is:

$$\lambda^2 + \frac{\mu\beta}{\mu+\gamma}\lambda + \mu(\mu+\gamma)\frac{\beta-\gamma-\mu}{\mu+\gamma} = 0$$

Alternatively, it can be written as:

$$\lambda^2 + \frac{\mu\beta}{\mu+\gamma}\lambda + \mu(\beta-\gamma-\mu) = 0$$

Using Rauth Hurwitz criterion ,a second degree polynomial with all coefficients positive will obviously have negative roots

$$\begin{aligned} \mu(\beta-\gamma-\mu) > 0 &\iff \\ \beta-\gamma-\mu > 0 &\iff \\ \frac{\beta}{\mu+\gamma} > 1 &\iff \\ R_0 > 1 & \end{aligned}$$

also,

$$\frac{\mu\beta}{\mu+\gamma} > 0$$

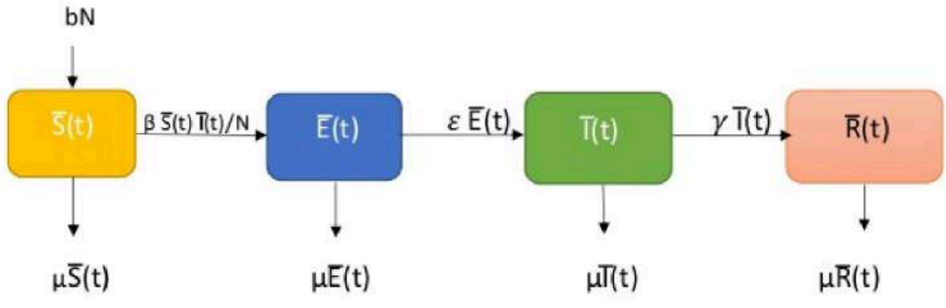
$\therefore$ Endemic equilibrium point is locally asymptotically stable iff $R_0 > 1$,otherwise it is unstable.

3.3 Mathematical Modelling of Tuberculosis through SEIR model

3.3.1 Formation of model

Assuming the total population remains constant over time which implies that the birth rate and death rate are same.

This model contains four compartments susceptible,exposed,infected,recovered. susceptible are those individuals who are not infected but they can get the infection,exposed are those individuals who are infected but symptoms are not seen and they cannot spread disease as of now,infected are those individuals who has the tb infection and can transmit the disease,Recovered are those who are recovered from tb and are immune to the disease. Schematic diagram of the model is:



$\therefore$ Nonlinear system of differential equations is

$$\begin{aligned}
\frac{d\bar{S}(t)}{dt} &= bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)\\
\frac{d\bar{E}(t)}{dt} &= \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\varepsilon+\mu)\bar{E}(t)\\
\frac{d\bar{I}(t)}{dt} &= \varepsilon\bar{E}(t) - (\gamma+\mu)\bar{I}(t)\\
\frac{d\bar{R}(t)}{dt} &= \gamma\bar{I}(t) - \mu\bar{R}(t)
\end{aligned} \tag{3.7}$$

with initial conditions

$\bar{S}(0) \geq 0, \bar{E}(0) \geq 0, \bar{I}(0) \geq 0,\ \bar{R}(0) \geq 0$

where,

$N =$ Total population

$\bar{S}(t) =$ number of susceptible individuals

$\bar{E}(t) =$ number of exposed individuals

$\bar{I}(t) =$ number of infected individuals

$\bar{R}(t) =$ number of recovered individuals

$b =$ The birth rate of the population

$\mu =$ The death rate of the population

$\beta =$ The transmission rate

$\varepsilon =$ rate at which the exposed individuals become infected

$\gamma =$ The recovery rate

Total population is denoted by $N = \bar{S}(t) + \bar{E}(t) + \bar{I}(t) + \bar{R}(t)$

β is the rate at which disease is transmitted from infected individuals to susceptible.γ is the rate at which infected individuals are recovered.ε is the rate at which exposed individuals become infected.vertical transmission are not taken into account in this model.

3.3.2 Positivity of solutions

Theorem 3.3.2.1. *If* $\bar{S}(0) \geq 0$, $\bar{E}(0) \geq 0$, $\bar{I}(0) \geq 0$, $\bar{R}(0) \geq 0$, *then the solutions of the system of equations* $\bar{S}(t)$,$\bar{E}(t)$, $\bar{I}(t)$, $\bar{R}(t)$ *are positive for all* $t \geq 0$.

Proof. :

$$\frac{d\bar{S}(t)}{dt} = bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)$$

$$\frac{d\bar{S}(t)}{dt} \geq -\frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)$$

$$\frac{d\bar{S}(t)}{dt} \geq -(\frac{\beta\bar{I}(t)}{N} + \mu)\bar{S}(t)$$

$$\frac{d\bar{S}(t)}{\bar{S}(t)} \geq -(\frac{\beta\bar{I}(t)}{N} + \mu)dt$$

$$\ln\bar{S}(t) \geq -(\frac{\beta\bar{I}(t)}{N} + \mu)t + c^*$$

$$\bar{S}(t) \geq c\exp(-(\frac{\beta\bar{I}(t)}{N} + \mu)t)$$

$$\therefore \boxed{\bar{S}(t) \geq 0}$$

$$\frac{d\bar{E}(t)}{dt} = \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\varepsilon + \mu)\bar{E}(t)$$

$$\frac{d\bar{E}(t)}{dt} \geq -(\varepsilon+\mu)\bar{E}(t)$$
$$\frac{d\bar{E}(t)}{\bar{E}(t)} \geq -(\varepsilon+\mu)dt$$
$$\ln\bar{E}(t) \geq -(\varepsilon+\mu)t+c^*$$
$$\bar{E}(t) \geq c\exp(-(\varepsilon+\mu)t)$$
$$\therefore \boxed{\bar{E}(t) \geq 0}$$

$$\frac{d\bar{I}(t)}{dt} = \varepsilon\bar{E}(t)-(\gamma+\mu)\bar{I}(t)$$
$$\frac{d\bar{I}(t)}{dt} \geq -(\gamma+\mu)\bar{I}(t)$$
$$\frac{d\bar{I}(t)}{\bar{I}(t)} \geq -(\gamma+\mu)dt$$
$$\ln\bar{I}(t) \geq -(\gamma+\mu)t+c^*$$
$$\bar{I}(t) \geq c\exp(-(\gamma+\mu)t)$$
$$\therefore \boxed{\bar{I}(t) \geq 0}$$

$$\frac{d\bar{R}(t)}{dt} = \gamma\bar{I}(t)-\mu\bar{R}(t)$$
$$\frac{d\bar{R}(t)}{dt} \geq -\mu\bar{R}(t)$$
$$\frac{d\bar{R}(t)}{\bar{R}(t)} \geq -\mu dt$$
$$\ln\bar{R}(t) \geq -\mu t+c^*$$
$$\bar{R}(t) \geq c\exp(-\mu t)$$
$$\therefore \boxed{\bar{R}(t) \geq 0}$$

$$\therefore \bar{S}(t) \geq 0, \bar{E}(t) \geq 0, \bar{I}(t) \geq 0, \bar{R}(t) \geq 0$$

□

3.3.3 Boundedness of the solution

Theorem 3.3.3.1. *All feasible solutions $\bar{S}(t)$,$\bar{E}(t)$, $\bar{I}(t)$, $\bar{R}(t)$ of the system of equations are bounded by the region $\Omega = \{(\bar{S}(t), \bar{E}(t), \bar{I}(t), \bar{R}(t)) \in \mathbb{R}^4_+;\ N = constant\}$.*

Proof. :

$$\frac{dN}{dt} = \frac{d\bar{S}(t)}{dt} + \frac{d\bar{E}(t)}{dt} + \frac{d\bar{I}(t)}{dt} + \frac{d\bar{R}(t)}{dt}$$

$$\frac{dN}{dt} = bN - \frac{\beta \bar{S}(t)\bar{I}(t)}{N} - \mu \bar{S}(t) + \frac{\beta \bar{S}(t)\bar{I}(t)}{N} - (\varepsilon + \mu)\bar{E}(t) + \varepsilon \bar{E}(t) - (\gamma + \mu)\bar{I}(t)$$

$$+ \gamma \bar{I}(t) - \mu \bar{R}(t)$$

$$\frac{dN}{dt} = bN - \mu(\bar{S}(t) + \bar{E}(t) + \bar{I}(t) + \bar{R}(t))$$

$$\frac{dN}{dt} = bN - \mu N$$

$$\frac{dN}{dt} = (b - \mu)N$$

$$\frac{dN}{dt} = 0$$

$$\therefore N = constant$$

$\therefore \Omega = \{(\bar{S}(t), \bar{E}(t), \bar{I}(t), \bar{R}(t)) \in \mathbb{R}^4_+; N = constant\}$is feasible region.

□

3.3.4 Reduction of system of differential equations

Substituting,

$$S(t) = \frac{\bar{S}(t)}{N}, \quad E(t) = \frac{\bar{E}(t)}{N}, \quad I(t) = \frac{\bar{I}(t)}{N}, \quad R(t) = \frac{\bar{R}(t)}{N} \tag{3.8}$$

we get,

$$1 = S(t) + E(t) + I(t) + R(t)$$

Consider the equation,

$$\frac{d\bar{S}(t)}{dt} = bN - \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - \mu\bar{S}(t)$$

then substituting (3.8) in the above equation:

$$\begin{aligned}
\frac{d(NS(t))}{dt} &= bN - \frac{\beta(NS(t))(NI(t))}{N} - \mu NS(t) \\
\frac{S(t)}{dt} &= b - \beta S(t)I(t) - \mu S(t) \\
\frac{S(t)}{dt} &= \mu - \beta S(t)I(t) - \mu S(t) \\
\frac{S(t)}{dt} &= -\beta S(t)I(t) + \mu(1 - S(t))
\end{aligned}$$

Consider the equation,

$$\frac{d\bar{E}(t)}{dt} = \frac{\beta\bar{S}(t)\bar{I}(t)}{N} - (\varepsilon + \mu)\bar{E}(t)$$

then substituting (3.8) in the above equation:

$$\begin{aligned}
\frac{dNE(t)}{dt} &= \frac{\beta(NS(t))(NI(t))}{N} - (\varepsilon + \mu)NE(t) \\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - (\varepsilon + \mu)E(t)
\end{aligned}$$

Consider the equation,

$$\frac{d\bar{I}(t)}{dt} = \varepsilon\bar{E}(t) - (\gamma+\mu)\bar{I}(t)$$

then substituting (3.8) in the above equation:

$$\begin{aligned}\frac{dNI(t)}{dt} &= \varepsilon NE(t) - (\gamma+\mu)NI(t)\\ \frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma+\mu)I(t)\end{aligned}$$

Consider the equation,

$$\frac{d\bar{R}(t)}{dt} = \gamma\bar{I}(t) - \mu\bar{R}(t)$$

then substituting (3.8) in the above equation:

$$\begin{aligned}\frac{d(NR(t))}{dt} &= \gamma(NI(t)) - \mu(NR(t))\\ \frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t)\\ \frac{d(1-S(t)-I(t))}{dt} &= \gamma I(t) - \mu(1-S(t)-I(t))\\ -\frac{dS(t)}{dt} - \frac{dI(t)}{dt} &= \gamma I(t) - \mu + \mu S(t) + \mu I(t)\\ \beta S(t)I(t) - \mu(1-S(t)) - \beta S(t)I(t) + (\gamma+\mu)I(t) &= \gamma I(t) - \mu + \mu S(t) + \mu I(t)\\ \therefore 0 &= 0\end{aligned}$$

Therefore, the system is reduced to

$$\begin{aligned}\frac{dS(t)}{dt} &= -\beta S(t)I(t) + \mu(1-S(t)) \\ \frac{dE(t)}{dt} &= \beta S(t)I(t) - (\varepsilon+\mu)E(t) \\ \frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma+\mu)I(t)\end{aligned} \tag{3.9}$$

3.3.5 Reproduction number

We use next generation matrix method to find reproduction number.

In F we put those terms from equation (3.9) which helps in growing secondary infection and in V we put all other terms with opposite signs.

$$F = \begin{bmatrix} \beta S(t)I(t) \\ 0 \end{bmatrix}, \; V = \begin{bmatrix} (\varepsilon+\mu)E(t) \\ -\varepsilon E(t) + (\gamma+\mu)I(t) \end{bmatrix}$$

$$J_F = \begin{bmatrix} 0 & \beta S(t) \\ 0 & 0 \end{bmatrix}, \; J_V = \begin{bmatrix} \varepsilon+\mu & 0 \\ -\varepsilon & \gamma+\mu \end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, \; J_{V(D_1)} = \begin{bmatrix} \varepsilon+\mu & 0 \\ -\varepsilon & \gamma+\mu \end{bmatrix}$$

$$J^{-1}_{V(D_1)} = \begin{bmatrix} \frac{1}{\varepsilon+\mu} & 0 \\ \frac{\varepsilon}{(\varepsilon+\mu)(\gamma+\mu)} & \frac{1}{\gamma+\mu} \end{bmatrix}$$

$$A = J_{F(D_1)} J_{V(D_1)}^{-1} = \begin{bmatrix} \frac{\varepsilon\beta}{(\varepsilon+\mu)(\gamma+\mu)} & \frac{\beta}{\gamma+\mu} \\ 0 & 0 \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(A - \lambda I) = \left| A - \lambda I \right| = 0$$

Therefore,

$$\begin{vmatrix} \frac{\varepsilon\beta}{(\varepsilon+\mu)(\gamma+\mu)} - \lambda & \frac{\beta}{\gamma+\mu} \\ 0 & -\lambda \end{vmatrix} = 0$$

Then the characteristic equation of the above matrix we get,

$$\lambda^2 - \frac{\varepsilon\beta}{(\varepsilon+\mu)(\gamma+\mu)}\lambda = 0$$

$$\lambda(\lambda - \frac{\varepsilon\beta}{(\mu+\gamma)(\varepsilon+\mu)}) = 0$$

$$\therefore \lambda = 0 \quad \text{or} \quad \frac{\varepsilon\beta}{(\mu+\gamma)(\varepsilon+\mu)}$$

So, the reproduction number (R_0) is determined by the greatest eigenvalue:

$$R_0 = \frac{\varepsilon\beta}{(\mu+\gamma)(\varepsilon+\mu)}$$

3.3.6 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\frac{dS(t)}{dt} = \frac{dE(t)}{dt} = \frac{dI(t)}{dt} = 0$$

$$-\beta S(t)I(t) + \mu(1 - S(t)) = 0$$

$$-\beta S(t)I(t) + \mu - \mu S(t) = 0$$

$$\mu = S(t)(\beta I(t) + \mu)$$

$$\therefore S(t) = \frac{\mu}{\beta I(t) + \mu} \tag{3.10}$$

also,

$$I(t) = \frac{\mu(1 - S(t))}{\beta S(t)} \tag{3.11}$$

$$\beta S(t)I(t) - (\varepsilon + \mu)E(t) = 0$$

$$E(t) = \frac{\beta S(t)I(t)}{\varepsilon + \mu}$$

$$E(t) = \frac{\beta(\frac{\mu}{\beta I(t) + \mu})I(t)}{\varepsilon + \mu}$$

$$E(t) = \left(\frac{\beta\mu}{\varepsilon + \mu}\right)\left(\frac{I(t)}{\beta I(t) + \mu}\right) \tag{3.12}$$

$$\varepsilon E(t) - (\gamma + \mu) I(t) = 0$$

$$E(t) = \frac{(\gamma + \mu) I(t)}{\varepsilon} \tag{3.13}$$

$$\text{also}, \quad I(t) = \frac{\varepsilon E(t)}{\gamma + \mu}$$

$$I(t) = \left(\frac{\varepsilon}{\gamma + \mu}\right)\left(\frac{\beta\mu}{\varepsilon + \mu}\right)\left(\frac{I(t)}{\beta I(t) + \mu}\right)$$

$$\therefore I(t) = 0 \text{ or } 1 = \left(\frac{\varepsilon}{\gamma + \mu}\right)\left(\frac{\beta\mu}{\varepsilon + \mu}\right)\left(\frac{1}{\beta I(t) + \mu}\right) \tag{3.14}$$

$$\therefore I(t) = 0 \quad \text{or} \quad 1 = \frac{R_0 \mu}{\beta I(t) + \mu}$$

$$\therefore I(t) = 0 \quad \text{or} \quad I(t) = \frac{\mu(R_0 - 1)}{\beta} \tag{3.15}$$

$$\text{If } I(t) = 0, \text{ then}$$

$$S(t) = 1, \quad E(t) = 0$$

$\therefore$ disease-free equilibrium point is $D_1 = (1, 0, 0)$

$$\text{If} \quad I(t) = \frac{\mu(R_0 - 1)}{\beta}, \text{ then}$$

$$E(t) = \frac{(\gamma+\mu)\left(\frac{\mu(R_0-1)}{\beta}\right)}{\varepsilon}$$

$$E(t) = \frac{\mu(R_0-1)}{\frac{\beta\varepsilon(\varepsilon+\mu)}{(\gamma+\mu)(\varepsilon+\mu)}}$$

$$\therefore E(t) = \frac{\mu(R_0-1)}{R_0(\varepsilon+\mu)} \tag{3.16}$$

$$S(t) = \frac{\mu}{\beta(\frac{\mu(R_0-1)}{\beta})+\mu}$$

$$\therefore S(t) = \frac{1}{R_0} \tag{3.17}$$

$\therefore$ endemic equilibrium point is $D_2 = (S_2^*, E_2^*, I_2^*)$

where $\quad S_2^* = \frac{1}{R_0}, \quad E_2^* = \frac{\mu(R_0-1)}{R_0(\varepsilon+\mu)}, {}_2^* = \frac{\mu(R_0-1)}{\beta}$

$\therefore$ we get two equilirium points $D_1 = (S_1^*, E_1^*, I_1^*) = (1,0,0)$ and $D_2 = (S_2^*, E_2^*, I_2^*)$

3.3.7 Linearisation of system with respect to equilibrium points

Jacobian of the linearised system is denoted as J:

$$J = \begin{bmatrix} -\beta I(t) - \mu & 0 & -\beta S(t) \\ \beta I(t) & -(\varepsilon + \mu) & \beta S(t) \\ 0 & \varepsilon & -\gamma - \mu \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_1 = (S_1^*, E_1^*, I_1^*) = (1,0,0)$ yields the Jacobian matrix:

$$J_1 = \begin{bmatrix} -\mu & 0 & -\beta \\ 0 & -(\varepsilon + \mu) & \beta \\ 0 & \varepsilon & -\gamma - \mu \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_2 = (S_2^*, E_2^*, I_2^*)$ yields the Jacobian matrix:

$$J_2 = \begin{bmatrix} -\beta\left(\frac{\mu(R_0-1)}{\beta}\right) - \mu & 0 & -\beta\left(\frac{1}{R_0}\right) \\ \beta\left(\frac{\mu(R_0-1)}{\beta}\right) & -(\varepsilon + \mu) & \beta\left(\frac{1}{R_0}\right) \\ 0 & \varepsilon & -\gamma - \mu \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -\mu R_0 & 0 & -\dfrac{\beta}{R_0} \\ \mu(R_0 - 1) & -(\varepsilon + \mu) & \dfrac{\beta}{R_0} \\ 0 & \varepsilon & -\gamma - \mu \end{bmatrix}$$

3.3.8 Stability Analysis

In order for the equilibrium point of the system to be stable, the eigenvalues must be negative or have a negative real part.

Denote $J_1 = B = \begin{bmatrix} -\mu & 0 & -\beta \\ 0 & -(\varepsilon+\mu) & \beta \\ 0 & \varepsilon & -\gamma-\mu \end{bmatrix}$

We can get the characteristic equation by solving

$$\det(B-\lambda I) = \left|B-\lambda I\right| = 0$$

Therefore,

$$\begin{vmatrix} -\mu-\lambda & 0 & -\beta \\ 0 & -(\varepsilon+\mu)-\lambda & \beta \\ 0 & \varepsilon & -\gamma-\mu-\lambda \end{vmatrix} = 0$$

$$(\mu+\lambda)((\lambda+\mu+\varepsilon)(\lambda+\mu+\gamma)-\beta\varepsilon) = 0$$

is the characteristic equation.

Therefore, $\lambda = -\mu$ or $(\lambda+\mu+\varepsilon)(\lambda+\mu+\gamma)-\beta\varepsilon = 0$

Therefore, $\lambda = -\mu$ or $\lambda^2+(\varepsilon+2\mu+\gamma)\lambda+\varepsilon\gamma+\varepsilon\mu+\mu\gamma+\mu^2-\beta\varepsilon = 0$

Therefore, $\lambda = -\mu$ or $\lambda^2+(\varepsilon+2\mu+\gamma)\lambda+\gamma(\varepsilon+\mu)+\mu(\varepsilon+\mu)-\beta\varepsilon = 0$

Therefore, $\lambda = -\mu$ or $\lambda^2+(\varepsilon+2\mu+\gamma)\lambda+(\varepsilon+\mu)(\gamma+\beta)-(\varepsilon+\mu)(\gamma+\beta)R_0 = 0$

$$\therefore \lambda = -\mu \text{ or } \lambda^2+(\varepsilon+2\mu+\gamma)\lambda+(\varepsilon+\mu)(\gamma+\beta)(1-R_0) = 0 \tag{3.18}$$

Using the Routh–Hurwitz criterion, a second degree polynomial with all coefficients positive will obviously have negative roots:

$$\begin{aligned} \varepsilon + 2\mu + \gamma &> 0 \\ \text{also } (\varepsilon+\mu)(\gamma+\beta)(1-R_0) &> 0 \iff \\ 1 - R_0 &> 0 \iff \\ R_0 &< 1 \end{aligned}$$

Therefore, the disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$; otherwise, it is unstable.

$$\text{Denote } J_2 = C = \begin{bmatrix} -\mu R_0 & 0 & -\dfrac{\beta}{R_0} \\ \mu(R_0-1) & -(\varepsilon+\mu) & \dfrac{\beta}{R_0} \\ 0 & \varepsilon & -\gamma-\mu \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(C - \lambda I) = \left| C - \lambda I \right| = 0$$

Therefore,

$$\begin{vmatrix} -\mu R_0 - \lambda & 0 & -\dfrac{\beta}{R_0} \\ \mu(R_0-1) & -(\varepsilon+\mu+\lambda) & \dfrac{\beta}{R_0} \\ 0 & \varepsilon & -(\gamma+\mu+\lambda) \end{vmatrix} = 0$$

The characteristic equation is:

$$-(\mu R_0+\lambda)((\varepsilon+\mu+\lambda)(\gamma+\mu+\lambda)-\frac{\varepsilon\beta}{R_0})-\frac{\beta}{R_0}(\mu\varepsilon(R_0-1))=0$$

$$-(\mu R_0+\lambda)((\varepsilon+\mu+\lambda)(\gamma+\mu+\lambda)-(\varepsilon+\mu)(\gamma+\mu))-\mu(\varepsilon+\mu)(\gamma+\mu)(R_0-1)=0$$

$$(\mu R_0+\lambda)((\varepsilon+\mu+\lambda)(\gamma+\mu+\lambda)-(\varepsilon+\mu)(\gamma+\mu))+\mu(\varepsilon+\mu)(\gamma+\mu)(R_0-1)=0$$

$$(\mu R_0+\lambda)((\varepsilon+\mu)(\gamma+\mu)+(\varepsilon+\mu)\lambda+(\gamma+\mu)\lambda+\lambda^2-(\varepsilon+\mu)(\gamma+\mu))+\mu(\varepsilon+\mu)(\gamma+\mu)(R_0-1)=0$$

$$(\mu R_0+\lambda)((\varepsilon+\mu)(\gamma+\mu)+(\varepsilon+\mu)\lambda+(\gamma+\mu)\lambda+\lambda^2-(\varepsilon+\mu)(\gamma+\mu))+\mu(\varepsilon+\mu)(\gamma+\mu)(R_0-1)=0$$

$$(\mu R_0+\lambda)(\lambda^2+(\varepsilon+2\mu+\gamma)\lambda)+\mu(\varepsilon+\mu)(\gamma+\mu)(R_0-1)=0$$

$$\lambda^3+(\varepsilon+2\mu+\gamma)\lambda^2+\mu R_0\lambda^2+\mu R_0(\varepsilon+2\mu+\gamma)\lambda+\mu(\mu+\varepsilon)(\mu+\gamma)(R_0-1)=0$$

$$\lambda^3+(\varepsilon+\gamma+\mu(2+R_0))\lambda^2+\mu R_0(\varepsilon+2\mu+\gamma)\lambda+\mu(\mu+\varepsilon)(\mu+\gamma)(R_0-1)=0 \quad (3.19)$$

Theorem 3.3.8.1. *(Routh-Hurwitz Stability Criterion) Given the polynomial,*

$$P(\lambda)=\lambda^n+a_1\lambda^{n-1}+\ldots+a_{n-1}\lambda+a_n;$$

where the coefficients a_i are real constants ($i=1,\ldots,n$), define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1=\begin{bmatrix}a_1\end{bmatrix};$$

$$H_2=\begin{bmatrix}a_1 & 1\\ a_3 & a_2\end{bmatrix};$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix};$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(\lambda)$ are negative or have a negative real part if and only if the determinants of all Hurwitz matrices are positive;

$$det(H_j) > 0, \quad j = 1, 2, \dots, n.$$

$\therefore$ Routh-Hurwitz Stability Criterion for $n = 3$

$$\det(H_1) = \begin{vmatrix} a_1 \end{vmatrix} = a_1 > 0$$

$$\det(H_2) = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_3 > 0$$

$$\det(H_3) = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_3(a_1 a_2 - a_3) > 0$$

Here $a_1 = \varepsilon + \gamma + \mu(2 + R_0)$

$$a_2 = \mu R_0(\varepsilon + 2\mu + \gamma)$$

$$a_3 = \mu(\mu+\varepsilon)(\mu+\gamma)(R_0-1)$$

$$\det(H_1) = \begin{vmatrix} a_1 \end{vmatrix} = a_1 = \varepsilon+\gamma+\mu(2+R_0) > 0$$

$$\det(H_2) = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_3 = (\varepsilon+\gamma+\mu(2+R_0))\mu R_0(\varepsilon+2\mu+\gamma) - \mu(\mu+\varepsilon)(\mu+\gamma)(R_0-1)$$

$$= \mu R_0((\varepsilon+2\mu+\gamma)^2 + \mu R_0(\varepsilon+2\mu+\gamma)) - \mu(\mu+\varepsilon)(\mu+\gamma)R_0 + \mu(\mu+\varepsilon)(\mu+\gamma)$$

$$= \mu R_0((\mu+\varepsilon)^2 + (\mu+\gamma)^2) + 2\mu(\mu+\varepsilon)(\mu+\gamma)R_0 - \mu(\mu+\varepsilon)(\mu+\gamma)R_0 + \mu(\mu+\varepsilon)(\mu+\gamma)$$

$$= \mu R_0((\mu+\varepsilon)^2 + (\mu+\gamma)^2) + \mu(\mu+\varepsilon)(\mu+\gamma)R_0 + \mu(\mu+\varepsilon)(\mu+\gamma) > 0$$

$$\det(H_3) = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_3(a_1a_2 - a_3)$$

$$\therefore \det(H_3) = a_3(a_1a_2 - a_3) > 0$$

$$\iff a_3 > 0$$

$$\iff \mu(\mu+\varepsilon)(\mu+\gamma)(R_0-1) > 0$$

$$\iff R_0 - 1 > 0$$

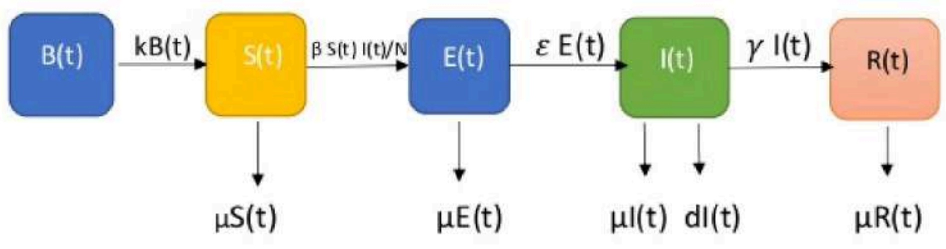
$$\iff R_0 > 1$$

$\therefore$ Endemic equilibrium point is locally asymptotically stable iff $R_0 > 1$, otherwise it is unstable.

3.4 Mathematical Modelling of Tuberculosis through BSEIR model

3.4.1 Formation of model

Here we are investigating the action of BCG vaccination on TB disease. This model consists of five compartments: BCG vaccinated, susceptible, exposed, infected, and recovered. BCG vaccinated individuals are those who have been vaccinated and are within the BCG vaccination period. Susceptible individuals are those who have not been infected but are at risk of contracting the infection. Exposed individuals are infected but do not exhibit symptoms and are not currently able to transmit the disease. Infected individuals have active TB infection and can spread the disease. Recovered individuals are those who have recuperated from TB and are immune to the disease.
Schematic diagram of the model is :



Nonlinear system of differential equations is

$$\begin{aligned}
\frac{dB(t)}{dt} &= \Lambda p - kB(t) \\
\frac{dS(t)}{dt} &= kB(t) + \Lambda(1-p) - \frac{\beta S(t)I(t)}{N} - \mu S(t) \\
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - (\varepsilon + \mu)E(t) \\
\frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma + \mu + d)I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t)
\end{aligned} \tag{3.20}$$

with initial conditions

$B(0) \geq 0;\quad S(0) \geq 0;\quad E(0) \geq 0;\quad I(0) \geq 0;\quad R(0) \geq 0$

where,

$N =$ Total population

$B(t) =$ number of BCG vaccinated individuals who are in the BCG protection period

$S(t) =$ number of susceptible individuals

$E(t) =$ number of exposed individuals

$I(t) =$ number of infected individuals

$R(t) =$ number of recovered individuals

$p =$ probability of the newborns vaccinated successfully $(0 < p < 1)$

$\Lambda =$ recrutement rate at which newborns are vaccinated

$k =$ The rate at which vaccinated individuals become susceptible again

$b =$ The birth rate of the population

$\mu =$ The natural death rate of the population

$d =$ The disease induced death rate of the population

$\beta =$ The transmission rate

$\varepsilon =$ rate at which the exposed individuals become infected

$\gamma =$ The recovery rate

Total population is denoted by $N = B(t) + S(t) + E(t) + I(t) + R(t)$
The positive effect of BCG vaccination is limited ,the vaccinated successfully individuals become suscetibe again by rate k. β is the rate at which disease is transmitted from infected individuals to susceptible.γ is the rate at which infected individuals are recovered.ε is the rate at which exposed individuals become infected.vertical transmission are not taken into account in this model.
During the BCG protection period, individuals will remain uninfected even upon exposure to infected individuals, as the vaccination confers immunity to all of them.
Given that the effectiveness of the vaccination is believed to last for approximately 10 to 15 years and the mortality rate among children is around 1%, we neglect the natural death rate within the BCG vaccinated subgroup.

3.4.2 Positivity of solutions

Theorem 3.4.2.1. *If $B(0) \geq 0$,$S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, then the solutions of the system of equations $B(t)$,$S(t)$,$E(t)$, $I(t)$, $R(t)$ are positive for all $t \geq 0$.*

Proof. :

$$\begin{aligned}
\frac{dB(t)}{dt} &= \Lambda p - kB(t) \\
\frac{dB(t)}{dt} &\geq -kB(t) \\
\frac{dB(t)}{B(t)} &\geq -kdt \\
\ln B(t) &\geq -kt + c^* \\
B(t) &\geq c\exp(-kt)
\end{aligned}$$

$$\therefore \boxed{B(t) \geq 0}$$

$$\begin{aligned}
\frac{dS(t)}{dt} &= kB(t) + \Lambda(1-p) - \frac{\beta S(t)I(t)}{N} - \mu S(t) \\
\frac{dS(t)}{dt} &\geq -\frac{\beta S(t)I(t)}{N} - \mu S(t) \\
\frac{dS(t)}{dt} &\geq -(\frac{\beta I(t)}{N} + \mu)S(t) \\
\frac{dS(t)}{S(t)} &\geq -(\frac{\beta I(t)}{N} + \mu)dt \\
\ln S(t) &\geq -(\frac{\beta I(t)}{N} + \mu)t + c^* \\
S(t) &\geq c\exp(-(\frac{\beta I(t)}{N} + \mu)t)
\end{aligned}$$

$$\therefore \boxed{S(t) \geq 0}$$

$$\begin{aligned}
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - (\varepsilon + \mu)E(t) \\
\frac{dE(t)}{dt} &\geq -(\varepsilon + \mu)E(t) \\
\frac{dE(t)}{E(t)} &\geq -(\varepsilon + \mu)dt
\end{aligned}$$

$$\ln E(t) \geq -(\varepsilon+\mu)t + c^*$$

$$E(t) \geq c\exp(-(\varepsilon+\mu)t)$$

$$\therefore \boxed{E(t) \geq 0}$$

$$\frac{dI(t)}{dt} = \varepsilon E(t) - (\gamma+\mu+d)I(t)$$

$$\frac{dI(t)}{dt} \geq -(\gamma+\mu+d)I(t)$$

$$\frac{dI(t)}{I(t)} \geq -(\gamma+\mu+d)dt$$

$$\ln I(t) \geq -(\gamma+\mu+d)t + c^*$$

$$I(t) \geq c\exp(-(\gamma+\mu+d)t)$$

$$\therefore \boxed{I(t) \geq 0}$$

$$\frac{dR(t)}{dt} = \gamma I(t) - \mu R(t)$$

$$\frac{dR(t)}{dt} \geq -\mu R(t)$$

$$\frac{dR(t)}{R(t)} \geq -\mu dt$$

$$\ln R(t) \geq -\mu t + c^*$$

$$R(t) \geq c\exp(-\mu t)$$

$$\therefore \boxed{R(t) \geq 0}$$

$$\therefore B(t) \geq 0, S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, R(t) \geq 0$$

□

3.4.3 Boundedness of the solution

Theorem 3.4.3.1. *All feasible solutions* $B(t)$,$S(t)$,$E(t)$, $I(t)$, $R(t)$ *of the system of equations are bounded by the region* $\Omega = \{(B(t),S(t),E(t),I(t),R(t)) \in \mathbb{R}^5_+;\ N = B(t)+S(t)+E(t)+I(t)+R(t) \leq \frac{\Lambda}{\mu}\}$.

Proof. :

$$
\begin{aligned}
\frac{dN}{dt} &= \frac{dB(t)}{dt} + \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} \\
\frac{dN}{dt} &= \Lambda p - kB(t) + kB(t) + \Lambda(1-p) \\
&\quad - \frac{\beta S(t)I(t)}{N} - \mu S(t) + \frac{\beta S(t)I(t)}{N} \\
&\quad - (\varepsilon+\mu)E(t) + \varepsilon E(t) \\
&\quad - (\gamma+\mu+d)I(t) + \gamma I(t) - \mu R(t) \\
\frac{dN}{dt} &= \Lambda - \mu(S(t)+E(t)+I(t)+R(t)) - dI(t) \\
\frac{dN}{dt} &\leq \Lambda - \mu(S(t)+E(t)+I(t)+R(t)) \\
\frac{dN}{dt} &\leq \Lambda - \mu N \\
\frac{dN}{\Lambda-\mu N} &\leq dt \\
\frac{\ln|\Lambda-\mu N|}{-\mu} &\leq t + c^{'} \\
-\ln|\Lambda-\mu N| &\leq \mu t + c^{''} \\
\ln|\Lambda-\mu N| &\geq -\mu t - c^{''} \\
|\Lambda-\mu N| &\geq c^{*}\exp(-\mu t) \\
case(i): \Lambda-\mu N &\geq c^{*}\exp(-\mu t)
\end{aligned}
$$

$$\Lambda - \exp(-\mu t) \geq c^{*}\mu N$$
$$\frac{\Lambda}{\mu} - c\exp(-\mu t) \geq N$$
$$\text{At } t = 0:$$
$$\frac{\Lambda}{\mu} - c \geq N(0)$$
$$\text{i.e } N(0) \leq \frac{\Lambda}{\mu} - c$$
$$N(0) - \frac{\Lambda}{\mu} \leq -c \leq c$$
$$\text{i.e } N(0) - \frac{\Lambda}{\mu} \leq c$$
$$-(N(0) - \frac{\Lambda}{\mu}) \geq -c$$
$$\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \geq \frac{\Lambda}{\mu} - c\exp(-\mu t) \geq N$$
$$\therefore N \leq \frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t)$$
$$\leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t)$$

$$N \leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \tag{3.21}$$

case(ii):

$$\Lambda - \mu N \leq -c^{*}\exp(-\mu t)$$
$$\Lambda + c^{*}\exp(-\mu t) \leq \mu N$$
$$\frac{\Lambda}{\mu} + c\exp(-\mu t) \leq N$$
$$N \geq \frac{\Lambda}{\mu} + c\exp(-\mu t)$$
$$\text{At } t = 0:$$

$$
\begin{aligned}
N(0) &\geq \frac{\Lambda}{\mu} + c \\
N(0) - \frac{\Lambda}{\mu} &\geq c \geq -c \\
\therefore N(0) - \frac{\Lambda}{\mu} &\geq -c \\
-(N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) &\leq c\exp(-\mu t) \\
\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) &\leq \frac{\Lambda}{\mu} + c\exp(-\mu t) \\
&\leq N
\end{aligned}
$$

$$
\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \leq N \tag{3.22}
$$

but we want only upperbound ,therefore

$$
\begin{aligned}
\lim_{t\to\infty} N(t) &\leq \lim_{t\to\infty} \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \\
\therefore \lim_{t\to\infty} \sup N(t) &\leq \frac{\Lambda}{\mu} \\
\therefore N = B(t) + S(t) + E(t) + I(t) + R(t) &\leq \frac{\Lambda}{\mu}\}
\end{aligned}
$$

Therefore we get the region which is given by the set
$\Omega = \{(B(t), S(t), E(t), I(t), R(t)) \in \mathbb{R}^5_+;\ N = B(t)+S(t)+E(t)+I(t)+R(t) \leq \frac{\Lambda}{\mu}\}$ which is a positively invariant set. $\square$

3.4.4 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\frac{dB(t)}{dt} = \frac{dS(t)}{dt} = \frac{dE(t)}{dt} = \frac{dI(t)}{dt} = \frac{dR(t)}{dt} = 0$$

$$\Lambda p - kB(t) = 0$$

$$B(t) = \frac{\Lambda p}{k} \tag{3.23}$$

$$kB(t) + \Lambda(1-p) - \frac{\beta S(t)I(t)}{N} - \mu S(t) = 0$$

$$k\frac{\Lambda p}{k} + \Lambda(1-p) - \frac{\beta S(t)I(t)}{N} - \mu S(t) = 0$$

$$\Lambda - S(t)(\frac{\beta I(t)}{N} + \mu) = 0$$

$$S(t) = \frac{\Lambda}{\frac{\beta}{N}I(t) + \mu}$$

$$S(t) = \frac{N\Lambda}{\beta I(t) + N\mu} \tag{3.24}$$

$$\frac{\beta}{N}S(t)I(t) - (\varepsilon + \mu)E(t) = 0$$

$$E(t)=\frac{\frac{\beta}{N}S(t)I(t)}{\varepsilon+\mu}$$

$$E(t)=\frac{\frac{\beta}{N}(\frac{\Lambda}{\frac{\beta}{N}I(t)+\mu})I(t)}{\varepsilon+\mu}$$

$$E(t)=\frac{\beta\Lambda I(t)}{(\beta I(t)+N\mu)(\varepsilon+\mu)} \tag{3.25}$$

$$\varepsilon E(t)-(\gamma+\mu+d)I(t)=0$$

$$\varepsilon(\frac{\beta\Lambda I(t)}{(\beta I(t)+N\mu)(\varepsilon+\mu)})-(\gamma+\mu+d)I(t)=0$$

$$\therefore I(t)=0 \quad \text{or} \quad \frac{\varepsilon\beta\Lambda}{(\beta I(t)+N\mu)(\varepsilon+\mu)}-(\gamma+\mu+d)=0 \tag{3.26}$$

$$\therefore I(t)=0 \quad \text{or} \quad \varepsilon\beta\Lambda-(\beta I(t)+N\mu)(\varepsilon+\mu)(\gamma+\mu+d)=0$$

$$\therefore I(t)=0 \quad \text{or} \quad \varepsilon\beta\Lambda-\beta I(t)(\varepsilon+\mu)(\gamma+\mu+d)+N\mu(\varepsilon+\mu)(\gamma+\mu+d)=0$$

$$\therefore I(t)=0 \quad \text{or} \quad I(t)=\frac{\varepsilon\beta\Lambda-N\mu(\varepsilon+\mu)(\gamma+\mu+d)}{\beta(\varepsilon+\mu)(\gamma+\mu+d)} \tag{3.27}$$

$$\gamma I(t)-\mu R(t)=0$$

$$\therefore R(t) = \frac{\gamma I(t)}{\mu} \tag{3.28}$$

$$\begin{aligned}
\text{If} \quad I(t) &= 0, \text{ then} \\
B(t) &= \frac{\Lambda p}{k}, \\
s(t) &= \frac{\Lambda}{\mu}, \\
E(t) &= 0, \\
R(t) &= 0
\end{aligned}$$

$\therefore$ disease-free equilibrium point is $D_1 = (b_1^*, s_1^*, e_1^*, i_1^*, r_1^*) = (\frac{\Lambda p}{k}, \frac{\Lambda}{\mu}, 0, 0, 0)$

$$\begin{aligned}
\text{If } I(t) &= \frac{\varepsilon\beta\Lambda - N\mu(\varepsilon+\mu)(\gamma+\mu+d)}{\beta(\varepsilon+\mu)(\gamma+\mu+d)} = i_2^*, \text{ then} \\
B(t) &= \frac{\Lambda p}{k}, \\
S(t) &= \frac{N\Lambda}{\beta i_2^* + N\mu}, \\
E(t) &= \frac{\beta\Lambda i_2^*}{(\beta i_2^* + N\mu)(\varepsilon+\mu)} \\
R(t) &= \frac{\gamma i_2^*}{\mu}
\end{aligned}$$

$\therefore$ endemic equilibrium point is

$$D_2 = (b_2^*, s_2^*, e_2^*, i_2^*, r_2^*) = (\frac{\Lambda p}{k}, \frac{N\Lambda}{\beta i_2^* + N\mu}, \frac{\beta\Lambda i_2^*}{(\beta i_2^* + N\mu)(\varepsilon+\mu)}, i_2^*, \frac{\gamma i_2^*}{\mu})$$

Therefore, we get two equilibrium points $D_1 = (b_1^*, s_1^*, e_1^*, i_1^*, r_1^*)$ and $D_2 = (b_2^*, s_2^*, e_2^*, i_2^*, r_2^*)$.

$$\text{Also,}\quad \frac{dN}{dt} = \frac{dB(t)}{dt} + \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt}$$

$$= \Lambda p - kB(t) + kB(t) + \Lambda(1-p)$$

$$- \frac{\beta S(t)I(t)}{N} - \mu S(t) + \frac{\beta S(t)I(t)}{N}$$

$$-(\varepsilon+\mu)E(t) + \varepsilon E(t)$$

$$-(\gamma+\mu+d)I(t) + \gamma I(t) - \mu R(t)$$

$$= \Lambda - \mu(S(t)+E(t)+I(t)+R(t)) - dI(t)$$

$$= \Lambda - \mu N + \mu B(t) - dI(t)$$

$$\text{Now putting } \frac{dN}{dt} = 0$$

$$\therefore \Lambda - \mu N + \mu B(t) - dI(t) = 0$$

Substituting D_1 in above we get,

$$\therefore \Lambda - \mu N + \mu \frac{\Lambda p}{k} = 0$$

$$\therefore N = \frac{\Lambda}{\mu}\left(\frac{k + \mu p}{k}\right)$$

3.4.5 Reproduction number

We use next generation matrix method to find reproduction number. In F we put those terms from equation (3.20) which helps in growing secondary infection and in V we put all other terms with opposite signs.

$$F = \begin{bmatrix} \frac{\beta}{N} S(t) I(t) \\ 0 \end{bmatrix}, \; V = \begin{bmatrix} (\varepsilon + \mu) E(t) \\ -\varepsilon E(t) + (\gamma + \mu + d) I(t) \end{bmatrix}$$

$$J_F = \begin{bmatrix} 0 & \frac{\beta}{N} S(t) \\ 0 & 0 \end{bmatrix}, \; J_V = \begin{bmatrix} \varepsilon + \mu & 0 \\ -\varepsilon & \gamma + \mu + d \end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix} 0 & \frac{\beta \Lambda}{N \mu} \\ 0 & 0 \end{bmatrix}, \; J_{V(D_1)} = \begin{bmatrix} \varepsilon + \mu & 0 \\ -\varepsilon & \gamma + \mu + d \end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix} 0 & \dfrac{\beta\Lambda}{\dfrac{\Lambda}{\mu}(\dfrac{k+\mu p}{k})\mu} \\ 0 & 0 \end{bmatrix}, \ J_{V(D_1)} = \begin{bmatrix} \varepsilon+\mu & 0 \\ -\varepsilon & \gamma+\mu+d \end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix} 0 & \dfrac{\beta k}{k+\mu p} \\ 0 & 0 \end{bmatrix}, \ J_{V(D_1)} = \begin{bmatrix} \varepsilon+\mu & 0 \\ -\varepsilon & \gamma+\mu+d \end{bmatrix}$$

$$J^{-1}_{V(D_1)} = \frac{1}{(\varepsilon+\mu)(\gamma+\mu+d)} \begin{bmatrix} \gamma+\mu+d & 0 \\ \varepsilon & \varepsilon+\mu \end{bmatrix}$$

$$J^{-1}_{V(D_1)} = \begin{bmatrix} \dfrac{1}{\varepsilon+\mu} & 0 \\ \dfrac{\varepsilon}{(\varepsilon+\mu)(\gamma+\mu+d)} & \dfrac{1}{\gamma+\mu+d} \end{bmatrix}$$

$$A = J_{F(D_1)} J^{-1}_{V(D_1)} = \begin{bmatrix} \dfrac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)} & \dfrac{\beta k}{(\mu p+k)(\gamma+\mu+d)} \\ 0 & 0 \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(A-\lambda I) = \left| A-\lambda I \right| = 0$$

Therefore, we have

$$\begin{vmatrix} \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)}-\lambda & \frac{\beta k}{(\mu p+k)(\gamma+\mu+d)} \\ 0 & -\lambda \end{vmatrix} = 0$$

then the characteristic equation of the above matrix we get,

$$\lambda^2 - \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)}\lambda = 0$$

$$\lambda\left(\lambda - \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)}\right) = 0$$
$$\therefore \lambda = 0 \quad \text{or} \quad \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)}$$

So, the reproduction number (R_0) is determined by the greatest eigenvalue:

$$R_0 = \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)}$$

3.4.6 Linearisation of system with respect to equilibrium point

Jacobian of the linearised system is denoted as J:

$$J = \begin{bmatrix} -k & 0 & 0 & 0 & 0 \\ k & -\frac{\beta}{N}I(t)-\mu & 0 & -\frac{\beta}{N}S(t) & 0 \\ 0 & \frac{\beta}{N}I(t) & -(\varepsilon+\mu) & \frac{\beta}{N}S(t) & 0 \\ 0 & 0 & \varepsilon & -(\gamma+\mu+d) & 0 \\ 0 & 0 & 0 & \gamma & -\mu \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_1 = (b_1^*, s_1^*, e_1^*, i_1^*, r_1^*) = (\frac{\Lambda p}{k}, \frac{\Lambda}{\mu}, 0, 0, 0)$ yields the Jacobian matrix:

$$C = \begin{bmatrix} -k & 0 & 0 & 0 & 0 \\ k & -\mu & 0 & \frac{\beta\Lambda}{N\mu} & 0 \\ 0 & 0 & -(\varepsilon+\mu) & \frac{\beta\Lambda}{N\mu} & 0 \\ 0 & 0 & \varepsilon & -(\gamma+\mu+d) & 0 \\ 0 & 0 & 0 & \gamma & -\mu \end{bmatrix}$$

3.4.7 Stability Analysis

In order for the equilibrium point of the system to be stable, the eigenvalues must be negative or have a negative real part.

We can get the characteristic equation by solving

$$\det(C - \lambda I) = \left| C - \lambda I \right| = 0$$

Therefore,

$$\begin{vmatrix} -k-\lambda & 0 & 0 & 0 & 0 \\ k & -\mu-\lambda & 0 & \frac{\beta\Lambda}{N\mu} & 0 \\ 0 & 0 & -(\varepsilon+\mu+\lambda) & \frac{\beta\Lambda}{N\mu} & 0 \\ 0 & 0 & \varepsilon & -(\gamma+\mu+d+\lambda) & 0 \\ 0 & 0 & 0 & \gamma & -\mu-\lambda \end{vmatrix} = 0$$

$$(k+\lambda)(\mu+\lambda)^2((\varepsilon+\mu+\lambda)(\gamma+\mu+d+\lambda) - \frac{\beta\varepsilon\Lambda}{N\mu}) = 0$$

is the characteristic equation.

Therefore, $\lambda_1 = -k$ or $\lambda_2 = -\mu$ or $\lambda_3 = -\mu$ or $(\varepsilon+\mu+\lambda)(\gamma+\mu+d+\lambda) - \frac{\beta\varepsilon\Lambda}{N\mu} = 0$

$$\begin{aligned}(\varepsilon+\mu+\lambda)(\gamma+\mu+d+\lambda) - \frac{\beta\varepsilon\Lambda}{N\mu} &= 0\\ (\varepsilon+\mu)(\gamma+\mu+d) + (\varepsilon+\mu)\lambda + (\gamma+\mu+d)\lambda + \lambda^2 - \frac{\beta\varepsilon\Lambda}{N\mu} &= 0\\ \lambda^2 + (\gamma+2\mu+d+\varepsilon)\lambda + (\varepsilon+\mu)(\gamma+\mu+d) - \frac{\beta\varepsilon\Lambda}{N\mu} &= 0\end{aligned}$$

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{3.29}$$

where , $a_1 = \gamma+2\mu+d+\varepsilon, a_2 = (\varepsilon+\mu)(\gamma+\mu+d) - \frac{\beta\varepsilon\Lambda}{N\mu}$

Using the Routh–Hurwitz criterion, a second degree polynomial with all coefficients positive will obviously have negative roots:

$$\begin{aligned}a_1 = \gamma+2\mu+d+\varepsilon > 0 \quad &\text{and}\\ a_2 = (\varepsilon+\mu)(\gamma+\mu+d) - \frac{\beta\varepsilon\Lambda}{N\mu} > 0 \; &\Longleftrightarrow\\ (\varepsilon+\mu)(\gamma+\mu+d) > \frac{\beta\varepsilon\Lambda}{N\mu} \; &\Longleftrightarrow\\ (\varepsilon+\mu)(\gamma+\mu+d) > \frac{\beta\varepsilon\Lambda}{(\frac{\Lambda}{\mu}(\frac{k+\mu p}{k})\mu} \; &\Longleftrightarrow\\ (\varepsilon+\mu)(\gamma+\mu+d) > \frac{\beta\varepsilon k}{k+\mu p} \; &\Longleftrightarrow\\ 1 > \frac{\varepsilon\beta k}{(\mu p+k)(\varepsilon+\mu)(\gamma+\mu+d)} \; &\Longleftrightarrow\\ 1 > R_0 \; &\Longleftrightarrow\\ &R_0 < 1\end{aligned}$$

Therefore, the disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$; otherwise, it is unstable.

Chapter 4

Analysis of covid 19

4.1 Introduction

In late 2019, the emergence of the novel Coronavirus, named COVID-19, posed a significant threat to the international community due to its rapid spread and high mortality rate. The World Health Organization (WHO) was alerted to cases of idiopathic pulmonary infections in Wuhan, China, on December 31, 2019. By January 7th, Chinese authorities identified the cause as a new virus, 2019-nCoV. Mathematical modeling of COVID-19 is crucial for understanding its spread and evaluating containment measures like quarantine. Given China's high population density, it serves as a significant case study. Various mathematical models have been developed to analyze infectious diseases, including COVID-19. These models help in reducing the number of infections by predicting spread patterns.

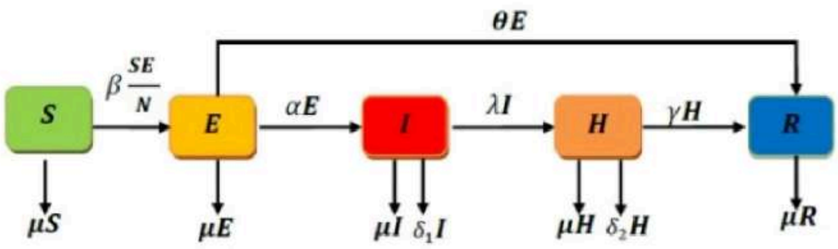
Specifically, the review mentions studies by M. Tahir et al.[23] and Zhi-Qiang Xia et al.[24], which developed mathematical models for MERS-CoV and analyzed factors contributing to disease spread, such as the basic reproduction number (R_0) and control measures. Additionally, A. Naheed et al.'s [25]research on population models for SARS examined the impact of diffusion on disease transmission and stability analysis of numerical solutions.

The model considers human-to-human transmission via direct contact with infected individuals. Local and global stability analyses of the model are conducted. The paper is structured with sections on model formulation, equilibrium analysis, stability analysis, conclusion.

4.2 Mathematical Modelling of Covid 19 through SEIHR model

4.2.1 Formation of model

This model consists of five compartments: susceptible, asymptomatic infected or having mild infection, infected,hospital and recovered. Susceptible individuals are those who have not been infected but are at risk of contracting the infection.Asymptomatic infected or having mild infection are at the initial stage og infection and can spread the disease. Infected individuals have active covid virus and are quarantined themselves or have hospitalised . Recovered individuals are those who have recuperated from covid and are immune to the disease.



Nonlinear system of differential equations is

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - \beta\frac{SE}{N} - \mu S\\ \frac{dE(t)}{dt} &= \beta\frac{SE}{N} - (\mu+\alpha+\theta)E\\ \frac{dI(t)}{dt} &= \alpha E - (\mu+\lambda+\delta_1)I\\ \frac{dH(t)}{dt} &= \lambda I - (\mu+\gamma+\delta_2)H\\ \frac{dR(t)}{dt} &= \gamma H + \theta E - \mu R\end{aligned} \tag{4.1}$$

with initial conditions

$S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, $H(0) \geq 0$, and $R(0) \geq 0$

where,

$N =$ Total population

$S(t) =$ number of susceptible individuals

$E(t) =$ number of asymptomatic infected or mildly infected individuals

$I(t) =$ number of infected individuals

$R(t) =$ number of recovered individuals

$\Lambda =$ new birth rate of the population

$\delta_1 =$ death of infected individuals due to covid 19

$\delta_2 =$ death of hospitalised individuals due to covid 19

$\mu =$ The natural death rate of the population

$\beta =$ The transmission rate

$\theta =$ rate at which asymptotic individual recover

$\alpha =$ rate at which asymptotic individual become infected

$\lambda =$ rate at which infected individuals hospitalised

$\gamma =$ The recovery rate of hospitalised individuals

Total population is denoted by $N = S(t) + E(t) + I(t) + H(t) + R(t)$

In the model, Λ denotes the birth rate of susceptible individuals, reflecting the rate at which new individuals enter the population. The parameter β signifies the transmission rate from susceptible individuals to those who become asymptomatic or develop mild symptoms due to interactions and contacts. The rate of natural mortality across all compartments is represented by μ. α denotes the rate at which asymptomatic or mildly symptomatic individuals transmit the infection to others who develop symptoms. λ signifies the transmission rate from individuals with symptoms to those who require hospitalization. γ represents the transmission rate from hospitalized individuals to those who eventually recover. θ captures the rate at which asymptomatic or mildly symptomatic individuals develop immunity and recover without requiring hospitalization. The parameters δ_1 and δ_2 correspond to the death rates of infected individuals and hospitalized cases, respectively. In this model it is assumed that infected people are not recovering unless they hospitalise themselves.Also it is assumed that once people recover they will not get infection back again.

4.2.2 Positivity of solutions

Theorem 4.2.2.1. *If $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$,$H(0) \geq 0$, $R(0) \geq 0$, then the solutions of the system of equations $S(t)$,$E(t)$, $I(t)$, $H(t)$,$R(t)$ are positive for all $t \geq 0$.*

Proof. :

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta\frac{SE}{N} - \mu S \\
\frac{dS(t)}{dt} &\geq -\beta\frac{SE}{N} - \mu S \\
\frac{dS(t)}{dt} &\geq -S(\beta\frac{E}{N} + \mu) \\
\frac{dS(t)}{S} &\geq -(\beta\frac{E}{N} + \mu)dt \\
\ln S(t) &\geq -(\beta\frac{E}{N} + \mu)t + c^* \\
S(t) &\geq c\exp\left(-(\beta\frac{E}{N} + \mu)t\right)
\end{aligned}$$

$$\therefore \boxed{S(t) \geq 0}$$

$$\begin{aligned}
\frac{dE(t)}{dt} &= \beta\frac{SE}{N} - (\mu + \alpha + \theta)E \\
\frac{dE(t)}{dt} &\geq -(\mu + \alpha + \theta)E \\
\frac{dE(t)}{E(t)} &\geq -(\mu + \alpha + \theta)dt \\
\ln E(t) &\geq -(\mu + \alpha + \theta)t + c^* \\
E(t) &\geq c\exp(-(\mu + \alpha + \theta)t)
\end{aligned}$$

$$\therefore \boxed{E(t) \geq 0}$$

$$\frac{dI(t)}{dt} = \alpha E - (\mu + \lambda + \delta_1)I$$
$$\frac{dI(t)}{dt} \geq -(\mu + \lambda + \delta_1)I$$
$$\frac{dI(t)}{I(t)} \geq -(\mu + \lambda + \delta_1)dt$$
$$\ln I(t) \geq -(\mu + \lambda + \delta_1)t + c^*$$
$$I(t) \geq c\exp(-(\mu + \lambda + \delta_1)t)$$
$$\therefore \boxed{I(t) \geq 0}$$

$$\frac{dH(t)}{dt} = \lambda I - (\mu + \gamma + \delta_2)H$$
$$\frac{dH(t)}{dt} \geq -(\mu + \gamma + \delta_2)H$$
$$\frac{dH(t)}{H(t)} \geq -(\mu + \gamma + \delta_2)dt$$
$$\ln H(t) \geq -(\mu + \gamma + \delta_2)t + c^*$$
$$H(t) \geq c\exp(-(\mu + \gamma + \delta_2)t)$$
$$\therefore \boxed{H(t) \geq 0}$$

$$\frac{dR(t)}{dt} = \gamma H + \theta E - \mu R$$
$$\frac{dR(t)}{dt} \geq -\mu R$$
$$\frac{dR(t)}{R(t)} \geq -\mu dt$$
$$\ln R(t) \geq -\mu t + c^*$$
$$R(t) \geq c\exp(-\mu t)$$
$$\therefore \boxed{R(t) \geq 0}$$

$$\therefore S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, H(t) \geq 0, R(t) \geq 0$$

□

4.2.3 Boundedness of the solution

Theorem 4.2.3.1. *All feasible solutions $S(t)$,$E(t)$, $I(t)$,$H(t)$, $R(t)$ of the system of equations are bounded by the region* $\Omega = \{(S(t),E(t),I(t),H(t),R(t)) \in \mathbb{R}^5_+;\ N = S(t)+E(t)+I(t)+H(t)+R(t) \leq \frac{\Lambda}{\mu}\}$.

Proof. :

$$
\begin{aligned}
\frac{dN}{dt} &= \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dH(t)}{dt} + \frac{dR(t)}{dt} \\
\frac{dN}{dt} &= \Lambda - \beta\frac{SE}{N} - \mu S \\
&\quad + \beta\frac{SE}{N} - (\mu+\alpha+\theta)E \\
&\quad + \alpha E - (\mu+\lambda+\delta_1)I \\
&\quad + \lambda I - (\mu+\gamma+\delta_2)H \\
&\quad + \gamma H + \theta E - \mu R \\
\frac{dN}{dt} &= \Lambda - \mu(S+E+I+H+R) - \delta_1 I - \delta_2 H \\
\frac{dN}{dt} &\leq \Lambda - \mu(S+E+I+H+R) \\
\frac{dN}{dt} &\leq \Lambda - \mu N \\
\frac{dN}{\Lambda-\mu N} &\leq dt \\
\frac{\ln|\Lambda-\mu N|}{-\mu} &\leq t + c'
\end{aligned}
$$

$$-\ln|\Lambda-\mu N| \leq \mu t + c^{''}$$

$$\ln|\Lambda-\mu N| \geq -\mu t - c^{''}$$

$$|\Lambda-\mu N| \geq c^{*}\exp(-\mu t)$$

$$case(i): \Lambda-\mu N \geq c^{*}\exp(-\mu t)$$

$$\Lambda-\exp(-\mu t) \geq c^{*}\mu N$$

$$\frac{\Lambda}{\mu} - c\exp(-\mu t) \geq N$$

$$\text{At } t=0:$$

$$\frac{\Lambda}{\mu} - c \geq N(0)$$

$$\text{i.e } N(0) \leq \frac{\Lambda}{\mu} - c$$

$$N(0) - \frac{\Lambda}{\mu} \leq -c \leq c$$

$$\text{i.e } N(0) - \frac{\Lambda}{\mu} \leq c$$

$$-(N(0) - \frac{\Lambda}{\mu}) \geq -c$$

$$\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \geq \frac{\Lambda}{\mu} - c\exp(-\mu t) \geq N$$

$$\therefore N \leq \frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t)$$

$$\leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t)$$

$$N \leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \tag{4.2}$$

case(ii):

$$\Lambda-\mu N \leq -c^{*}\exp(-\mu t)$$

$$\Lambda + c^{*}\exp(-\mu t) \leq \mu N$$

$$\frac{\Lambda}{\mu} + c\exp(-\mu t) \leq N$$
$$N \geq \frac{\Lambda}{\mu} + c\exp(-\mu t)$$
$$\text{At } t = 0:$$
$$N(0) \geq \frac{\Lambda}{\mu} + c$$
$$N(0) - \frac{\Lambda}{\mu} \geq c \geq -c$$
$$\therefore N(0) - \frac{\Lambda}{\mu} \geq -c$$
$$-(N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \leq c\exp(-\mu t)$$
$$\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \leq \frac{\Lambda}{\mu} + c\exp(-\mu t)$$
$$\leq N$$

$$\frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t) \leq N \tag{4.3}$$

but we want only upperbound ,therefore

$$\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu} + (N(0) - \frac{\Lambda}{\mu})\exp(-\mu t)$$
$$\therefore \lim_{t\to\infty} \sup N(t) \leq \frac{\Lambda}{\mu}$$
$$\therefore N = S(t) + E(t) + I(t) + H(t) + R(t) \leq \frac{\Lambda}{\mu}\}$$

Therefore we get the region which is given by the set

$\Omega = \{(S(t), E(t), I(t), H(t), R(t)) \in \mathbb{R}^5_+;\ N = S(t) + E(t) + I(t) + H(t) + R(t) \leq \frac{\Lambda}{\mu}\}$

which is a positively invariant set. □

4.2.4 Reduction of the system

Since first three equations in system (4.1) are independents of the variables H and R .Hence, the dynamics of equation system (4.1) is equivalent to the dynamics of equation system:

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - \beta\frac{SE}{N} - \mu S\\ \frac{dE(t)}{dt} &= \beta\frac{SE}{N} - (\mu+\alpha+\theta)E\\ \frac{dI(t)}{dt} &= \alpha E - (\mu+\lambda+\delta_1)I\end{aligned} \tag{4.4}$$

4.2.5 Reproduction number

We use next generation matrix method to find reproduction number.

In F we put those terms from equation (4.4) which helps in growing secondary infection and in V we put all other terms with opposite signs.

$$F = \begin{bmatrix}\beta\frac{SE}{N}\\ 0\end{bmatrix},\ V = \begin{bmatrix}(\mu+\alpha+\theta)E\\ -\alpha E + (\mu+\lambda+\delta_1)I\end{bmatrix}$$

Then by taking the jacobian of above we get,

$$J_F = \begin{bmatrix}\beta\frac{S}{N} & 0\\ 0 & 0\end{bmatrix},\ J_V = \begin{bmatrix}\mu+\alpha+\theta & 0\\ -\alpha & \mu+\lambda+\delta_1\end{bmatrix}$$

$$J_{F(D_1)} = \begin{bmatrix}\frac{\beta\Lambda}{N\mu} & 0\\ 0 & 0\end{bmatrix},\ J_{V(D_1)} = \begin{bmatrix}\mu+\alpha+\theta & 0\\ -\alpha & \mu+\lambda+\delta_1\end{bmatrix}$$

$$J^{-1}_{V(D_1)} = \frac{1}{(\mu+\alpha+\theta)(\mu+\lambda+\delta_1)} \begin{bmatrix} \mu+\lambda+\delta_1 & 0 \\ \alpha & \mu+\alpha+\theta \end{bmatrix}$$

$$J^{-1}_{V(D_1)} = \begin{bmatrix} \frac{1}{\mu+\alpha+\theta} & 0 \\ \frac{\alpha}{(\mu+\alpha+\theta)(\mu+\lambda+\delta_1)} & \frac{1}{\mu+\lambda+\delta_1} \end{bmatrix}$$

$$B = J_{F(D_1)} J^{-1}_{V(D_1)} = \begin{bmatrix} \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)} & 0 \\ 0 & 0 \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(B-\lambda I) = \begin{vmatrix} B-\lambda I \end{vmatrix} = 0$$

Therefore, we have

$$\begin{vmatrix} \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)} - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

then the characteristic equation of the above matrix we get,

$$\lambda^2 - \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)}\lambda = 0$$

$$\lambda\left(\lambda - \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)}\right) = 0$$

$$\therefore \lambda = 0 \quad \text{or} \quad \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)}$$

So, the reproduction number (R_0) is determined by the greatest eigenvalue:

$$R_0 = \frac{\beta\Lambda}{N\mu(\mu+\alpha+\theta)}$$

4.2.6 Equilibrium points

To find equilibrium points of the system of differential equations, we equate the derivatives to zero:

$$\frac{dS(t)}{dt} = \frac{dE(t)}{dt} = \frac{dI(t)}{dt} = 0$$

$$\Lambda - \beta\frac{SE}{N} - \mu S = 0$$

$$\Lambda = (\beta\frac{E}{N} + \mu)S \tag{4.5}$$

$$S = \frac{\Lambda}{\beta\frac{E}{N} + \mu} \tag{4.6}$$

$$\Longrightarrow \frac{\Lambda}{S} = \beta\frac{E}{N} + \mu$$

$$\frac{\Lambda}{S} - \mu = \beta\frac{E}{N}$$

$$E = \frac{N}{\beta}(\frac{\Lambda}{S} - \mu) \tag{4.7}$$

$$\beta\frac{SE}{N} - (\mu+\alpha+\theta)E = 0$$

$$E(\beta\frac{S}{N} - (\mu+\alpha+\theta)) = 0$$

$$E = 0 \quad \text{or} \quad \beta \frac{S}{N} - (\mu + \alpha + \theta) = 0$$

$$E = 0 \quad \text{or} \quad S = \frac{N}{\beta}(\mu + \alpha + \theta) \tag{4.8}$$

$$\alpha E - (\mu + \lambda + \delta_1) I = 0$$

$$I = \frac{\alpha E}{\mu + \lambda + \delta_1} \tag{4.9}$$

If $E = 0$, then

$$S = \frac{\Lambda}{\mu},$$

$$I = 0,$$

$\therefore$ disease-free equilibrium point is $D_1 = (s_1^*, e_1^*, i_1^*) = (\frac{\Lambda}{\mu}, 0, 0)$

If $S = \frac{N}{\beta}(\mu + \alpha + \theta)$, then

$$E = \frac{N}{\beta}\left(\frac{\Lambda}{(\frac{N}{\beta}(\mu + \alpha + \theta))} - \mu\right)$$

$$E = \frac{\Lambda}{\mu + \alpha + \theta} - \frac{\mu}{\beta} N \tag{4.10}$$

$$I = \frac{\alpha}{\mu + \alpha + \delta_1} \left(\frac{\Lambda}{\mu + \alpha + \theta} - \frac{\mu}{\beta} N \right) \tag{4.11}$$

consider

$$S = \frac{N}{\beta}(\mu + \alpha + \theta) = \frac{N}{\beta} \frac{\Lambda \beta}{\mu R_0}$$

$$S=\frac{\Lambda}{\mu R_0} \tag{4.12}$$

consider

$$E=\frac{\Lambda}{\mu+\alpha+\theta}-\frac{\mu}{\beta}N=\frac{N\mu R_0}{\beta}-\frac{\mu}{\beta}N$$

$$E=N\frac{\mu}{\beta}(R_0-1) \tag{4.13}$$

consider

$$I=\frac{\alpha}{\mu+\lambda+\delta_1}\left(\frac{\Lambda}{\mu+\alpha+\theta}-\frac{\mu}{\beta}N\right)$$

$$I=\frac{\mu N\alpha}{\beta(\mu+\lambda+\delta_1)}(R_0-1) \tag{4.14}$$

$\therefore$ endemic equilibrium point is

$$D_2=(s_2^*,e_2^*,i_2^*)=\left(\frac{\Lambda}{\mu R_0},N\frac{\mu}{\beta}(R_0-1),\frac{\mu N\alpha}{\beta(\mu+\lambda+\delta_1)}(R_0-1)\right)$$

Therefore, we get two equilibrium points $D_1=(s_1^*,e_1^*,i_1^*)$ and $D_2=(s_2^*,e_2^*,i_2^*)$.

4.2.7 Linearisation of system with respect to equilibrium points

Jacobian of the linearised system is denoted as J:

$$J=\begin{bmatrix} -\frac{\beta}{N}E-\mu & \frac{\beta}{N}S & 0 \\ \frac{\beta}{N}E & \frac{\beta}{N}S-(\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_1 = (s_1^*, e_1^*, i_1^*) = (\frac{\Lambda}{\mu}, 0, 0)$ yields the Jacobian matrix:

$$J_1 = \begin{bmatrix} -\mu & -\frac{\beta\Lambda}{N\mu} & 0 \\ 0 & \frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$$

Linearisation of the system with respect to equilibrium point $D_2 = (s_2^*, e_2^*, i_2^*)$ yields the Jacobian matrix:

$$J_2 = \begin{bmatrix} -\frac{\beta}{N}(N\dfrac{\mu}{\beta}(R_0-1)) - \mu & \frac{\beta}{N}(\frac{N}{\beta}(\mu+\alpha+\theta)) & 0 \\ \frac{\beta}{N}(N\dfrac{\mu}{\beta}(R_0-1)) & \frac{\beta}{N}(\frac{N}{\beta}(\mu+\alpha+\theta)) - (\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -\mu R_0 & -(\mu+\alpha+\theta) & 0 \\ \mu(R_0-1) & 0 & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$$

4.2.8 Stability Analysis

Local Stability:

Theorem 4.2.8.1. *The COVID-19 disease-free equilibrium $D_1 = (s_1^*, e_1^*, i_1^*)$ of the system is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. : In order for the equilibrium point of the system to be stable, the eigenvalues must be negative or have a negative real part.

$$\text{Denote } J_1 = B = \begin{bmatrix} -\mu & -\frac{\beta\Lambda}{N\mu} & 0 \\ 0 & \frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta) & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$$

We can get the characteristic equation by solving

$$\det(B-\zeta I) = \left|B-\zeta I\right| = 0$$

Therefore,

$$\begin{vmatrix} -\mu-\zeta & -\frac{\beta\Lambda}{N\mu} & 0 \\ 0 & \frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta)-\zeta & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1)-\zeta \end{vmatrix} = 0$$

Therefore,

$$(\mu+\zeta)(\frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta)-\zeta)((\mu+\lambda+\delta_1)+\zeta) = 0$$

is the characteristic equation.

Therefore, $\zeta_1 = -\mu$ or $\zeta_2 = \frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta)$ or $\zeta_3 = -(\mu+\lambda+\delta_1)$

$$\begin{aligned} \text{Consider } \zeta_2 &= \frac{\beta\Lambda}{N\mu} - (\mu+\alpha+\theta) \\ &= R_0(\mu+\alpha+\theta) - (\mu+\alpha+\theta) \\ &= (\mu+\alpha+\theta)(R_0-1) \end{aligned}$$

Here ζ_1 and ζ_2 are clearly negative and

$$\begin{aligned}
\zeta_2 < 0 &\iff \\
(\mu+\alpha+\theta)(R_0-1) < 0 &\iff \\
R_0 - 1 < 0 &\iff \\
R_0 < 1
\end{aligned}$$

Therefore, the disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$; otherwise, it is unstable. □

Theorem 4.2.8.2. *The COVID-19 disease-present equilibrium (endemic equilibrium) $D_2 = (s_2^*, e_2^*, i_2^*)$ of the system is asymptotically stable if $R_0 > 1$ and unstable if $R_0 < 1$.*

Proof. : Denote $J_2 = C = \begin{bmatrix} -\mu R_0 & -(\mu+\alpha+\theta) & 0 \\ \mu(R_0-1) & 0 & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1) \end{bmatrix}$

We can get the characteristic equation by solving

$$\det(C-\zeta I) = \left| C - \zeta I \right| = 0$$

Therefore,

$$\begin{vmatrix} -\mu R_0 - \zeta & -(\mu+\alpha+\theta) & 0 \\ \mu(R_0-1) & -\zeta & 0 \\ 0 & \alpha & -(\mu+\lambda+\delta_1)-\zeta \end{vmatrix} = 0$$

The characteristic equation is:

$$(\mu+\lambda+\delta_1+\zeta)((\mu R_0+\zeta)\zeta+(\mu+\alpha+\theta)\mu(R_0-1))=0$$

Therefore $\zeta_1=-(\mu+\lambda+\delta_1)$ or $(\mu R_0+\zeta)\zeta+(\mu+\alpha+\theta)\mu(R_0-1)=0$

Therefore $\zeta_1=-(\mu+\lambda+\delta_1)$ or $\zeta^2+\mu R_0\zeta+(\mu+\alpha+\theta)\mu(R_0-1)=0$

Now consider $\zeta^2+\mu R_0\zeta+(\mu+\alpha+\theta)\mu(R_0-1)=0$

Using Rauth Hurwitz criterion ,a second degree polynomial with all coefficients positive will obviously have negative roots.

We can clearly see that $\mu R_0>0$ and

$$\begin{aligned}(\mu+\alpha+\theta)\mu(R_0-1)>0 &\iff\\ R_0-1>0 &\iff\\ R_0>1 &\iff\end{aligned}$$

$\therefore$ Endemic equilibrium point is locally asymptotically stable iff $R_0>1$, otherwise it is unstable. □

Global Stability:

Theorem 4.2.8.3. *The COVID-19 disease-free equilibrium $D_1 = (s_1^*, e_1^*, i_1^*)$ is globally asymptotically stable in Ω if $R_0 < 1$ and unstable otherwise.*

Proof. : Let the following Lyapunov function:

$V : \Gamma \to \mathbb{R}$

$V(S,E) = \frac{1}{2}((S - S_0) + E)^2 + \frac{N}{\beta}(2\mu + \alpha + \theta)E$

where $\Gamma = \{(S,E) \in \Gamma / S > 0, E > 0\}$.

Firstly, to show that $V(S,E)$ is positive definite:

Case (i): when $(S - S_0) + E < 0$ or > 0

then

$$\frac{1}{2}((S - S_0) + E)^2 > 0$$

$$\therefore V(S,E) > 0$$

Case (ii): when $(S - S_0) + E = 0$

then

$$\frac{1}{2}((S - S_0) + E)^2 = 0$$

but

$$\frac{N}{\beta}(2\mu + \alpha + \theta)E > 0$$

$$\therefore V(S,E) > 0$$

And at the disease-free equilibrium point $S = S_0$ and $E = 0$,

$$\therefore V(S,E) = 0 \iff S = S_0 \text{ and } E = 0$$

$$\therefore V(S,E) \text{ is positive definite.}$$

Then, the time derivative of the Lyapunov function is given by:

$$\begin{aligned}
\frac{dV(S,E)}{dt} &= [(S-S_0)+E]\left(\frac{dS(t)}{dt}+\frac{dE(t)}{dt}\right)+\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= [(S-S_0)+E]\left(\Lambda-\beta\frac{SE}{N}-\mu S+\beta\frac{SE}{N}-(\mu+\alpha+\theta)E\right) \\
&\quad +\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= [(S-S_0)+E](\Lambda-\mu S-(\mu+\alpha+\theta)E)+\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= [(S-S_0)+E](\mu S_0-\mu S-(\mu+\alpha+\theta)E)+\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= [(S-S_0)+E](-\mu(S-S_0)-(\mu+\alpha+\theta)E)+\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= -\mu(S-S_0)^2-(\mu+\alpha+\theta)(S-S_0)E-\mu(S-S_0)E-(\mu+\alpha+\theta)E^2 \\
&\quad +\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= -\mu(S-S_0)^2-(\mu+\alpha+\theta)E^2-(2\mu+\alpha+\theta)(S-S_0)E \\
&\quad +\frac{N}{\beta}(\mu+\alpha+\theta)\frac{dE(t)}{dt} \\
&= -\mu(S-S_0)^2-(\mu+\alpha+\theta)E^2 \\
&\quad +(2\mu+\alpha+\theta)\left[\frac{\Lambda}{\mu}E-SE+\frac{N}{\beta}\left(\frac{\beta}{N}-(\mu+\alpha+\theta)E\right)\right] \\
&= -\mu(S-S_0)^2-(\mu+\alpha+\theta)E^2 \\
&\quad +(2\mu+\alpha+\theta)\left[\frac{\Lambda}{\mu}E-SE+SE-\frac{N}{\beta}(\mu+\alpha+\theta)E\right] \\
&= -\mu(S-S_0)^2-(\mu+\alpha+\theta)E^2 \\
&\quad +(2\mu+\alpha+\theta)E\left[\frac{\Lambda}{\mu}-\frac{N}{\beta}(\mu+\alpha+\theta)\right]
\end{aligned}$$

$$= -\mu(S-S_0)^2 - (\mu+\alpha+\theta)E^2$$
$$+ (2\mu+\alpha+\theta)E[R_0(\mu+\alpha+\theta)\frac{N}{\beta} - \frac{N}{\beta}(\mu+\alpha+\theta)]$$
$$= -\mu(S-S_0)^2 - (\mu+\alpha+\theta)E^2 + (2\mu+\alpha+\theta)E(\mu+\alpha+\theta)\frac{N}{\beta}(R_0-1)$$

$$\therefore \frac{dV(S,E)}{dt} < 0 \quad \text{for} \quad R_0 - 1 < 0$$
$$\text{i.e} \; \frac{dV(S,E)}{dt} < 0 \quad \text{for} \quad R_0 < 1$$
$$\frac{dV}{dt} = 0 \quad \text{if and only if} \quad S = S_0 \text{ and } E = 0.$$

Hence, by LaSalle's invariance principle, the disease-free equilibrium point is globally asymptotically stable in Ω. □

Chapter 5

Conclusion

In **chapter two** we constructed a continuous mathematical framework, labeled as the transmission model without immunity, to simulate the advancement of Dengue disease within a population. Within this framework, we highlighted a critical parameter known as the basic reproduction number, represented as R_0. This parameter serves to quantify the disease's potential for dissemination and provides insights into the dynamics of the system. we found out that when the equilibrium point is situated at the origin, it signifies local stability if the reproduction number is below one. In such instances, the disease tends to diminish over time and eventually dies out. Conversely, if the reproduction number exceeds one, the endemic equilibrium point indicates local stability. This suggests that the disease persists within the population at an endemic level.

In **Chapter Three**, we delve into the transmission patterns of tuberculosis (TB), employing three distinct epidemic models: the SIR, SEIR, and BSEIR models. Initially, the research focuses on a modified version of the SIR model, where population size remains constant and birth and death rates are equal. Subsequently, the SEIR model, incorporating an exposed group, is examined. Finally, a more comprehensive model, the BSEIR, is explored, incorporating vaccination rates, offering enhanced predictive capabilities compared to the other models. The findings suggest that the BSEIR model yields more precise predictions. The research also entails mathematical analyses of all three models, including stability assessments and the calculation of the reproduction number (R_0). The R_0 plays a pivotal role in characterizing the dynamic behavior of the models. When the equilibrium point resides at the origin, achieving local stability requires an R_0 below one, indicating a decline and eventual extinction of TB over time. Conversely, an R_0 exceeding one suggests local stability at an endemic equilibrium point, indicating the persistence of TB within the population at a steady level.

In **chapter four** we developed a continuous mathematical framework, denoted as SEIHR, to model the progression of the COVID-19 disease within a population. One key parameter we identified is the basic reproduction number, denoted as R_0, which characterizes the potential spread of the disease and informs us about the system's behavior. By employing stability analysis techniques for nonlinear systems, we investigated the behavior of the COVID-19 disease model. Specifically, we examined both local and global stability properties. Local asymptotic stability at the disease-free equilibrium is achieved when the value of R_0 does not exceed unity . Conversely, if R_0 exceeds 1, the equilibrium representing the presence of COVID-19 becomes locally asymptotically stable. To demonstrate the global stability of the disease-free equilibrium, we utilized a Lyapunov function. Our analysis revealed that the disease-free state is globally asymptotically stable under the condition that $R_0 < 1$. This implies that in scenarios where the basic reproduction number remains below or equal to 1, the COVID-19 epidemic is effectively controlled, and the population tends towards a state free of the disease.

Chapter 6

Further scope

Utilizing data gathered from hospitals and relevant institutions, we can conduct numerical simulations to obtain precise outcomes aligned with the data utilized. This approach proves beneficial in formulating accurate predictions aimed at effectively managing and containing outbreaks within specific regions.

In particularly we can take data from Goa for diseases like dengue ,tuberculosis ,covid-19 and can separately find analysis of the models . Which will help us to know more about the these disease in Goa and to control these diseases.

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