Exploring Domination In Graph Structures: A Comprehensive Analysis

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by

Ms. ANUJA ANIL NAIK

22P0410004

ABC ID : 843-974-843-614

201902148

Under the Supervisor of

Dr. JESSICA FERNANDES E PEREIRA

School of Physical & Applied Sciences

Mathematics Discipline

GOA UNIVERSITY

APRIL 2024

Examined by: Seal of the School

DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Exploring Domination In Graph Structures: A Comprehensive Analysis" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Jessica Fernandes e Pereira and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given in the dissertation.

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Student Name: Anuja Anil Naik

Seat no: 22P0410004

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Exploring Domination In Graph Structures: A Comprehensive Analysis" is a bonafide work carried out by Ms. Anuja Anil Naik under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

Signarure : ______________________

Supervisor : Dr. Jessica Fernandes e Pereira

Date: MAY 08, 2024

School of Physical and Applied Sciences Goa University

Signature of HoD of the Dept

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PREFACE

This Project Report has been created as a part of the MAT-651 Discipline Specific Dissertation for the M.Sc. in Mathematics program during the academic year 2023-2024. It focuses on the topic "Exploring domination in graph structures: a comprehensive analysis" and is divided into four chapters, each systematically covering different aspects of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of graph domination and its historical context.

SECOND CHAPTER:

The chapter delves into the domination number of various graphs, including Fan [$F_{m,2}$], Diamond Snake [D_n], Banana tree [$B(m,n)$], and Coconut tree [$CT(m,n)$], to Firecracker [$F(m,n)$]. It also examines trees without duplicated leaves, establishing the minimum and maximum orders of such trees ensuring a certain domination number. Furthermore, it discusses Mycielski graphs $\mu(G)$, Crib graphs $C(G)$, and Modified Mycielski graphs $\mu^*(G)$.

THIRD CHAPTER:

The chapter discusses total domination in graph theory, exploring different graph operators such as $S(G)$ and $R(G)$ and to determine the total domination number of new graphs. It also introduces the concept of total co-independent domination and total co-independent dominating sets. The chapter further discusses total domination in Generalized Petersen graphs $P(n,1)$ and $P(n,2)$ and introduces minimal total dominating sets of these graphs. Additionally, it covers total equitable domination and determines the total domination number and total equitable domination number of certain path-related graphs.

FOURTH CHAPTER:

The Chapter delves into specific cases of domination numbers within graph structures. It examines the domination number of the snare graph of a graph $S(G)$ and the Lollipop graph $L_{m,n}$, providing insights into their structural properties and the minimum number of vertices needed to dominate these graphs effectively. Additionally, it explores the total domination number of a special case of the Lollipop graph denoted as $L_{m,1}$ shedding light on its unique characteristics and the implications for total domination within this graph family. Furthermore, the chapter explores the total domination number of tadpole graphs $T_{m,n}$ and the crib graph of the path P_n.

ACKNOWLEDGEMENTS

I wish to extend heartfelt gratitude to our mentor, Prof. Dr. Jessica Fernandes e Pereira, whose unwavering guidance served as a constant wellspring of inspiration throughout my journey. With her encouragement to think creatively and pursue our work with conviction, she played an instrumental role in our success. Leveraging her profound expertise and extensive experience in Graph theory, she equipped me with the tools necessary to triumphantly complete this project. Without her invaluable support and supervision, this endeavor would have remained beyond my reach. I am truly fortunate to have had such an exceptional mentor accompany me in my academic pursuits.

ABSTRACT

Graph theory is an active field in modern mathematics, with domination theory at its forefront, representing a key area of exploration. A set D of vertices in a graph $G = (V,E)$ is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of the dominating set in G. A set $D \subseteq V$ is a total Dominating Set of G if every vertex of G has at least a neighbour in D. The total domination number denoted by $\gamma_t(G)$ is the cardinality of the smallest total dominating set in G. A subset D of $V(G)$ is called an **Equitable Dominating Set** if for every $v \in V(G) - D$ $\exists$ a vertex $u \in D \ni uv \in E(G)$ and $|d(v) - d(u)| \leq 1$, where $d(v)$ and $d(u)$ denotes the degree of vertex v and u respectively. The minimum cardinality of such dominating set is called the equitable domination number of G which is denoted by $\gamma^e(G)$. A dominating set that satisfies both criteria is termed a total equitable dominating set, and the total equitable domination number, denoted by $\gamma_t{}^e(G)$, is the smallest cardinality of such a set in G.

The main aim of this article is, in general, to prove some basic results concerning the above-defined domination concepts. In particular, the article seeks to compute the domination number and total domination number for various families of graphs.

Keywords: Dominating set, Domination number, Total dominating set, Total domination number, Equitable domination, Total equitable dominating set and Total equitable domination number.

Table of contents

List of figures

Notations and Abbreviations

$V(G)$	vertex set of G
$E(G)$	edge set of G
$\Delta(G)$	maximum degree
P_n	path on n vertices
$\delta(G)$	minimum degree
C_n	cycle on n vertices
K_n	complete graph
$N(v)$	open Neighborhood of v
$N[v]$	closed Neighborhood of v
$F(m,n)$	firecracker graph
$F_{m,2}$	fan graph
D_n	dimond snake graph
$B(m,n)$	banana tree graph
$CT(m,n)$	coconut tree graph
$\mu(P_n)$	mycielski graph of P_n
$C(P_n)$	crib graph of P_n
$\mu^*(P_n)$	modified mycielski graph of P_n
$\gamma(G)$	domination number of G
$\overline{K}_m$	complement of a complete graph
$\gamma_t(G)$	total domination number of G
$\gamma^e(G)$	equitable domination number of G
$\gamma_t{}^e(G)$	total equitable domination number of G
W_n	wheel graph on n vertices
$\gamma_{t,coi}$	total co-independent domination number of G
$\alpha(G)$	independence number of G
G^2	square of G
$DS(G)$	degree split graph of G
$P(n,k)$	generalized petersen graph
$S(P_n)$	snare graph of P_n
$L_{m,n}$	lollipop graph
$T_{m,n}$	tadpole graph

Chapter 1

INTRODUCTION

1.1 Background

Graph theory, a vibrant branch of modern mathematics offers a rich tapestry of concepts and applications that permeate diverse fields. From engineering and the physical sciences to social dynamics and biology, its relevance is undeniable. Within this realm, one finds a plethora of intriguing areas, including graph coloring, matching theory, domination theory, graph labeling, and algebraic graph theory, each contributing to the theoretical depth and practical utility of the discipline.

At its core, domination theory has emerged as a focal point of research, embodying the essence of graph theoretic exploration. Its origins can be traced back to the illustrious Euler, who, in 1736, etched his name in history by solving the **Konigsberg Bridge Problem**, thereby laying the groundwork for both graph theory and topology. Since then, domination theory has evolved into a cornerstone of modern graph theory, captivating the minds of mathematicians.

The inception of dominating sets, a fundamental concept within domination theory, can be attributed to Claude Berge in 1958. These sets, defined within the context of graphs, represent subsets of vertices with the remarkable property that every vertex is either included in the set or adjacent to it. While the historical lineage of domination theory finds resonance in the realms of chess, with its roots intertwined in Queens Problem which captivated minds in the 1850s.

Queen's Problem : (Mention by Ore)

In 1850s, Several chess players were interested in the minimum number of queens such that every square on the chess board either contains a queen or is attacked by a queen (recall that a queen can move any number of squares horizontally, vertically, or diagonally on the chess board).

The question arose was "How to place a minimum number of queens on a chessboard so that each square is controlled by at least one queen ?"

Using graph theory to model this problem, the Queens graph (figure 1) is formed by representing each of the 64 (8×8) squares of the chessboard as a vertex of a graph G. Two vertices (squares) are adjacent in G if each square can be reached by a queen on the other square in a single move. Obviously, to solve the queen's problem we are looking for the minimum number of queens that dominate all the squares of the chessboard that is domination number. (Note that many variations on this problem are formed by considering different chess pieces and different size chessboards).

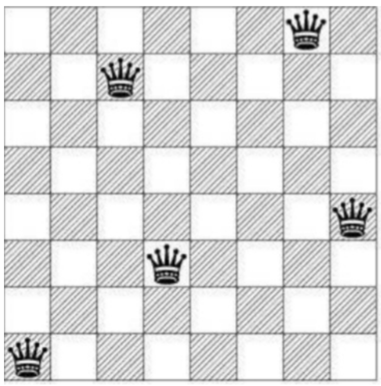

Figure 1.1: **Queen's Dominating the Chessboard**

This problem laid the groundwork for the exploration of domination theory, offering a tangible application for its principles. By employing graph theory to model such scenarios, we unveil the elegance of its applicability in addressing real-world challenges.

For another motivation of this concept, consider a bipartite graph (Figure 1.2) where one part represents people, the other part represents jobs, and the edges represent the skills of each person. Each person may take more than one job. One is interested to find the minimum number of people such that jobs are occupied. As shown in Figure 1.2, {X, W} form a minimum size dominating set.

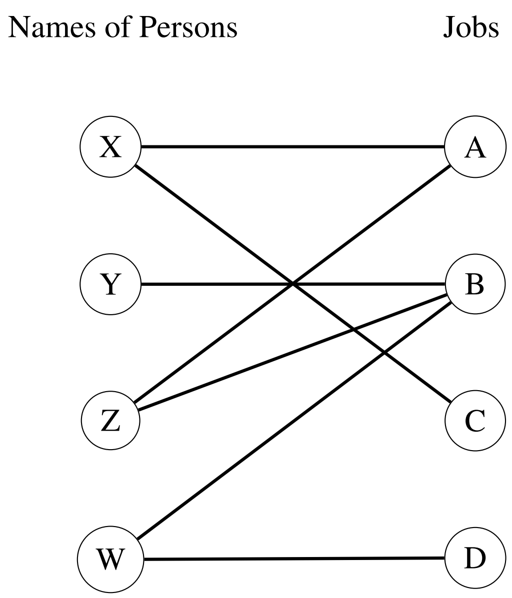

Figure 1.2: **Dominated Jobs**

Yet, the allure of domination theory extends far beyond the confines of chess and job allocation. Its tendrils reach into myriad domains, from facility location problems in logistics, where the optimization of travel distance or service coverage is paramount, to representative selection in political science and network monitoring in telecommunications. Indeed, the versatility and breadth of domination theory underscore its significance in addressing real-world challenges and illuminating the intricate connections that permeate our world.

In the late 1950s and 1960s, the study of domination in graphs saw significant development, with Claude Berge's groundbreaking book on graph theory introducing the coefficient of external stability, now known as the domination number. Concurrently, Oystein Ore pioneered the terms "dominating set" and "domination number," denoted by δ(G), in his 1962 work. Building on these foundations, Cockayne and Hedetniemi, in 1977, established the notation γ(G) to represent the domination number. These luminaries, including Berge, Ore, Cockayne, and Hedetniemi, profoundly influenced the landscape of graph theory, leaving an indelible mark on mathematical inquiry through their innovative concepts and enduring contributions.

1.2 Definitions:

Definition 1.2.0.1. A set D of vertices in a graph $G = (V,E)$ is called a ***dominating set*** of G, if every vertex in $V - D$ is adjacent to some vertex in D.

Definition 1.2.0.2. The ***domination number*** $\gamma(G)$ of a graph G is the minimum cardinality of the dominating set in G.

Definition 1.2.0.3. The ***floor function*** of a real number x is the greatest integer less than or equal to x and is denoted by $\lfloor x \rfloor$.
Suppose that $n \leq$ x $< n+1$, where n is an integer, then $\lfloor x \rfloor$ = n.

Definition 1.2.0.4. The ***ceiling function*** of a real number x is the lowest integer greater than or equal to x and is denoted by $\lceil x \rceil$.
Suppose that $n-1 <$ x $\leq n$, where n is an integer, then $\lceil x \rceil$ = n.

Definition 1.2.0.5. The cardinality of $V(G)$ is called the ***order*** of G and is denoted by $|G|$.

Definition 1.2.0.6. The ***(open) neighbourhood*** $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G and the ***(close) neighbourhood*** $N_G[v]$ is $N_G[v] = N(v) \cup \{v\}$.

Definition 1.2.0.7. The ***union of two disjoint graphs*** G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 1.2.0.8. A ***forest*** is a graph with no cycles and a ***tree*** is a connect forest.

Definition 1.2.0.9. The ***degree of*** v is the cardinality of $N_G(v)$ and is denoted by $deg_G(v)$.

Definition 1.2.0.10. A vertex is said to be a ***leaf*** if $deg_G(v) = 1$.

Definition 1.2.0.11. A set $D \subseteq V$ is a ***Total Dominating Set*** of G if every vertex of G has at least a neighbour in D.

Definition 1.2.0.12. ***The Total Domination Number*** of G is denoted by $\gamma_t(G)$ is the cardinality of minimum TD - set of G.

Definition 1.2.0.13. A subset D of $V(G)$ is called an ***Equitable Dominating Set*** if for every $v \in V(G) - D \ \exists$ a vertex $u \in D \ni uv \in E(G)$ and $|d(v) - d(u)| \leq 1$, where $d(v)$ and $d(u)$ denotes the degree of vertex v and u respectively.

Definition 1.2.0.14. The minimum cardinality of equitable dominating set is called the ***Equitable Domination Number*** of G and is denoted by $\gamma^e(G)$.

Definition 1.2.0.15. A dominating set which is both ***Total and Equitable*** is called total equitable dominating set and is denoted by $\gamma_t{}^e(G)$.

Observation 1.2.0.16. *(ref.[1]) It is known that*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1; \ if\, n \equiv 2(mod4) \\ \lceil \frac{n}{2} \rceil; \ otherwise \end{cases}$$

Theorem 1.2.0.17. *(ref. [2]) For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Chapter 2

DOMINATION NUMBER

2.1 Domination Number of Different Type of Graphs

Definition 2.1.0.1. A ***Fan graph*** $F_{m,2} = \overline{K}_m + P_2$ where $\overline{K}_m$ is the empty graph (consisting of m isolated nodes with no edges) and P_2 is the path graph on two nodes.

Definition 2.1.0.2. A ***Fire Cracker*** $F(m,n)$ is a graph obtained by the series of interconnected m copies of n stars by linking one leaf from each.

Definition 2.1.0.3. The graph G consists of collection of n cycles C_4, these cycles are connected in such a way that any two adjacent cycles share a common vertex, the resulting graph is called a ***diamond snake graph*** and it is denoted by D_n. A diamond snake has $3n+1$ vertices and $4n$ edges, where n is the number of blocks in the diamond snake. A snake is an Eulerian path that has no chords.

Definition 2.1.0.4. A ***banana tree*** $B(m,n)$ is a graph obtained by connecting one leaf of each of m copies of n-star graph with a single root vertex 'v'. Note that edges contain pendant nodes are called tree leaves.

Definition 2.1.0.5. A ***coconut tree*** $CT(m,n)$ is the graph obtained from the path P_m by appending n new pendant edges at an end vertex of P_m.

Theorem 2.1.0.6. *The domination number of any fan graph $F_{m,2}$ is 1, where $m \geq 1$.*

Proof. let $G \cong F_{m,2}$ be a fan graph on $m+2$ vertices with $2m+1$ edges and let D be a minimum dominating set of graph G.

By definition of the fan graph, the graph $G = \overline{K}_m + P_2$ where $\overline{K}_m$ is the empty graph (consisting of m isolated nodes with no edges) and P_2 is the path graph on two nodes.

Let $V(P_2) = \{u, v\}$. There are 2 nodes available in path P_2 of fan graph, so that if we choose any one vertex from the path P_2 then all the other vertices of G are dominated by our chosen vertex.

So we will get a minimum dominating set and its cardinality is the domination number of the graph and hence the Dominating Set D of $G = \{u\}$ or $\{v\}$.

$\therefore$ the domination number of graph G is 1. i.e. $\gamma(G) = 1$. □

Example: 2.1.0.7. *The fan graph $F_{4,2}$ is shown in Figure 2.1 below.*

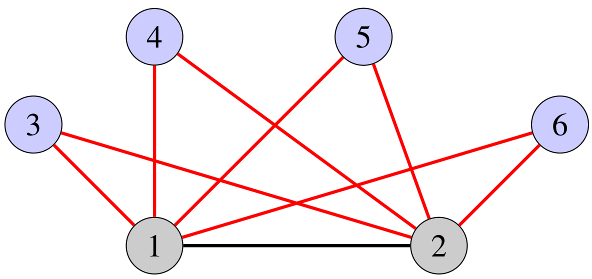

Figure 2.1: **Fan Graph** $F_{4,2}$
$D = \{1\}$ or $\{2\}$ and $\gamma(F_{4,2}) = 1$

Theorem 2.1.0.8. *For any firecracker graph $F(m,n)$, the domination number is m, where $n \geq 2$.*

Proof. Let $G \cong F(m,n)$ be a firecracker graph on mn vertices with $(mn) - 1$ edges and let D be a minimum dominating set of graph G.

By definition of firecracker graph, the graph is obtained from series of interconnected m copies of n stars by linking one leaf from each.

For each of the n starts, if we choose all of the central vertices as one set, it will dominate all the other vertices of G.

So we will get a minimum dominating set and its cardinality is the domination number of graph G.

$\therefore$ the domination number of G is m. That is $\gamma(G) = m$. $\square$

Example: 2.1.0.9. *The dominating set and the domination number of the firecracker graph $F(2,4)$ is shown in Figure 2.2 below.*

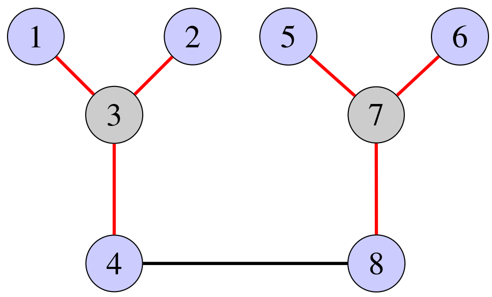

Figure 2.2: **Firecracker Graph F(2,4)**
D=$\{3,7\}$ and $\gamma(F(2,4)) = 2$

Theorem 2.1.0.10. *For any diamond snake graph D_n, the domination number is $n+1$, where $n \geq 1$.*

Proof. Let $G \cong D_n$ be a diamond snake graph on $3n+1$ vertices with $4n$ edges and let D be a minimum dominating set of graph G.

By definition of the diamond snake graph, the graph G consists of collection of n cycles C_4, these cycles are connected in such a way that any two adjacent cycles share a common vertex, where n is the number of blocks in the diamond snake.

If we choose any one of the vertices of degree 2 from the first and the last copies of G, then we choose all common vertices which are shared by consecutive cycles of G.

So we will get a minimum dominating set and its cardinality is the domination number of graph G.

$\therefore$ the domination number of G is $n+1$.

That is $\gamma(G) = n+1$. □

Example: 2.1.0.11. *The dominating set and the domination number of the diamond snake graph D_2 is shown in Figure 2.3 below.*

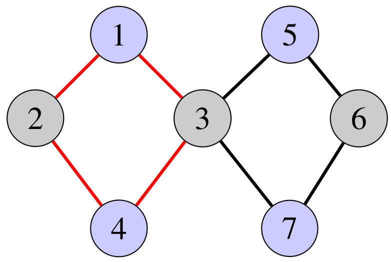

Figure 2.3: **Diamond Snake Graph** D_2
D ={2,3,6} and $\gamma(D_2) = 3$

Theorem 2.1.0.12. *For any Banana tree $B(m,n)$, the domination number is $\gamma(G) = m+1$, where $m \geq 1$, $n \geq 3$.*

Proof. Let $G \cong B(m,n)$ be a banana tree on $(mn)+1$ vertices with mn edges and let D be a minimum dominating set of graph G.

By definition of the banana tree, the graph is obtained by connecting one leaf of each of m copies of a n star graph with a new single root vertex 'v'.

We distinguish 3 cases to obtain the domination number of graph G.

Case 1: The domination number of banana tree graph $B(m,1)$ is shown in Figure 2.4 below.

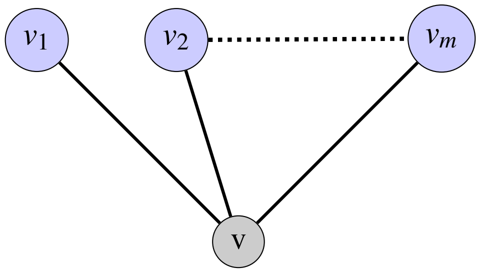

Figure 2.4: **Banana Tree Graph** $B(m,1)$
$D = \{\text{v}\}$ and $\gamma(G) = 1$

Case 2: The domination number of banana tree graph $B(m,2)$ which is shown in Figure 2.5 below.

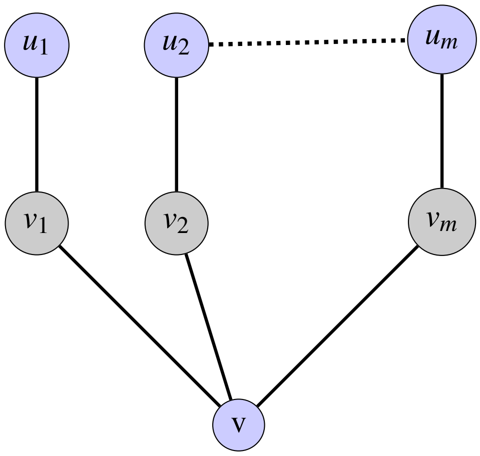

Figure 2.5: **Banana Tree Graph** $B(m,2)$
$D = \{v_1,v_2,\ldots,v_m\}$ and $\gamma(G) = m$.

Case 3: Let $G \cong B(m,n)$, where $n \geq 3$.

Moreover $\deg(v) = \Delta(G)$. So 'v' must be included in any minimum dominating set of G.

For G, if we choose the apex vertices of every star graph it will dominate all the other vertices except single root vertex.

Hence $\gamma(G) = m + |\{v\}| = m+1$.

$\therefore$ the domination number of G is $m+1$. $\square$

Example: 2.1.0.13. *The dominating set and the domination number of the banana tree graph $B(2,5)$ is shown in Figure 2.6 below.*

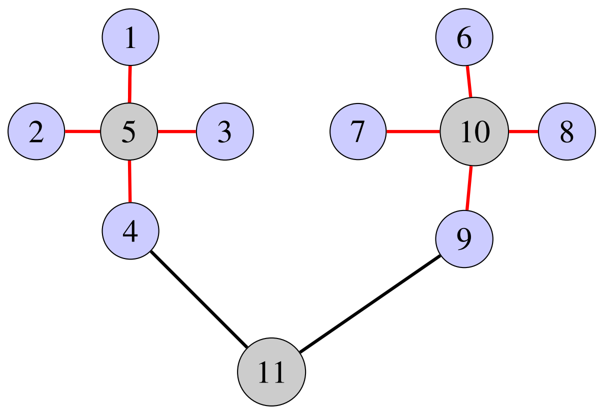

Figure 2.6: **Banana Tree Graph** $B(2,5)$

D ={5,10,11} and $\gamma(B(2,5)) = 3$

Theorem 2.1.0.14. *For any coconut tree $CT(m,n)$, the domination number is 1 + $\lceil \frac{m-2}{3} \rceil$, where $m \geq 1$, $n \geq 1$.*

Proof. Let $G \cong CT(m,n)$ be a coconut tree on $m+n$ vertices with $m+(n-1)$ edges and let D be a minimum dominating set of graph G.

By definition of coconut tree, the graph is obtained from the path P_m by appending n new pendent edges at an end vertex v_1 (say) of P_m.

Clearly the vertex v_1 must be included in any minimum dominating set, since deg(v_1) = $\Delta(CT(m,n))$.

In G, v_1 dominates all pendant vertices attached with v_1.

$\therefore$ the domination number of G is $1 + \gamma(P_{m-2}) = 1 + \lceil \frac{m-2}{3} \rceil$.

Hence γ(G) $= 1 + \lceil \frac{m-2}{3} \rceil$. □

Example: 2.1.0.15. *The dominating set and the domination number of the coconut tree graph $CT(4,7)$ is shown in Figure 2.7 below.*

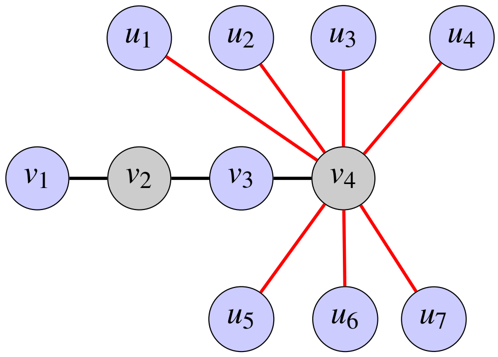

Figure 2.7: **Coconut Tree** $CT(4,7)$
$D = \{v_2, v_4\}$ and $\gamma(CT(4,7)) = 2$

2.2 Domination Number of Trees

Definition 2.2.0.1. Two distinct vertices u and v are called ***duplicated*** in G if $N_G(u) = N_G(v)$.

Definition 2.2.0.2. A vertex of G is a ***support vertex*** if it is adjacent to a leaf in G.

Definition 2.2.0.3. A dominating set of cardinality $\gamma(G)$ in G is said to be a ***γ-set***.

Definition 2.2.0.4. A γ-set containing all support vertices of G is called a γ_U - set.

Notation: $L(G)$ and $U(G)$ are the collection of all leaves and support vertices of G respectively.

Lemma 2.2.0.5. *If uv is an edge of a connected graph G and $G-uv = G_1 \cup G_2$ then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$.*

Proof. Suppose uv is an edge of a connected graph G and $G-uv = G_1 \cup G_2$.

let S_i be a γ-set of G_i for $i = 1,2$.

Suppose $S = S_1 \cup S_2$.

Then $N[S] = N[S_1 \cup S_2] = N[S_1] \cup N[S_2] = V(G)$.

So S is a dominating set of G.

Thus $\gamma(G) \leq |S| = |S_1| + |S_2| = \gamma(G_1) + \gamma(G_2)$. □

Lemma 2.2.0.6. *If G is a graph with at least three vertices then there exists a γ_U-set of G.*

Proof. Suppose G is a graph with at least three vertices and let S be a γ-set of G.

If S is a γ_U-set of G, then we are done.

So we assume that $A = U(T) - S \neq \emptyset$, and let $B = L(T) \cap N(A)$.

Then $B \subseteq S$ and $|B| \geq |A|$.

Let $S' = (S-B) \cup A$.

Then $N[S'] = V(G)$.

So S' is a dominating set of G.

Thus $|S| = \gamma(G) \leq |S'| = |S|$ - $|B|$ + $|A| \leq |S|$, the equalities hold and S' is a γ_U-set of G. □

Lemma 2.2.0.7. *If x and x' are two duplicated leaves adjacent to the same support vertex in a graph G then $\gamma(G-x') = \gamma(G)$.*

Proof. Suppose x and x' are two duplicated leaves adjacent to the same support vertex in a graph G and let $G' = G - x$.

If S is a γ_U-set of G then S is a γ_U-set set of G'.

So $\gamma(G-x') = \gamma(G') = |S| = \gamma(G)$. □

Observation 2.2.0.8. *Lemma 2.2.0.7 implies that the maximum order of a tree T with $\gamma(T) = k$ is infinite. Consequently, attention is directed towards trees devoid of duplicated leaves. The objective becomes determining the minimum order of such trees with $\gamma(T) = k$ and subsequently characterizing them.*

Theorem 2.2.0.9. *If T is a tree with at least two vertices and $\gamma(T) = k$ where $k \geq 1$ then $|T| \geq 2k$.*

Proof. Suppose T is a tree with at least two vertices and $\gamma(T) = k$ where $k \geq 1$.
let S be a γ - set of T.

Then $N[S] = V(G) = N[V(T) - S]$.

So $S' = V(T) - S$ is a dominating set of T.

Hence $|S'| \geq k$ and $|T| = |S| + |S'| \geq k + k = 2k$. □

Lemma 2.2.0.10. *Let T be a tree with at least two vertices and $\gamma(T) = k$ where $k \geq 1$. if $|T| = 2k$ then T has no duplicated leaf.*

Proof. Let T be a tree of order $2k$ and $\gamma(T) = k$ where $k \geq 1$.
suppose that there exist two distinct leaves x and x' adjacent to y in T, by Lemma 2.2.0.7.
then $\gamma(T - x') = \gamma(T) = k$.

Note that $T' = T - x'$ is a tree.

By theorem 2.2.0.9. $|T| \geq 2k$.

Thus $|T| = |T'| + 1 \geq 2k + 1$.

This is a contradiction, we complete the proof. □

Theorem 2.2.0.11. *Let T be a tree with at least two vertices and $\gamma(T) = k$, where $k \geq 1$. If $|T| = 2k$, then $V(T) = U(T) \cup L(T)$ and $|U(T)| = k$.*

Proof. We prove this theorem by induction on $k \geq 1$.

If $k = 1$, then $T = P_2$ and If $k = 2$, then $T = P_4$.

Its true for $k = 1$ and $k = 2$.

Let $k \geq 3$. Assume that it's true for all $k' < k$.

Suppose that T is a tree of order $2k$ and $\gamma(T) = k$.

Let $P_i : x_i, y_i, z, w, u, ...$ be a longest path of T , where $|P_i| = m \geq 5$ and $i = 1, ..., a$.

By Lemma 2.2.0.10. , then $|N(y_i) \cap L(T)| = 1$ for every i.

Let $A = \{y_1, ..., y_a\}$.

If $m = 5$, then $a = k - 2$. Thus $z \in U(T)$ and $U(T) = A \cup \{z, w\}$.

So its true for $m = 5$.

Thus we assume that $m \geq 6$.

Claim 1: $z \in U(T)$.

Suppose that $z \notin U(T)$, then $N(z) = A \cup \{w\}$ and

$H = T - N[A]$ is a tree of order $|H| = |T| - (2a + 1) = 2(k - a) - 1 \geq 3$.

By Theorem 2.2.0.9. , $\gamma(H) \leq k - a - 1$.

Note that $z \in N(A)$.

By Lemma 2.2.0.5. , $k = \gamma(T) \leq |A| + \gamma(H) \leq a + (k - a - 1) = k - 1$.

This is a contradiction , so $z \in U(T)$.

Let z' be the leaf of z in T and $T' = T$ - $(N[A] \cup \{z, z'\})$ and $T' = T - V(T')$.

Then T is a tree of order $|T'| = |T|$ - $(2a - 2) = 2(k - a - 1) \geq 3$.

By Theorem 2.2.0.9. , $\gamma(T') \leq k - a - 1$.

Note that zw is an edge of T such that $T - zw = T^* \cup T$.

by Lemma 2.2.0.5, $K = \gamma(T) \leq \gamma(T*) + \gamma(T) \leq (a + 1) + (k - a - 1) = k$.

The equality hold, $\gamma(T') = k - a - 1$.

Hence T' is a tree of order $|T'| = 2(k-a-1)$ and $\gamma(T') = k-a-1$, by induction hypothesis $V(T') = U(T') \cup L(T')$.

Claim 2: $w \in U(T')$

Suppose that $w \notin U(T')$, then $w \in U(T')$ and $T" = T' - \{w\}$ is a tree of order $|T''|$ = $|T'|$ - 1 = $2(k-a-1)$ - 1 ≥ 2. Hence by Theorem 2.2.0.9. , we have that $\gamma(T'') \leq k-a-2$.

Note that $w \in N(z)$ and $T - wu = (T - V(T'')) \cup T''$. By Lemma 2.2.0.5. , $K = \gamma(T) \leq \gamma\,(T - V(T'')) + \gamma(T'') \leq (a+1) + (k-a-2) = k-1$.

This is a contradiction, so $w \in U(T)$.

□

Observation 2.2.0.12. *The maximum order of trees T without duplicated leaves and $\gamma(T) = k$ has been established. Subsequently, the tree of maximum orders have been characterized.*

Definition 2.2.0.13. Let $\Omega(k)$ be the collection of trees T which holds the following Properties:

- T has no duplicated leaves.
- $\gamma(T) = |U(T)| = k$.
- For each $v \in U(T)$, $\delta(v) = \min\,\{d(u,v) : u \in U(T)\} = 3$, where $d(u,v)$ is the distance between u and v.

Lemma 2.2.0.14. *Suppose $T \in \Omega(k)$, then T is a tree without duplicated leaves of order $|T| = 4k-2$ and $U(T)$ is a γ_U set of T, where $\gamma(T) = |U(T)| = k$.*

Theorem 2.2.0.15. *Suppose T is a tree without duplicated leaves and $\gamma(T) = k$, where k $\geq$ 1. Then $|T| \leq 4k-2$. The inequality holds iff $T \in \Omega(k)$.*

Proof. Proof by induction on k.

It's true for $k = 1$, so we assume that $k \geq 2$.

Suppose that T is a tree without duplicated leaves and $\gamma(T) = k$ such that $|T|$ is as large as possible.

By lemma 2.2.0.14. , then we obtain that $|T| \geq 4k-2$.

Let S be a γ_U - set of T.

Since $|T|$ is as large as possible, we obtain that $S = U(T)$ and $N[u] \cap N[v] = \phi$ for $u \neq v$ in S.

Thus $|V(T) - \{U(T) \cup L(T)\}| \leq 2(|S| - 1) = 2k-2$.

Hence $4k-2 \leq |T| = |U(T) \cup L(T)| + |V(T) - \{U(T) \cup L(T)\}| \leq 2k + 2(|S| - 1) = 4k-2$.

The inequalities hold, $\gamma(T) = |U(T)| = |L(T)| = k$ and $|V(T) - \{U(T) \cup L(T)\}| = 2(|S| - 1) = 2k-2$.

That is $T \in \Omega(k)$. □

2.3 Domination Number of Certain Mycielski Type of Graphs

2.3.1 Mycielski Graph:

The **Mycielski graph** of a graph G with vertex set $V(G) = \{v_1,v_2,...,v_n\}$ is the graph which is obtained by applying the following steps:

1. Corresponding to each vertex v_i in $V(G)$, introduce a new vertex u_i and let $U = \{u_i : 1 \leq \text{i} \leq n\}$. Add edges from each vertex u_i of U to the vertex v_j if $v_iv_j \in E(G)$.
2. Take another vertex w and add edges from w to all vertices in U.

The New graph thus obtained is called the Mycielski graph of G and is denoted by $\mu(G)$.

Theorem 2.3.1.1. *For any mycielskian graph with 2n+1 vertices $\gamma[\mu(P_n)] = \gamma(P_n) + 1$ for $n \geq 2$.*

Proof. Proof by induction on n.

for n = 2; $\gamma[\mu(P_2)] = 2 = \gamma(P_2) + 1$.

Assume the statement holds for $n = k$; $k \geq 2$. i.e $\gamma[\mu(P_k)] = \gamma(P_k) + 1$.

We want to show that the result holds for $n = k+1$.

Consider the mycielski graph $\mu(P_{k+1})$.

This graph is obtained by adding one vertex to P_k and adding a vertex u_{K+1} to the V($\mu(P_k)$).

So the V($\mu(P_{k+1})$) = {v_1,v_2,...,v_{k+1},u_1,u_2,...,u_{k+1},w}

The domination number of $\mu(P_{k+1})$ by considering the following 2 cases:-

Case 1: The added vertex is dominated by a vertex in P_k.

In this case the domination number remains the same.

i.e. $\gamma[\mu(P_k)] = \gamma(P_k) + 1$.

Case 2: The added vertex dominates a vertex in P_k.

In this case the domination number increases by 1 as compared to $\mu(P_k)$.

Since the added vertex contribute one to the dominating set.

hence $\gamma[\mu(P_{k+1})] = \gamma(P_k) + 1$ or $\gamma(P_{k+1}) + 1$ depending on which case applies.

Hence by induction, the statement holds $\forall$ $n \geq 2$. □

Example: 2.3.1.2. *The Mycielski graph of the path P_6 is shown in Figure 2.8. below.*

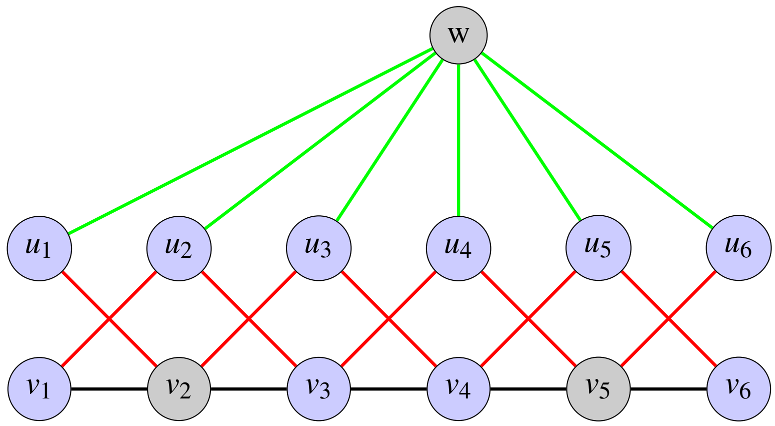

Figure 2.8: **Mycielski graph of (P_6)**

$\gamma[\mu(P_6)] = 3$

2.3.2 Crib Graph:

The **crib graph** of a given graph G with $V(G) = \{v_1, v_2, ..., v_n\}$ is the graph which is obtained by applying the following steps:

1. Corresponding to each vertex v_i in $V(G)$, introduce a new vertex u_i and let $U = \{u_i : 1 \leq i \leq n\}$. Add edges from each vertex u_i of U to the vertex v_j if $v_i v_j \in E(G)$.

2. Take another vertex w and add edges from w to all vertices in both U and V.

The new graph thus obtained is called the crib graph of G and is denoted by $C(G)$.

Theorem 2.3.2.1. *For any crib graph with n vertices domination number is 1.*
i.e. $\gamma(C(P_n)) = 1$.

Proof. Consider the structure of the crib graph $C(P_n)$ where P_n is a path on n vertices. In crib graph $C(P_n)$ we have $2n+1$ vertices that includes n vertices from the path P_n, n vertices from the set $U = \{u_i : 1 \leq \text{i} \leq \text{n}\}$ and one additional vertex w.

i.e. $V(C(P_n)) = \{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n,w\}$.

Note that w is connected to all vertices in the path P_n and to the vertices from the set U.

Thus $D = \{w\}$.

hence $\gamma(C(P_n)) = 1$. □

Example: 2.3.2.2. *The Crib graph of of the path P_5 is shown in Figure 2.9. below.*

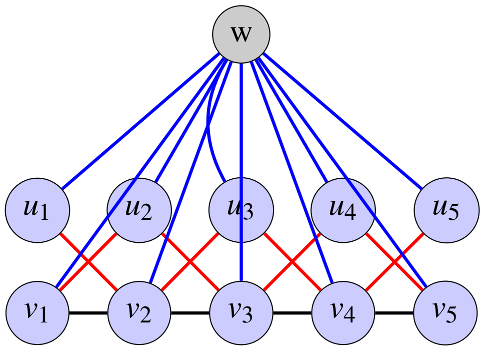

Figure 2.9: **Crib Graph of** P_5

$\gamma[C(P_5)] = 1$

2.3.3 Modified Mycielski Graph:

Let G be a triangle free graph with $V(G) = \{v_1,v_2,\ldots,v_n\}$. We define the **modified mycielski graph** $\mu^*(G)$ of G as the graph such that $V(\mu^*(G)) = \{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n,w_1,w_2,\ldots,w_n\}$ and with edges that follows the below mentioned rules:

1. $v_iv_j \in E(\mu^*(G)) \Longleftrightarrow v_iv_j \in E(G)$.
2. $w_iw_j \in E(\mu^*(G)) \Longleftrightarrow v_iv_j \in E(G)$.
3. $w_iu_j \in E(\mu^*(G)) \Longleftrightarrow u_iv_j \in E(G)$.
4. $\forall\ i = 1,2,\ldots,n;\ \ v_iv_j \in V(\mu^*(G))$.

Theorem 2.3.3.1. *For any modified mycielski graph* $\gamma_k(\mu^*(G)) = 2\ \gamma_k(G)$.

Proof. Proof by induction on k.

Let D be a dominating set of G.

Construct a dominating set D' for $\mu^*(G)$ as follows:

- For each vertex v_i in G, add both u_i and w_i to D'.
- For each vertex v_i in G, if v_i is in D, then add v_i to D'.

Clearly, $|D'| = 2|D|$, so $\gamma_1(\mu^*(G)) = 2\gamma_1(G)$.

Assume the statement holds for $k = n$, i.e., $\gamma_n(\mu^*(G)) = 2\gamma_n(G)$.

To Prove: $\gamma_{n+1}(\mu^*(G)) = 2\gamma_{n+1}(G)$.

Let D be a dominating set of size $\gamma_{n+1}(G)$.

We want to construct a dominating set D' of $\mu^*(G)$ such that $|D'| = 2\gamma_{n+1}(G)$.

Consider the construction of $\mu^*(G)$:

- Each vertex v_i in G has corresponding vertices u_i and w_i in $\mu^*(G)$.
- Let D' contain all the vertices from D that correspond to vertices in G.
- For each vertex v_i in G, add both u_i and w_i to D'.

Now, let's analyze D':

- Every vertex in G is dominated by D.
- For each v_i in G, u_i's are adjacent to v_i, so they dominate v_i.
- Additionally, u_i and w_i are adjacent to each other, ensuring that they dominate each other as well.
- Therefore, D' is a dominating set for $\mu^*(G)$, and $|D'| = 2\gamma_{n+1}(G)$.

Hence, by the principle of mathematical induction, the theorem holds ∀ positive integers k. Thus, $\gamma_k(\mu^*(G)) = 2\gamma_k(G)$ for any positive integer k. □

Example: 2.3.3.2. *The Modified Mycielski graph of P_5 is shown in Figure 2.10. below.*

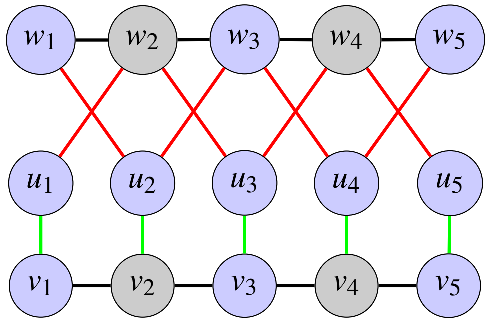

Figure 2.10: **Modified Mycielski Graph of** P_5

$\gamma_k(\mu^*(G)) = 4 = 2\ \gamma_k(G)$

Chapter 3

TOTAL DOMINATION NUMBER

3.1 Total Domination in Graph Operators

Definition 3.1.0.1. For a graph $G = (V,E)$: the ***Subdivision Operator***, denoted by $S(G)$, acts on G by replacing each of its edges by a path of length two.

NOTATION:

If $G = (V,E)$, where $V = \{v_1, \ldots, v_n\}$ then $S(G) = (V', E')$ where $V' = V \cup \{v^{i,j} \colon v_i, v_j \in E,\ i < j\}$ and $E' = \{v_i v^{i,j}, v_j v^{i,j} \colon v_i, v_j \in E,\ i < j\}$. To avoid writing all the time $i < j$, we will consider $v^{i,j} = v^{j,i}$.

Example: 3.1.0.2. *The graph G and S(G) of G is shown in Figure 3.1 and Figure 3.2 below respectively.*

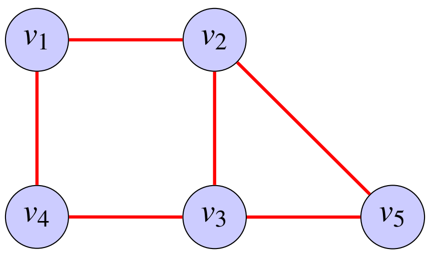

Figure 3.1: **Graph** G

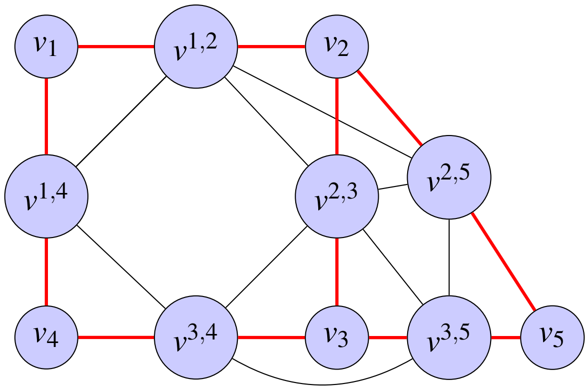

Figure 3.2: $S(G)$ **of** G

The following corollary might be useful when we do not know the independence number.

Corollary 3.1.0.3. *Let G be a graph with order n, which is not a complete graph then* $\gamma_t(S(G)) \leq \lfloor \frac{3n}{2} \rfloor$ *- 1.*

Observation 3.1.0.4.

$$\text{It is known that } \gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2}+1; & \text{if } n \equiv 2 (\text{mod } 4) \\ \lceil \frac{n}{2} \rceil; & \text{otherwise} \end{cases}$$

Since $S(C_n) = C_{2n}$, the total domination number of the transformation of this graph is $\gamma_t(S(C_n)) = n+1$ when $n \equiv 1 (\text{mod } 2)$ [if $n \equiv 3 (\text{mod } 4)$ then $2n \equiv 2 (\text{mod } 4)$] and $\gamma_t(S(C_n))$ = n.

Theorem 3.1.0.5. *For a wheel graph W_n with n vertices, we have $\gamma_t(S(W_n))$ = 2 + $\gamma_t(P_{2n-3})$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$, where v_1 is the center of wheel graph.

If D_o is a minimum total dominating set in the path with vertices $v^{2,3}, v_3, v^{3,4}, v_4, ...,$ $v_n, v^{n,2}$, then $D = \{v_1, v^{1,2}\} \cup D_o$ is a total dominating set in $S(W_n)$, so $\gamma_t(S(W_n)) \leq 2 + \gamma_t(P_{2n-3})$.

Let D be a minimum total dominating set in $S(W_n)$ such that $v_1, v^{1,2} \in D$.

Since this set must contain a total dominating set in the path in which its vertices are $v^{2,3}, v_3, v^{3,4}, v_4, \ldots, v_n, v^{n,2}$ then $|D| = \geq 2 + \gamma_t(P_{2n-3})$.

If v_1 does not belong to D then $v_2, v_3, \ldots, v_n$ must belong to D.

Moreover, as $\{v_2, V_3, \ldots, v_n\}$ is an independent set, we need $\lceil \frac{n-1}{2} \rceil$ more vertices, and then $|D| > 2 + \gamma_t(P_{2n-3})$. □

Example: 3.1.0.6. *The total domination number of $S(W_5)$ is shown in Figure 3.3 below.*

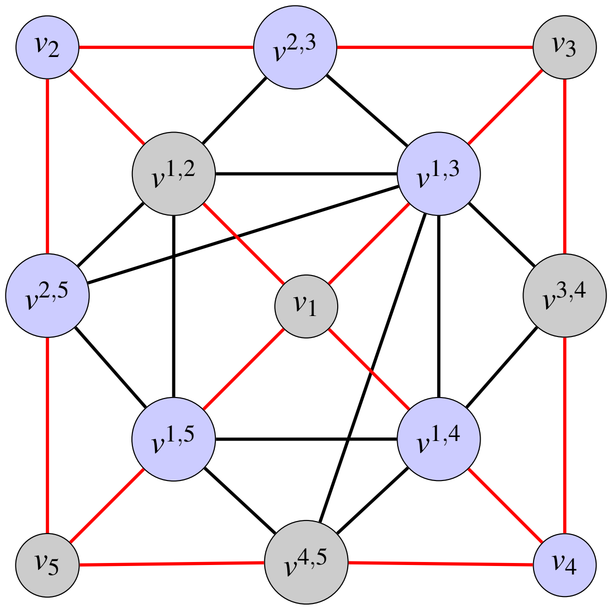

Figure 3.3: $S(W_5)$ **of** W_5

$\gamma_t(S(W_5))$ **= 6 = 2 +** $\gamma_t(P_{2n-3})$

Theorem 3.1.0.7. *For the complete graph K_n, we have $\gamma_t(S(K_n)) = \lceil \frac{3n}{2} \rceil - 1$.*

Proof. We denote $V(K_n) = \{v_1, \ldots, v_n\}$.

Now if n is an even number then the set $\{v^{1,2}, v_2, v_3, v^{3,4}, v_4, \ldots, v_{n-1}, v^{n-1,n}, v_n\}$ is a total

dominating set and if n is odd number, the set
$\{v^{1,2}, v_2, v_3, v^{3,4}, v_4, ..., v_{n-2}, v^{n-2,n-1}, v_{n-1}, v_n, v_{1,n}\}$ is a total dominating set.

Then $\gamma_t(\mathrm{S}(K_n)) \leq \lceil \frac{3n}{2} \rceil$ - 1.

Now let D be a minimum total dominating set in $S(K_n)$.

If $\exists\, v_i \notin D$, then to dominate $v^{s,i}$ with $s \neq i$, it is necessary to have $v_s \in D$ for $s \neq i$.

As v_i must be dominated by $D \; \exists\, j \neq i \ni v^{i,j} \in D$.

Since $v_s \in D$ for every $s \neq i$, and it is an independent set in $S(K_n)$ for every two of those vertices we need another vertex in D connecting them.

Therefore $|D| \geq n + \frac{n-2}{2} = \frac{3n}{2}$ - 1.

Finally if $\{v_1, ..., v_n\} \subseteq D$, since this is an independent set in $S(K_n)$, D must contain at least $\frac{n}{2}$ more vertices, so $|D| \geq n + \frac{n}{2} \geq \lceil \frac{3n}{2} \rceil + 1$. $\square$

Example: 3.1.0.8. *The total domination number of $S(K_4)$ is shown in Figure 3.4 below.*

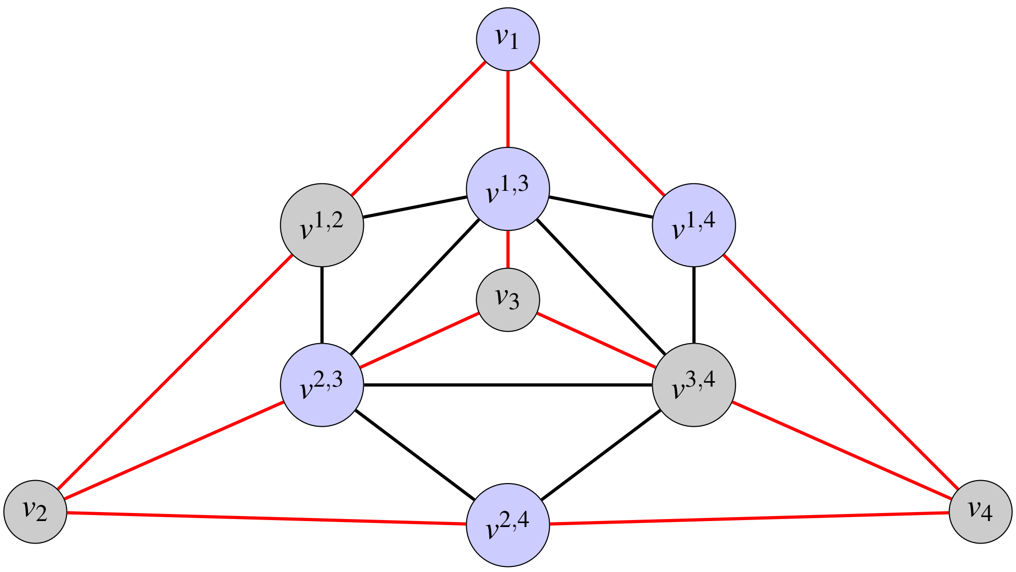

Figure 3.4: $S(K_4)$ **of** K_4

γ_t**(S(**K_4**))** = 5

Definition 3.1.0.9. For a graph $G = (V,E)$: the **Operator** $R(G)$, acts on G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

Example: 3.1.0.10. *The graph G and $R(G)$ of G is shown in Figure 3.5 and Figure 3.6 below.*

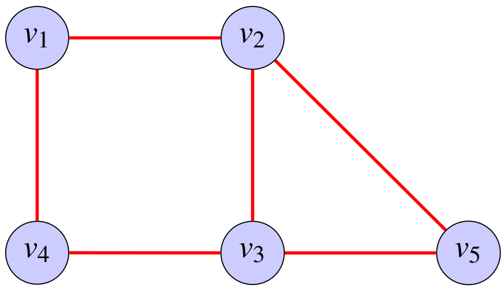

Figure 3.5: **Graph** G

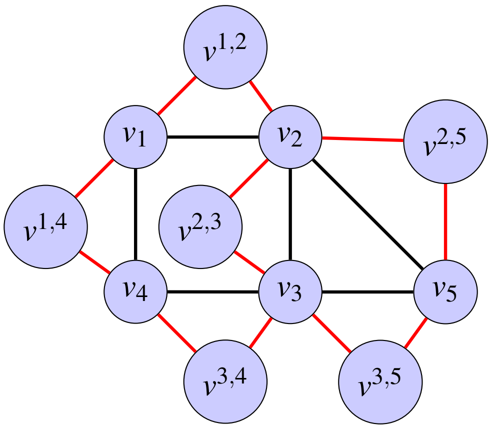

Figure 3.6: $R(G)$ **of** G

NOTATION:

If $G = (V,E)$, where $V = \{v_1,...,v_n\}$ then $R(G) = (V',E')$ where $V' = V \cup \{v^{i,j}\colon v_i,v_j \in E,\, i < j\}$ and $E' = E \cup \{v_i v^{i,j}, v_j v^{i,j}\colon v_i,v_j \in E,\, i < j\}$.

To avoid writing all the time $i < j$, we will consider $v^{i,j} = v^{j,i}$.

Definition 3.1.0.11. A total dominating set D of G is called a ***Total Co-independent*** dominating set if the set of vertices $V-D$ is a non-empty independent set.

Definition 3.1.0.12. The minimum cardinality of any total co-independent dominating set is the ***Total Co-independent domination number*** and is denoted by $\gamma_{t,coi}(G)$.

Theorem 3.1.0.13. *For any graph G with order $n \geq 3$, we have $\gamma_t(G)$ = $\gamma_{t,coi}(G)$.*

Proof. If D is a total co-independent dominating set in G, it is a total dominating set in G and any edge contains a vertex of D and then it is a total dominating set in $R(G)$.

$\therefore \gamma_t(G) \leq \gamma_{t,coi}(G)$.

Now, let D be a minimum total dominating set in $R(G)$.

If $v^{i,j} \in D$ for some $1 \leq i < j \leq n$, then $v^i \in D$ or $v^j \in D$.

If for instance $v^i \in D$, we take D' = (D - $\{v^{i,j}\}$) $\bigcup$ $\{v^j\}$, which is also a minimum total dominating set in $R(G)$.

Doing that with any vertex $v^{i,j} \in D$, we obtain a TD - set $D' \subseteq V$ in both graphs $R(G)$ and G such that any edge in E contains a vertex of D', so D' is a total co-independent dominating set of G.

$\gamma_{t,coi}(G) \leq D$ = $|D|$ = $\gamma_t(G)$. □

Proposition 3.1.0.14. *Let G be a graph with order n and independence number $\alpha(G)$. Then $n-\alpha(G) \leq \gamma_t(R(G)) \leq 2(n-\alpha(G))$.*

Proposition 3.1.0.15. *Let G be a graph with order n, size m, minimum degree δ and maximum degree Δ. Then,*

$$max\left\{\frac{n\delta}{\Delta+\delta-1}, \frac{2m+n\delta}{3\Delta+\delta-2}\right\} \leq \gamma_t(R(G)) \leq n-1.$$

Moreover $\gamma_t(R(G)) = n-1$ iff G is K_n, P_3, C_4, C_5.

Definition 3.1.0.16. A ***Hair*** in G is an edge $uv \in E(G)$ such that $d(u) = 2$ and $d(v) = 1$.

Proposition 3.1.0.17. *If $L(G)$ is the number of leaves of a graph G with order n, and $\exists$ a vertex v, such that it is not a leaf, nor a support vertex nor adjacent to a hair then $\gamma_t(R(G)) \leq n - L(G) - 1$.*

Proof. If $L = \{u \in V : d(u) = 1\}$, then the set $D = V - (L \cup \{v\})$ is a TD - set in G. Since v is neither a support vertex nor adjacent to any hair, and $L \cup \{v\}$ is an independent set, we have that D is a total co-independent dominating set in G.
$\therefore \gamma_t(R(G)) \leq |D| = \leq n - L(G) - 1$. □

Observation 3.1.0.18. *The upper bound in Proposition 3.1.0.17. is attained, for instance, if we take any graph G with minimum degree $\delta(G) \geq 2$, and we add a leaf to every vertex in G except one.*

Proposition 3.1.0.19. *Let G be a graph with order n, minimum degree $\delta \geq 2$ and independence number $\alpha(G)$. Then $\gamma_t(R(G)) \leq n - \delta + 1$ iff $\alpha(G) \geq \delta - 1$.*

Proof. If $\alpha \geq \delta - 1$ and we take an independent set $A = \{v_1, v_2, ..., v_{\delta-1}\}$ then every vertex $v \in V - A$ satisfies $d_{V-A}(v) \geq 1$, and then $V - A$ is a total co-independent dominating set in G, so $\gamma_t(R(G)) \leq n - \delta + 1$.

On the other hand if $\alpha \leq \delta - 2$,
by Proposition 3.1.0.14.
we have $\gamma_t(R(G)) \geq n - \alpha(G) \geq n - \delta + 2$. □

Corollary 3.1.0.20. *Let G be a graph with order n, minimum degree $\delta \geq 2$, and independence number $\alpha(G)$. The following conditions hold:*

1. *If $\alpha(G) = \delta(G) - 1$, then $\gamma_t(R(G)) = n - \alpha(G)$*

2. *If* $\alpha(G) \geq \delta$, *then* $\gamma_t(R(G)) = n - \delta + 1$.

Theorem 3.1.0.21. *Let G be a graph with order n, minimum degree* δ *and maximum degree* Δ*. If* $n \geq (\delta - 3)\Delta + \delta + 2$, *then* $\gamma_t(R(G)) \leq n - \delta + 1$.

Proof. Let v_1 be any vertex in the graph such that $\delta(v_1) = \delta$, we take any vertex $v_2 \in V - \{v_1\}$ adjacent to $N(v_1)$.

Note that $\{v_1, v_2\}$ is an independent set and $|N[v_1] \cup N[v_2]| \leq \delta + 1 + \Delta$.

Now since G is a connected graph, $\exists\, v_3 \in V - \{v_1, v_2\}$ adjacent to $N(v_1) \cup N(v_2)$ and then we have that $\{v_1, v_2, v_3\}$ is an independent set and $|N[v_1] \cup N[v_2] \cup N[v_3]| \leq \delta + 1 + 2\Delta$.

We can continue this process to obtain an independent set
$A = \{v_1, v_2, ..., v_{\delta-1}\}$.

Using Corollary (3.1.20.) we conclude that $\gamma_t(R(G)) \leq n - \delta + 1$. □

Observation 3.1.0.22.

1. *The upper bound in Theorem 3.1.0.21. is attained, for instance if we consider a complete graph* G_1 *with 5 vertices* $v_1, ..., v_5$, *a* $(2r-2)$ *- regular graph* G_2 *with vertices* $u_1, u_2, ..., u_{2r}$ *(*$r \geq 3$*), and the graph* $G = (V, E)$ *with order* $n = 2r + 5$ *such that* $V = V(G_1) \cup V(G_2)$ *and* $E(G) = E(G_1) \cup E(G_2) \cup \{v_1u_1, v_2u_2\}$*. In such a case* $\alpha(G) = 3$, $\delta(G) = 4$, $\Delta(G) = 2r - 1$, $n = (\delta(G) - 3)\Delta(G) + \delta(G) + 2$, *and* $\gamma_t(R(G)) = n - \delta(G) + 1$.

2. *For cycle graph* C_n*, wheel graph* W_n *and complete graph* K_n *with n vertices we have,*

$$\gamma_t(R(C_n)) = \lceil \tfrac{2n}{3} \rceil,$$
$$\gamma_t(R(W_n)) = \lceil \tfrac{n+1}{2} \rceil,$$
$$\gamma_t(R(K_n)) = n - 1.$$

3.2 Total Domination in Some Path Related Graphs

Definition 3.2.0.1. ***The square of a graph*** G denoted by G^2 has the same vertex set as that of G and the two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G.

Theorem 3.2.0.2.

$$If \;\; G = {P_n}^2 \;\; then \;\; \gamma_t(G) = \begin{cases} 2\lfloor \frac{n}{7} \rfloor + 1; \;\; if \;\; n \;\; \equiv 1 \;\; or \;\; 2(mod \;\; 7) \\ \\ 2\lceil \frac{n}{7} \rceil; \;\; if \;\; n \;\; \not\equiv 1 \;\; or \;\; 2(mod \;\; 7). \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\} = V(G)$ where $d_G(v_1) = d_G(v_n) = 2$, $d_G(v_2) = d_G(v_{n-1})$ = 3 and $d_G(v_i) = 4$, $\forall i \in \{3,4,5,...,n-2\}$.

If D is any total dominating set of G then it is obvious that $v_3 \in D$ as $d_G(v_3) = 4 = \Delta(G)$.

To prove the result we consider the following two cases:

Case I: $n \equiv 1$ or 2(mod 7).

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} \bigcup \{v_{n-1}\}; \;\; if \;\; n \equiv 1(mod \;\; 7) \\ \\ \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} \bigcup \{v_{n-2}\}; \;\; if \;\; n \equiv 2(mod \;\; 7) \end{cases}$$

Then $|D| = 2 \lfloor \frac{n}{7} \rfloor + 1$.

Also D is a total dominating set of G as D has no isolated vertex.

Further D is a total minimum dominating set of G.

Claim: $|D|$ is minimum.

Any $v \in D$ will dominate maximum number of distinct vertices of G as $d_G(v) = 4 = \Delta(G)$.

$\therefore$ any set containing the vertices less than that of $|D|$ cannot be a TD - set of G.

Hence $\gamma_t(G) = 2\lfloor \frac{n}{7} \rfloor + 1$, when $n \equiv 1$ or 2(mod 7).

Case II: $n \not\equiv 1$ or 2(mod 7).

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor\}; \ if \ \ n \ \ \equiv 0 \ \ or \ \ 5 \ \ or \ \ 6(mod \ \ 7) \\ \\ \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} \bigcup \{v_{n-1,v_n}\}; \ if \ \ n \equiv \ \ 3 \ \ or \ \ 4(mod \ \ 7) \end{cases}$$

Then $|D|$ =2 $\lceil \frac{n}{7} \rceil$.

Also D is a total dominating set of G as D has no isolated vertex.

Further D is a total minimum dominating set of G.

Claim: $|D|$ is minimum.

Any $v \in D$ will dominate maximum number of distinct vertices of G as $d_G(v) = 4 = \Delta(G)$.

$\therefore$ any set containing the vertices less than that of $|D|$ cannot be a TD - set of G.

Hence $\gamma_t(G) = 2 \lceil \frac{n}{7} \rceil$ □

Example: 3.2.0.3. *The square graph of P_7 and its total domination number is shown in figure 3.7. below.*

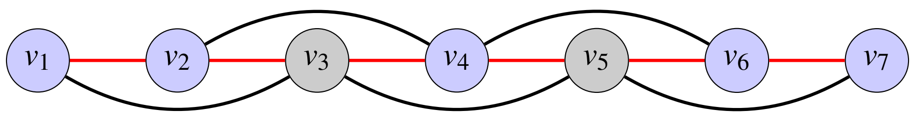

Figure 3.7: **The Graph $P_7{}^2$**

$\gamma_t(P_7{}^2) = 2$

Theorem 3.2.0.4.

$$If \ \ G = P_n{}^2 \ \ then \ \ \gamma_t{}^e(G) = \begin{cases} 2\lceil \frac{n}{7} \rceil + 1; \ \ if \ \ n \equiv 0 \ \ or \ \ 6(mod \ \ 7) \\ \\ 2\lceil \frac{n}{7} \rceil \quad ; \ otherwise. \end{cases}$$

Proof. If D is any total equitable dominating set of G then it is obvious that $v_2 \in D$, as $|d_G(v_1) - d_G(v_2)| = 1$ and $|d_G(v_2) - d_G(v_3)| = 1$.

To prove this result we consider the following cases:

Case I: $n \equiv 0$ or 6(mod 7).

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lceil \frac{n}{7} \rceil - 1\} \bigcup \{v_{n-1}\}; \ if \ \ n \equiv 0 (mod \ \ 7) \\ \\ \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lceil \frac{n}{7} \rceil - 1\} \bigcup \{v_n\}; \ if \ \ n \equiv 6 (mod \ \ 7) \end{cases}$$

Then $|D| = 2\lceil \frac{n}{7} \rceil + 1$.

Also D is a total dominating set of G as D has no isolated vertex and D is also an equitable dominating set of G as for any $v \in V(G) - D$ $\exists$ a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimum total equitable dominating set of G.

Claim: $|D|$ is minimum.

Any $v \in D$ will dominate maximum 4 vertices as $d_G(v_2) = 3$ and $d_G(v_i) = 4 = \Delta(G)$ $\forall$ 3 $\leq$ i $\leq n - 2$.

Hence $\gamma_t{}^e(G) = 2 \lceil \frac{n}{7} \rceil + 1$.

Case II: $n \not\equiv 0$ or 6(mod 7).

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{7} \lfloor -1\} \bigcup \{v_{n-2}, v_n\}; \ if \ \ n \equiv 1 \ \ or \ \ 2 \ \ or \ \ 3 \ \ or \ \ 4 (mod \ \ 7) \\ \\ \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor\}; \ if \ \ n \equiv 5 (mod \ \ 7) \end{cases}$$

Then $|D| = 2 \lceil \frac{n}{7} \rceil$

Also D is a total dominating set of G as D has no isolated vertex and D is also an equitable dominating set of G as for any $v \in V(G) - D$ $\exists$ a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimum total equitable dominating set of G.

Claim: $|D|$ is minimum.

Any $v \in D$ will dominate maximum 4 vertices as $d_G(v_2) = 3$ and $d_G(v_i) = 4 = \Delta(G)$ $\forall$ 3 $\leq$ i $\leq n-2$.

Hence $\gamma_t^e(G) = 2\left\lceil \frac{n}{7} \right\rceil$. □

Example: 3.2.0.5. *The graph $P_7{}^2$ and its total equitable domination is given in Figure 3.8. below.*

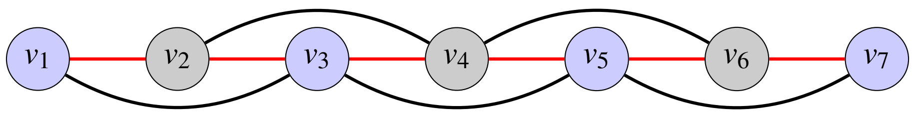

Figure 3.8: **The Graph $P_7{}^2$**

$\gamma_t^e(P_7{}^2) = 3$

Definition 3.2.0.6. Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$, where each S_i is a set of all the vertices having same degree (at least 2 vertices) and $T = V(G) - \bigcup_{i=1}^{n} S_i$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.

Theorem 3.2.0.7. *If G is the graph obtained by degree splitting of P_n then*

$$\gamma_t(G) = \begin{cases} n-1; & if \ \ n = 3, 4 \\ 4; & if \ \ n > 4. \end{cases}$$

Proof. The path P_n have two pendant vertices and remaining $n-2$ vertices of degree two. Then $V(P_n) = \{v_i : 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i : 2 \leq i \leq n-1\}$.

To obtain G from P_n add two vertices x and y corresponding to S_1 and S_2 respectively.

Thus $V(G) = V(P_n) \cup \{x,y\}$ and $E(G) = E(P_n) \cup \{xv_i \colon v_i \in S_1\} \cup \{yv_j \colon v_j \in S_2\}$.

Suppose D is any total dominating set of G. Then $y \in D$ as $\Delta(G) = n-2 = d_G(y)$.

$$D = \begin{cases} \{v_1, v_2\}; & if \ \text{ n} = 3 \\ \{v_1, v_2, v_3\}; & if \ \text{ n} = 4. \end{cases}$$

Then $|D| = n-1$. Also D is a minimum total dominating set of G.

Hence $\gamma_t(G) = n-1$, for $n \leq 4$.

For $n > 4$ we construct a dominating set $D = \{y, v_1, v_2, v_{n-1}\}$ with $|D| = 4$.

Hence D is a minimum total dominating set of G.

Claim: $|D|$ is minimum.

$\Delta(G) = n-2 = d_G(y)$ will dominate maximum number of vertices of G, $\{v_2, v_{n-1}\}$ dominates $\{v_1, v_n\}$ while v_1 and v_n dominates x.

Thus $\gamma_t(G) = 4$, $\forall\ n > 4$. □

Example: 3.2.0.8. *The degree split graph of P_7 and its total domination number is shown in Figure 3.9 below.*

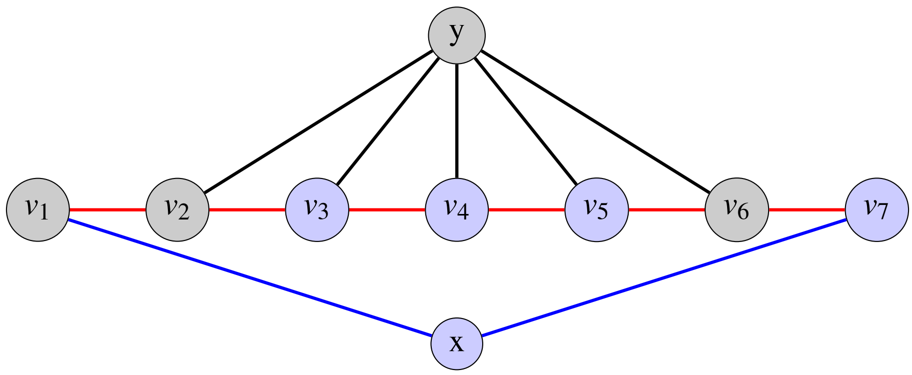

Figure 3.9: **The Degree Split Graph of P_7**

$\gamma_t(DS(P_7)) = 4$

Theorem 3.2.0.9.

$$\gamma_t{}^e(G) = \begin{cases} 3 & ;\textit{if } n = 4 \\ \lfloor \frac{n}{3} \rfloor + 2 & ;\textit{if } n \geq 5. \end{cases}$$

Proof. If D is any total equitable dominating set of G then it is obvious that $v_1 \in D$, as $|d_G(v_1) - d_G(v_2)| = 1$ and $|d_G(v_1) - d_G(x)| = 0$.

To prove this result we consider the following cases:

Case I: $n = 4$

We construct a set of vertices as $D = \{v_1, v_2, v_3\}$.

Then $|D| = 3$. Also D is a total dominating set of G as D has no isolated vertex and D is also an equitable dominating set of G as for any $v \in V(G) - D$ $\exists$ a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimum total equitable dominating set of G.

$\therefore$ $|D|$ is minimum and hence $\gamma_t{}^e(G) = 3$.

Case II: $n \geq 5$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{3i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\} \bigcup \{y, v_{n-1}, v_n\}; \ \ if \ \ n \equiv 0 \ \ or \ \ 2(mod \ \ 3) \\ \{v_{3i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\} \bigcup \{y, v_n\}; \ \ if \ \ n \equiv 1(mod \ \ 3) \end{cases}$$

Then $|D| = \lceil \frac{n}{3} \rceil + 2$.

Also D is a total dominating set of G as D has no isolated vertex and D is also an equitable dominating set of G as for any $v \in V(G) - D$ $\exists$ a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Claim: $|D|$ is minimum.

Assume that $D - \{v\}$ is a minimum total equitable dominating set of G for any $v \in D$.

For a vertex v, $|d_G(u) - d_G(v)| \geq 1$. This is a contradiction to $D - \{v\}$ is a total equitable dominating set of G.

Thus D is a minimum total equitable dominating set of G. $\square$

Example: 3.2.0.10. *The degree split graph of P_7 and and its total equitable domination number is shown in Figure 3.10 below.*

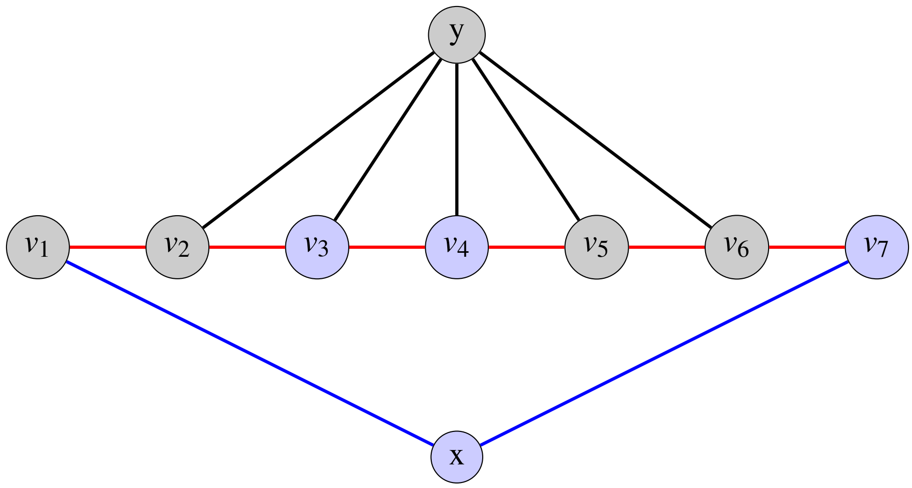

Figure 3.10: **The Degree Split Graph Of P_7**

$\gamma_t{}^e(DS(P_7)) = 5$

3.3 Total Domination in Generalized Petersen Graphs

Definition 3.3.0.1. Let n, k be the positive integers such that $n \geq 3$ and $1 \leq$ k $\leq \left[\frac{n}{2}\right]$. The ***Generalized Petersen Graph*** $P(n,k)$ is the graph whose vertex set is {a_i,b_i: $1 \leq$ i $\leq$ n} and whose edge set is $\{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+k}\} : 1 \leq i \leq n\}$.

NOTE: We take outer vertices as $u_1, u_2, ..., u_n$ and inner vertices as $v_1, v_2, ..., v_n$.

3.3.1 Total Dominating Set of the Generalized Petersen Graphs $P(n,1)$:

Theorem 3.3.1.1. *The minimum total dominating set for the generalized Petersen graphs $P(n,1)$ with $n \geq 3$ except $n = 7$ is given by*

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \leq i < \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ 0 \leq i < \lceil \frac{n}{3} \rceil. \end{cases}$$

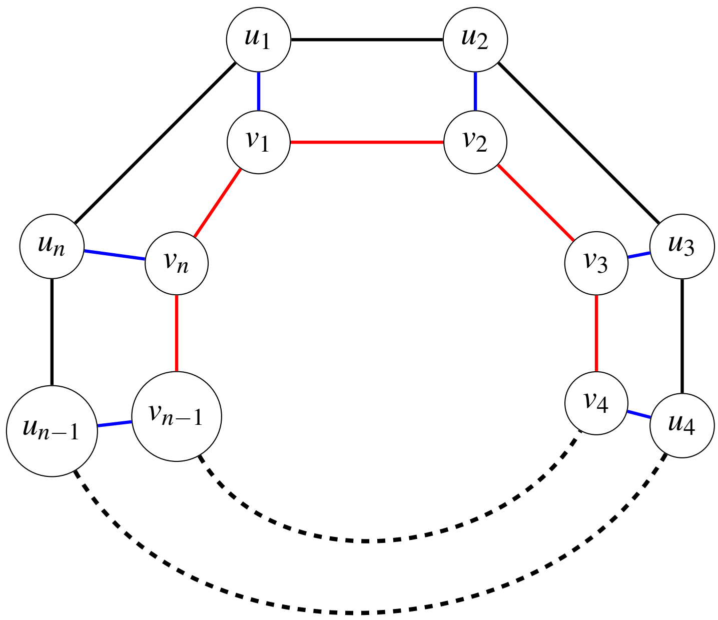

Figure 3.11: **Generalized Petersen Graph** $P(n,1)$

Proof. let $n \geq 3$ and $n \neq 7$.

The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \leq$ i $< \lceil \frac{n}{3} \rceil$ (addition modulo i); and the vertex v_{1+3i} dominates the vertices v_{3i}, v_{3i+2} and u_{1+3i} for $1 \leq i < \lceil \frac{n}{3} \rceil$ (addition modulo i).

For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_2, v_n and u_1.

As i ranges from 0 to $\lceil \frac{n}{3} \rceil$, the minimal TD - set thus obtained is

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \leq i < \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ \leq i < \lceil \frac{n}{3} \rceil. \end{cases}$$

□

Example: 3.3.1.2. *Consider the generalized Petersen graph $P(6,1)$. Let $u_1, u_2, ..., u_6$ be the outer vertices and $v_1, v_2, ..., v_6$ be the corresponding inner vertices. By applying Theorem 3.3.1., the minimum total dominating set of $P(6,1)$ is $\{u_1, u_4, v_1, v_4\}$.*

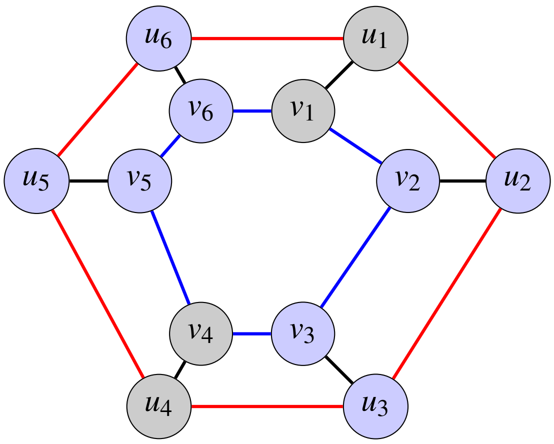

Figure 3.12: **Generalized Petersen Graph** $P(6,1)$

Example: 3.3.1.3. *Consider the generalized petersen graph $P(7,1)$ when $n = 7$, as shown in Figure 3.13. below. Let $u_1, u_2, ..., u_7$ be outer vertices and $v_1, v_2, ..., v_n$ be inner vertices. The vertex u_1 dominates the vertices u_2, u_7 and v_1; the vertex u_2 dominates the vertices u_1, u_3 and v_2; the vertex u_7 dominates the vertices u_1, u_6 and v_7; The vertex v_4 dominates the vertices v_3,v_5 and u_4; and the vertex v_5 dominated the vertices v_4, v_6 and u_5. Thus a set of vertices $\{u_1, u_2, u_7, v_4, v_5\}$ dominates every vertex of $P(7,1)$. Thus the minimum total dominating set is $\{u_1, u_2, u_7, v_4, v_5\}$.*

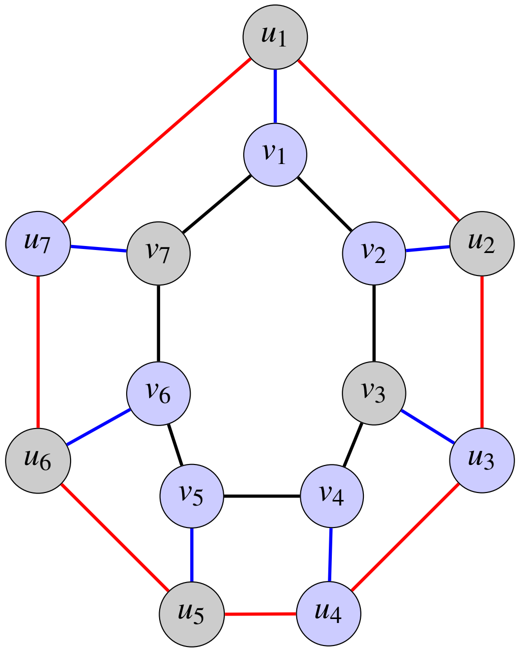

Figure 3.13: **Generalized Petersen Graph P(7,1)**

3.3.2 Total Dominating Set of the Generalized Petersen Graphs $P(n,2)$:

Theorem 3.3.2.1. *The minimum total dominating set for the generalized Petersen graph is given by*

1. *For n even , $n > 8$ there are two cases:*
   (a) $n \not\equiv 2(mod\ 6)$;

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ 0 \leq i \leq \lceil \frac{n}{3} \rceil \end{cases}$$

   (b) $n \equiv 2(mod\ 6)$;

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{n-2} \end{cases}$$

2. *For n odd, $n > 5$ there are two cases:*

   (a) $n \equiv 0 \pmod 3$ *or* $n \equiv 1 \pmod 3$;

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \le i \le \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ 0 \le i \le \lceil \frac{n}{3} \rceil \end{cases}$$

   (b) $n \equiv 2 \pmod 3$;

$$TD = \begin{cases} u_{1+3i}, & if \ \ 0 \le i \le \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \ \ 0 \le i \le \lceil \frac{n}{3} \rceil \\ v_{n-2} \end{cases}$$

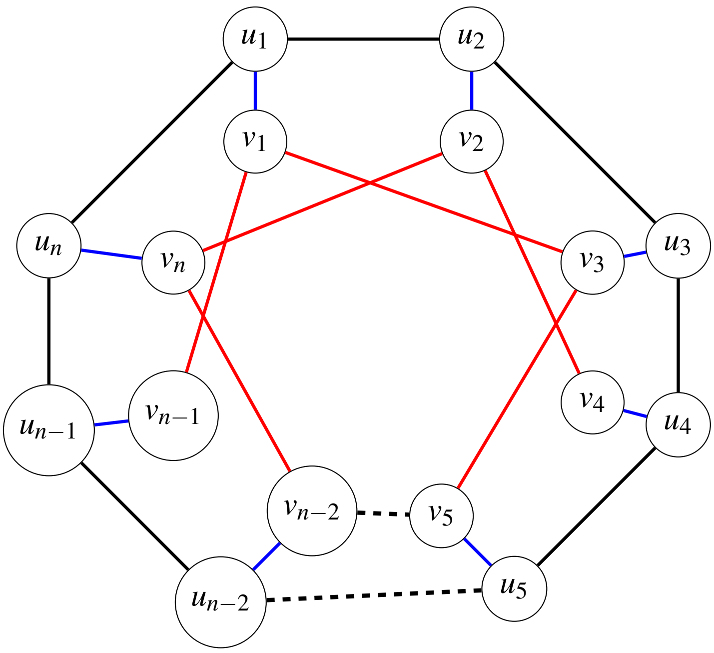

Figure 3.14: **Generalized Petersen Graph** $P(n,2)$

Proof. 1. For n even, $n > 8$ there are two cases:

case (a): $n \not\equiv 2(\text{mod } 6)$

The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i); and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i)

For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1.

We get the TD - set of $P(n,2)$ $\forall$ values of i, $0 \leq i \leq \lceil \frac{n}{3} \rceil$ as

$$TD = \begin{cases} u_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \end{cases}$$

case (b): $n \equiv 2(\text{mod } 6)$

The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i); and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \leq \text{i} \leq \lceil \frac{n}{3} \rceil$ (addition modulo i)

For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1.

and also the vertex v_{n-2} dominates the vertices v_{n-4}, v_n and u_{n-2}.

We get the TD - set of $P(n,2)$ for all values of i, $0 \leq i \leq \lceil \frac{n}{3} \rceil$ as

$$TD = \begin{cases} u_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{n-2} \end{cases}$$

2. Let n be odd and $n > 5$. There are two cases:

   case (a): $n \equiv 0(\text{mod } 3)$ or $n \equiv 1(\text{mod } 3)$

   The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i); and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i)

   For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1.
   We get the TD - set of $P(n,2)$ $\forall$ values of i, $0 \leq i \leq \lceil \frac{n}{3} \rceil$ as

$$TD = \begin{cases} u_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \end{cases}$$

   case (b): $n \equiv 2(\text{mod } 3)$

   The vertex u_{1+3i} dominates the vertices u_{3i}, u_{3i+2} and v_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i); and the vertex v_{1+3i} dominates the vertices v_{3i-1}, v_{3i+3} and u_{1+3i} for $1 \leq i \leq \lceil \frac{n}{3} \rceil$ (addition modulo i).

   For $i = 0$, the vertex u_1 dominates the vertices u_2, u_n and v_1; and the vertex v_1 dominates the vertices v_3, v_{n-1} and u_1; and also the vertex v_{n-2} dominates the vertices v_{n-4}, v_n and u_{n-2}.

   We get the TD - set of $P(n,2)$ for all values of i, $0 \leq i \leq \lceil \frac{n}{3} \rceil$ as

$$TD = \begin{cases} u_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{1+3i}, & if \;\; 0 \leq i \leq \lceil \frac{n}{3} \rceil \\ v_{n-2} \end{cases}$$

□

NOTE:

The values 5, 6 and 8 of n are not included in the above theorem. Here we have given separately the TD - set of $P(5,2)$, $P(6,2)$ and $P(8,2)$.

1. Consider the generalized Petersen graph $P(5,2)$ given in Figure 3.15 below. Let $u_1, u_2, \ldots, u_5$ be the outer vertices and $v_1, v_2, \ldots, v_5$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_5 and v_1; the vertex v_1 dominates the vertices v_3, v_4 and u_1; the vertex v_3 dominates the vertices v_1, v_5 and u_3 ; and the vertex v_4 dominates the vertices v_1, v_2 and v_4. Thus the set of vertices $\{u_1, v_1, v_3, v_4\}$ dominates every vertex of $P(5,2)$. Thus the minimal total dominating set is $\{u_1, v_1, v_3, v_4\}$.

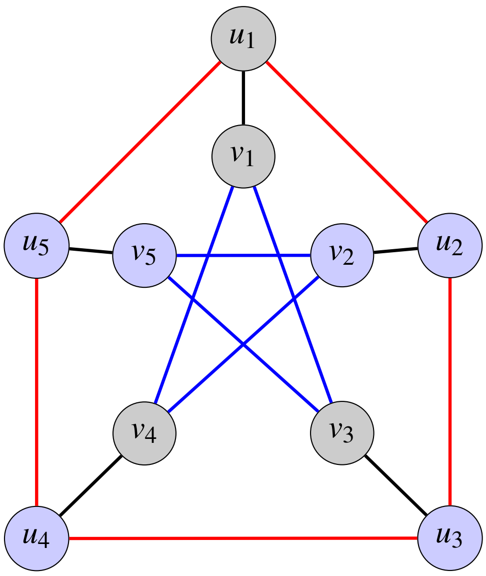

Figure 3.15: **Generalized Petersen Graph** $P(5,2)$

2. Consider the generalized Petersen graph $P(6,2)$ given in Figure 3.16 below. Let $u_1, u_2, \ldots, u_6$ be the outer vertices and $v_1, v_2, \ldots, v_6$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_6 and v_1; the vertex u_4 dominates

the vertices u_3, u_5 and v_4; the vertex v_1 dominates the vertices v_3, v_5 and u_1 ; and the vertex v_4 dominates the vertices v_6, v_2 and u_4. Thus the set of vertices $\{u_1, v_1, u_4, v_4\}$ dominates every vertex of $P(6,2)$. Thus the minimal total dominating set is $\{u_1, v_1, u_4, v_4\}$.

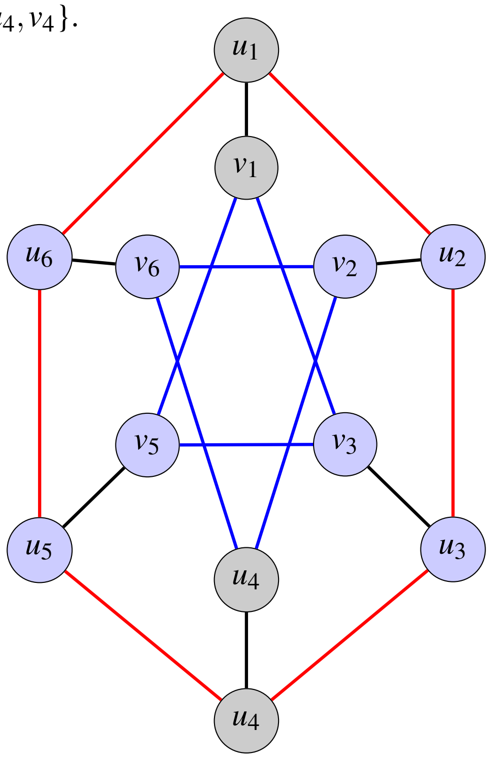

Figure 3.16: **Generalized Petersen Graph** $P(6,2)$

3. Consider the generalized Petersen graph $P(8,2)$ given in Figure 3.17 above. Let $u_1, u_2, ..., u_8$ be the outer vertices and $v_1, v_2, ..., v_8$ be the corresponding inner vertices. The vertex u_1 dominates the vertices u_2, u_8 and v_1; the vertex u_4 dominates the vertices u_3, u_5 and v_4; the vertex v_1 dominates the vertices v_3, v_7 and u_1 ; the vertex v_4 dominates the vertices v_6, v_2 and u_4; the vertex v_6 dominates the vertices v_4, v_8 and u_6; and the vertex v_7 dominates the vertices v_1, v_5 and u_7. Thus the set of vertices $\{u_1, v_1, u_4, v_4, v_6, v_7\}$ dominates every vertex of $P(8,2)$. Thus the minimal total dominating set is $\{u_1, v_1, u_4, v_4, v_6, v_7\}$.

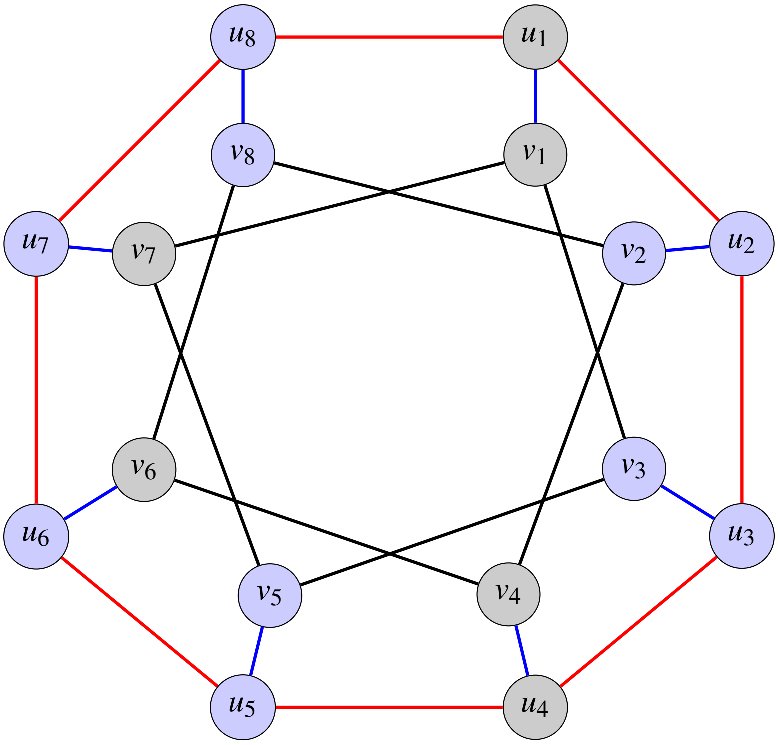

Figure 3.17: **Generalized Petersen Graph** $P(8,2)$

4. In the above Theorem 3.3.2.1., we note that the total dominating set of the cases a(1) and a(2) are the same and for the cases b(1) and b(2) also the total dominating sets are the same.

Example: 3.3.2.2. *Consider the generalized Petersen graph* $P(10,2)$ *in Figure 3.16 below, illustrate the Theorem 3.3.2.1. Let* $u_1, u_2, \ldots, u_{10}$ *be the outer vertices and* $v_1, v_2, \ldots, v_{10}$ *be the corresponding inner vertices. Here* $n = 10$ *by applying Theorem 3.3.2.1.(case a(1)), the minimal total dominating set of* $P(10,2)$ *is*
$\{u_1, u_4, u_7, u_{10}, v_1, v_4, v_7, v_{10}\}$.

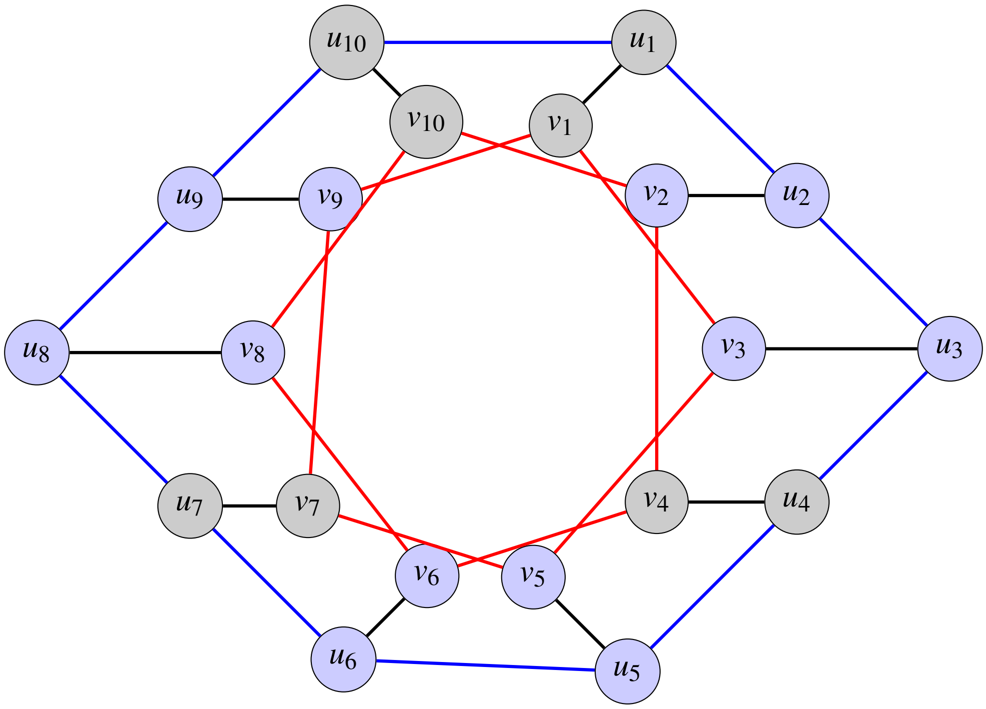

Figure 3.18: **Generalized Petersen Graph** $P(10,2)$

Chapter 4

PROPOSED THEOREMS

This chapter introduces and proves theorems concerning the domination number and total domination number of various graph structures. It explores the domination number of the Snare graph of the path P_n, as well as the domination number of a Lollipop graph $L_{m,n}$ and the total domination number of a specific case of Lollipop graph denoted as $L_{m,1}$, where $m \geq 3$. Additionally, the chapter discusses the total domination number of Tadpole graph $T_{m,n}$ where $m \geq 3$ and $n \geq 1$, as well as the total domination number of the crib graph of the graph G.

4.1 Domination Number & Total Domination Number of Lollipop Graph

Definition 4.1.0.1. The $(m,n)-Lollipop\ graph$ is the graph obtained by joining the complete graph K_m to the path P_n with a bridge.

Theorem 4.1.0.2. *The domination number for lollipop graph $L_{m,n}$ is $\gamma(L_{m,n})$ = $\lfloor \frac{n+1}{3} \rfloor$ + 1 for $m \geq 3$; $n \geq 1$.*

Proof. Let $G \cong L_{m,n}$ be a lollipop graph on $m+n$ vertices and $\frac{m(m-1)}{2}$ + n edges.
Let S be a dominating set of P_n .

Now consider the vertex v that is connected to all vertices in K_m adding v to S creates a set S' such that $|S'| = |S| + 1$.

Since S dominates all vertices in P_n and vertex v dominates all vertices in K_m. Hence S' dominates all vertices in $L_{m,n}$.

$\therefore \gamma(L_{m,n}) \leq \lfloor \frac{n+1}{3} \rfloor + 1$.

Now, let S be a dominating set of $L_{m,n}$.

Since S is a dominating set v must belong to S as v dominates vertices in K_m and removing v from S leaves a set S' that dominates path P_n making S' a dominating set of P_n.

Hence $|S'| = |S|$ - 1.

i.e. $|S| = |S'| + 1$.

Thus $\lfloor \frac{n+1}{3} \rfloor + 1 \leq \gamma(L_{m,n})$.

hence $\gamma(L_{m,n}) = \lfloor \frac{n+1}{3} \rfloor + 1$ for $m \geq 3; n \geq 1$. □

Example: 4.1.0.3. *The dominating set and the domination number of the lollipop graph $L_{4,2}$ is shown in Figure 4.1 below.*

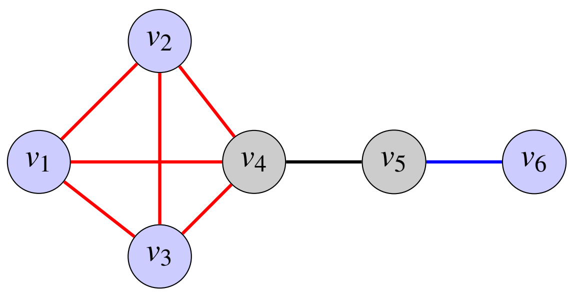

Figure 4.1: **Lollipop Graph** $L_{4,2}$

$D(L_{4,2}) = \{v_4, v_5\}$ and $\gamma(L_{4,2}) = 2$

Theorem 4.1.0.4. *Total domination number of a Lollipop graph $L_{m,1}$ is 2 for $m \geq 3$.*

Proof. Let $G \cong L_{m,1}$ be a lollipop graph on $m+1$ vertices and $\frac{m(m-1)}{2}$ + 1 edges.

Let D_t be the minimum TD - set of G.

By definition The $(m,1)-Lollipop\ graph$ is the graph obtained by joining the complete graph K_m to the path P_1 with a bridge.

So the vertex v where K_m is joined to P_1 by a bridge will have $\Delta(G)$.

$\therefore v \in D_t$ and hence $\gamma_t(L_{m,1})$ = 1 + $\gamma_t(P_1)$ = 1 + 1 = 2.

$\therefore \gamma_t(L_{m,1})$ = 2 for $m \geq 3$. □

Example: 4.1.0.5. *The total dominating set and the total domination number of $L_{4,1}$ is shown in the figure 4.2 below.*

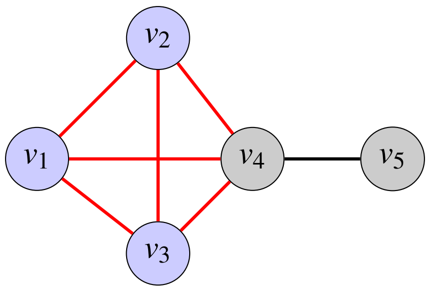

Figure 4.2: **Lollipop Graph** $L_{4,1}$

$D(L_{4,1}) = \{v_4, v_5\}$ and $\gamma_t(L_{4,1}) = 2$

4.2 Total Domination Number of Tadpole Graph

Definition 4.2.0.1. The $(m,n)-tadpole\ graph$ is the graph obtained by joining a cycle graph C_m to the path graph P_n by a bridge.

Theorem 4.2.0.2. *The total domination number of the tadpole graph $T_{m,n}$ is*

$$\gamma_t(T_{m,n}) = \begin{cases} \gamma_t(C_m) + (\frac{n+1}{2}); & n \equiv 3(mod\ \ 4) \\ \gamma_t(C_m) + \lfloor \frac{n}{2} \rfloor; & otherwise. \end{cases}$$

Proof. Let $G \cong T_{m,n}$ be a tadpole graph on $m+n$ vertices and $m+n$ edges.

We will prove the theorem by considering two cases based on the residue on n mod 4.

Case 1: $n \equiv 3(mod\ \ 4)$

In the dominating set of the cycle graph include the vertex where C_m is connected to P_n by a bridge.

Then construct a dominating set by selecting every two adjacent vertices along the cycle, leaving two vertices uncovered.

This ensures that all vertices of the cycle are dominated, resulting in $\gamma_t(C_m)$ vertices selected.

For the path graph P_n start the domination from the second vertex.

Then select every two consecutive vertices, along the path while skipping two vertices and maintain the pattern throughout the path.

This selection strategy guarantees that every vertex in the path has a neighbour in the selected set.

Thus we select $(\frac{n+1}{2})$ vertices from the path.

$\therefore \gamma_t(T_{m,n}) = \gamma_t(C_m) + (\frac{n+1}{2});\ n \equiv 3(\text{mod}\ \ 4)$

Case 2: $n \equiv 0$ or 1 or 2(mod 4)

The selection of vertices from the cycle graph is same as in case 1.

For the path graph start the domination from the third vertex of the path graph.

Then select every two consecutive vertices along the path while skipping two vertices and maintain this pattern throughout the path.

$\therefore \gamma_t(C_m) + \lfloor \frac{n}{2} \rfloor;$ otherwise. □

Example: 4.2.0.3. *The dominating set and the domination number of the tadpole graph $T_{6,3}$ is shown in Figure 4.3 below.*

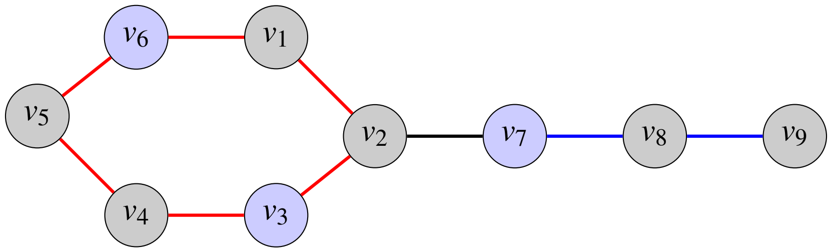

Figure 4.3: **Tadpole Graph** $T_{6,3}$

$D(T_{6,3}) = \{v_1, v_2, v_4, v_5, v_8, v_9\}$ and $\gamma(T_{6,3}) = 6 = \gamma_t(C_6) + (\frac{4+1}{2})$

4.3 Total Domination Number of Crib Graph

[ref. Sec. 2.3.2 for construction]

Theorem 4.3.0.1. *Let G be any graph of order n, then the total domination number of crib graph of G is 2. i.e. $\gamma_t(C(G)) = 2$.*

Proof. In the crib graph C(G) of G let u_i; i=$1, 2, \ldots, n$ be the new vertices introduced so that $u_i v_j \in E(C(G))$ iff $v_i v_j \in E(G)$.

Let w be the vertex of C(G) which is adjacent to all the vertices u_i and v_i; $i = 1, 2, \ldots, n$.

Define the TD-set to consist of vertex w and any other vertex of C(G) say v_1.

Since every vertex of C(G) has atleast one neighbour in D_t= $\{w, v_1\}$, D_t is a total dominating set of C(G).

Hence γ_t(C(G)) = 2. □

Example: 4.3.0.2. *The Crib graph of the path P_5 is shown in Figure 4.4 below.*

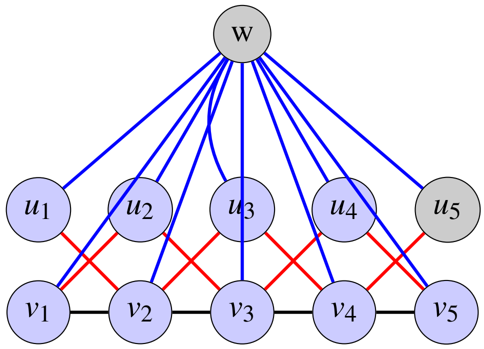

Figure 4.4: **Crib Graph of P_5**

γ_t[C(P_5)] = 2

4.4 Snare Graph of a Graph

The **Snare Graph** of a given graph G with $V(G) = \{v_1,v_2,...,v_n\}$ is the graph which is obtained by applying the following steps:

1. Corresponding to each vertex v_i in $V(G)$, introduce 2 new vertex u_i and w_i and let $U = \{u_i\colon 1 \leq i \leq n\}$ and $W = \{w_i\colon 1 \leq i \leq n\}$. Add edges from each vertex u_i of U and w_i of W to the vertex v_j if $v_iv_j \in E(G)$.
2. Take another vertex X and add edges from X to all vertices in both U and W.

The new graph thus obtained is called the Snare graph of G and is denoted by $S(G)$.

Theorem 4.4.0.1. *For any snare graph with $3n+1$ vertices $\gamma[S(P_n)] = \gamma(P_n) + 1$ for $n \geq 2$.*

Proof. Let D be a minimum dominating set of the path graph P_n, where $|D| = \gamma(P_n)$. In the snare graph $S(P_n)$, each vertex in D dominates its corresponding vertices in the sets U and W.

- Construction of Domination Set for Snare Graph:
  Start with the domination set D of P_n, the vertex X is added to the domination set, as it dominates all vertices in both U and W. Thus, the resulting domination set for $S(P_n)$ is $D' = D \cup X$, where $|D'| = \gamma(P_n) + 1$.
- Minimality of Domination Set for Snare Graph:
  Any vertex $v \notin D'$ must either reside in U, W, or be non-adjacent to any vertex in D'. If v belongs to U or W, it is dominated by its corresponding vertex in D'. If v is non-adjacent to any vertex in D', it must be adjacent to X, as X dominates all vertices in U and W. Therefore, D' is a dominating set of $S(P_n)$.
  Since D' is a dominating set of $S(P_n)$ with $|D'| = \gamma(P_n) + 1$, the proof establishes that $\gamma[S(P_n)] = \gamma(P_n) + 1$ for $n \geq 2$

□

Example: 4.4.0.2. *The Snare graph of P_5 is shown in Figure 4.5 below.*

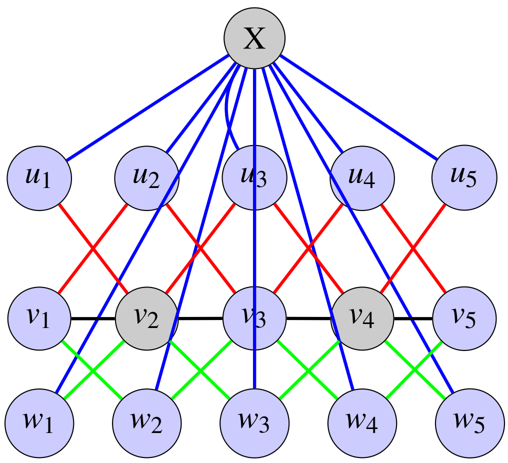

Figure 4.5: **Snare Graph of P_5**

$\gamma(P_5) = 3 = \gamma(P_5) + 1$

Chapter 5

ANALYSIS AND CONCLUSION

In **Chapter 2**, the domination numbers of diverse types of graphs is ascertained, all of which are finite, undirected, and devoid of loops and multiple edges. Additionally, we delved into the detailed analysis of domination numbers within trees without duplicated leaves, establishing both their minimum and maximum orders, thereby providing a characterization of trees at these extremities. Furthermore, we have explored the domination numbers of specific graph constructions such as the Myceilski graph, crib graph, and modified Myceilski graph. Notably, the proofs for the Myceilski graph and modified Myceilski graph were contributed by us.

In **Chapter 3**, we explored the application of graph operators in determining the total domination number of new graphs, specifically examining how operators $S(G)$ and $R(G)$ operate on given graphs to produce new ones. Additionally, the concept of total co-independent dominating sets and total co-independent domination number is discussed. We have also discussed the total domination number of generalized Petersen graphs and introduced the notion of minimum total dominating sets. Moreover, the total domination numbers in path-related graphs, the concept of total equitable domination arising from the combination of total domination and equitable domination, including the determination of the total equitable domination number of paths is also discussed. Further research directions, such as exploring the combination of parameters associated with domination theory using the studied unitary operators $S(G)$ and $R(G)$ is suggested. Furthermore, we also suggest focusing on excluded values of n in the context of total domination in generalized Petersen graphs and extending the analysis to other graph types. Another intriguing investigation would involve determining the total equitable domination number of different families of graphs.

In **Chapter 4**, we delved into the construction of Snare graph and examined the domination number of these graphs. Additionally, we investigated the domination number of Lollipop graphs and determined the total domination number of a specific class of Lollipop graphs denoted as $L_{m,1}$. Furthermore, the chapter explored the total domination number of tadpole graphs $T_{m,n}$ and the crib graph of the path P_n. It's noteworthy that all the theorems presented in this chapter were proved by us.

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