Study Of Mathematical Modelling Of Pest Control

A Dissertation for

MAT-651 Discipline Specific Dissertation

Credits: 16

Submitted in partial fulfilment of Masters Degree

M.Sc. in Mathematics

by

Miss. Chetali Damodar Naik

22P0410007

ABC ID : 164-551-955-847

201905599

Under the Supervision of

Dr. Mridini Gawas

School of Physical & Applied Sciences

Mathematics Discipline

GOA UNIVERSITY

APRIL 2024

Examined by: Seal of the School

DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Study Of Mathematical Modelling Of Pest Control" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the supervision of Dr. Mridini Gawas and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

I hereby authorize the University authorities to upload this dissertation on the dissertation repository or anywhere else as the UGC regulations demand and make it available to any one as needed.

Signature: Naik

Student Name: Chetali Damodar Naik

Seat no: 22P0410007

Date: 03/05/2024

Place: GOA UNIVERSITY

COMPLETION CERTIFICATE

This is to certify that the dissertation report "Study Of Mathematical Modelling Of Pest Control" is a bonafide work carried out by Miss. Chetali Damodar Naik under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

Signature : ____________________

Supervisor : Dr. Mridini Gawas

Date: 09/05/2024

School of Physical and Applied Sciences
Goa University

Signature of Program Director of the Dept

School

Stamp

Date: 10/5/2024

Place: Goa University

PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " Study Of Mathematical Modelling Of Pest Control". This research is divided into four chapters . Each chapter has its own relevance and importance. At the outset of each chapter, the abstract offers a concise summary of the research conducted. The central aim of this dissertation is to create and validate mathematical models for crops, pests, and their natural enemies, with a particular focus on understanding the adverse impact of pests on crops. Consequently, we can provide informed recommendations and propose effective pest management strategies to mitigate pest density while simultaneously improving agricultural productivity.

FIRST CHAPTER :

This chapter deals with motivation, biological background, mathematical model and the importance of functional responses .It also deals with the stability analysis.

SECOND CHAPTER:

In the second chapter, we delve into a mathematical model that explores the ecological dynamics between prey, pests, and their natural enemies. Our focus lies in understanding the existence and stability of steady-state conditions at various equilibrium points.

THIRD CHAPTER:

In this chapter we will form a mathematical model that explores the ecological dynamics between prey, pests (fertile and sterile) and their natural enemies. We will also see the different release rates of natural enemies and sterile pest insect to minimize the cost effective in controlling pest insect.

FOURTH CHAPTER:

In the subsequent sections, we present the conclusions drawn from our research efforts as outlined . Additionally, we delve into the potential avenues for future work based on the current findings.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my Mentor, Dr. Mridini Gawas, who was a continual source of inspiration and supported throughout my dissertation. She pushed me to think imaginatively and urged me to do this homework without hesitation. Her vast knowledge, extensive experience, and professional competence in Mathematical Modelling enabled me to successfully accomplish this project. This endeavour would not have been possible without her help and supervision.

I would also like to thank the dean Prof. Ramesh V. Pai , the program director Dr. M.Kunhanandan and all the faculty members of mathematics department who gave me the opportunity to do this dissertation.

I would like to extend my sincere appreciation to my family and friends for the camaraderie, late-night discussions. Your encouragement kept me going.

ABSTRACT

In this study , we construct a mathematical model representing an ecological system involving prey, pests, and their natural enemies. Our focus is on understanding the equilibrium points and assessing their stability. The goal is to develop an control strategy that minimizes crop loss while minimizing environmental impact. We explore various approaches, including chemical, biological, and physical controls.

Furthermore, we also delve into the intricate interactions between plants and pest insects, specifically examining the impact of natural enemies releases combined with sterile insect techniques. Our approach involves developing a system of nonlinear ordinary differential equations that model the control measures. These measures aim to simultaneously minimize pest density while implementing effective control efforts. We explore three distinct strategies related to the release rates of sterile insects and predator natural enemies: constant, proportional, and saturating proportional release rates. Among the strategies considered, the most cost-effective approach involves releasing sterile insects at a proportional rate and maintaining a constant release of natural enemies. This combined strategy achieves a remarkable reduction in pest population and increase in plant density during the control implementation.

Keywords: Pest control model; release rates; natural enemies; prey-predator model; holling type response.

Table of contents

List of figures

Notations and Abbreviations

Parameter	Description
$P_1(t)$	Density of crops (prey) at time t
$P_2(t)$	Density of pest at time t
$P_3(t)$	Density of natural enemies of pest at time t
r_1	Intrinsic growth rate of prey
K_1	The carrying capacity of prey
r_2	Intrinsic growth rate of pest
K_2	The carrying capacity of pest
m	Holling type-III functional response with consumption rate
α_1	Predation rate coefficient
α_2	Half saturation constant
α_3	Reproduction rate of pest per prey eaten
γ	Predator conversion rate
μ	The natural mortality rate of the predator
r_p	intrinsic growth rate of plant
r_h	intrinsic birth rate of fertile insects
a_1	plant consumption rate by fertile insects
a_2	plant consumption rate by sterile insects
b_F	fertile insect consumption rate by predator
b_S	sterile insect consumption rate by predator
d_F	death rate of fertile insect
d_S	death rate of sterile insect
d_E	death rate of natural enemy
α_F	death rate of fertile insect due to fertile-fertile interaction
α_S	death rate of fertile insect due to sterile-sterile interaction
β	death rate of fertile insect due to fertile-sterile interaction
α_E	death rate of natural enemy due to self-interaction
e_1	plant-to-fertile insect conversion factor
e_2	plant-to-sterile insect conversion factor
e_3	insect-to-natural enemy conversion factor

Chapter 1

INTRODUCTION

1.1 Motivation

Mathematical modeling finds its applications across diverse disciplines, benefiting from continuous interaction and refinement. System models have become an integral part of our cultural fabric. Biomathematics, also known as mathematical biology, employs mathematical modeling and computational techniques to analyze real-world challenges within biological systems and health domains.

In 2023, there was a huge decline the production of tomatoes.So to avoid such situation in future we carry out this research work. The field of biological sciences is intricate and demands collaboration across various disciplines. In contemporary agriculture, environmental protection principles delineate strict boundaries for implemented techniques. Simultaneously, addressing the global food supply challenge due to population growth remains a significant endeavor. Consequently, prioritizing resource-saving strategies is crucial for sustainable agricultural development. Agriculture, historically the backbone of economies, not only feeds the growing population but also supplies raw materials for industry. The integration of mathematics into agricultural growth has become both essential and advantageous.

Mathematical Modelling has numerous real-world applications in different disciplines including biology, computer science, biochemistry, chemistry, economics, electrical engineering, medicine,as well as in physics. Its use in various domains is growing and it is now an unique instrument for quantitative and qualitative analysis. By formulating the mathematical models we solve the real world problem . There are several methods to form the models but as per the condition of the problems the models may vary and these methods are very advantageous in the explanation of several real problems.

1.2 What is Mathematical Modelling ?

It is a process which consists of simplifying a real world problem, formulating a mathematical model, solving the model and interpreting the solutions in language of real world.

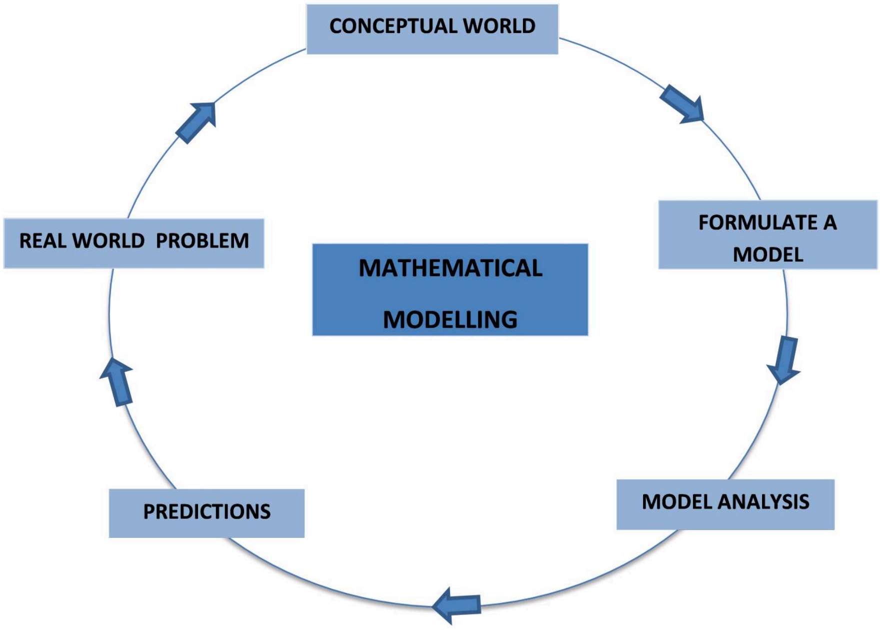

Figure 1.1: The Modelling Cycle

1.2.1 Classification Of Models

Static and Dynamic Model

Static model, also known as the steady-state model, describes a system in equilibrium

[11]. It remains unchanged over time, making it time-invariant. On the other hand, dynamic models are specifically designed to capture time-dependent changes within a system. These dynamic models are typically represented using difference equations or differential equations.

Linear and Non-linear Models

In mathematical modeling, we distinguish between linear and nonlinear systems. A system is linear if its behavior adheres to the principles of superposition and homogeneity. Specifically, linear systems exhibit properties such as proportionality, additivity, and constant coefficients. On the other hand, nonlinear systems deviate from these characteristics. They may involve interactions, non-proportional relationships, or variable coefficients. For example, consider a statistical model: while its relationship with certain parameters may be linear, the predictor variables themselves can introduce nonlinearity. Similarly, even though a differential equation may contain nonlinear expressions, it can often be expressed using linear differential operators.

In the realm of mathematical programming, we classify models based on the nature of their constraints and objective functions. A model is deemed linear when all its constraints and objectives can be expressed using linear equations. Conversely, a nonlinear model arises when the constraints or objectives involve nonlinear equations. Essentially, linearity refers to the simplicity of expressing relationships using straight lines, while nonlinearity introduces more complex, curved relationships.

Explicit and Implicit Models

Explicit Models: In an explicit model, we have complete knowledge of the input parameters. These models allow us to directly compute output parameters using a finite sequence of computations. Explicit models are like well-lit paths—the steps are clear, and we can readily calculate the results. Example: Imagine a straightforward algebraic equation where you can directly solve for the unknown variable.

Implicit Models: Implicit models often arise when solving complex systems or equations where direct computation is not feasible. Instead, we need to solve an equation involving both the current state and the desired outcome. Example: Newton's method, used for finding roots of nonlinear equations, where the solution emerges from an iterative process.

Deterministic and Probabilistic Model

In a deterministic model, each variable state is explicitly defined by specific parameters. These parameters consider a history of previous states of the variables. Consequently, the model behaves consistently for the same initial conditions.

On the other hand, in a probabilistic model, randomness plays a role. Variable states are not fixed to unique values; instead, they follow probability distributions. This type of model is also referred to as a statistical or stochastic model.

Discrete and Continuous Models

Discrete Models: In discrete models, we consider objects as distinct entities. These objects can represent various states in a statistical model or individual particles in a molecular model. The focus is on specific, separate instances.

Continuous Models: Conversely, continuous models represent objects as smoothly varying quantities. For instance, we describe stresses and temperatures in solids or apply electric fields uniformly across an entire model.

1.2.2 Steps of Modelling

In my opinion, anything in our life can be modeled Mathematically so we can always find a pattern in our life.

1) Start with the basic definition of your problem .If you define the problem , research

will become a lot easier and you will actually understand what you are doing rather than just mindlessly researching.
2) To simplify the scope of the problem we will need to make a few assumptions. If you We make assumptions we can get rid of some extraneous as insignificant factors .So Your model really only considers the most important variables.
3) Next define your variables. If you are trying to make a mathematical model,you are trying to create an equation of some sort that incorporates certain variables by defining those variables, by figuring out what those variables are, especially early on you know what you are looking for. Now its time to use the math you know to build the model .This is where you will see your solutions.
4) Getting a solution
5) Analyse your model to make sure it works, ask yourself what can I learn from my model, does it answer the original question and does the answer make sense.

1.3 Preliminaries

1.3.1 Logistic-Growth Model

When population is growing in limited space the density of population gradually increases until the presence of other organisms reduces fertility and longitivity of population. This reduces the growth rate and ultimately it stop to grow. The growth refined above is a sigmoid curve (or S-shaped curve) where density is plotted against time.
This curve was first suggested to describe the growth of human population by Verhults. This 'sigmoid curve' arises due to greater action of tremendous forces as population increases. The corresponding curve is also called as logistic curve and the equation is called the Logistic Equation.

Carrying Capacity

The carrying capacity of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.
We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Formulation of Model

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $N(t)$ represents the population of this organism as a function of time t, and the constant N_0 represents the initial population (population of the organism at time $t = 0$). Then the logistic differential equation is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right). \tag{1.1}$$

This differential equation can be coupled with the initial condition $N(0) = N_0$ to form an initial-value problem for $N(t)$.
Suppose that the initial population is small relative to the carrying capacity. Then $\frac{N}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of equation (1.1) is close to 1, and the right-hand side of this equation is close to rN. If $r > 0$, then the population grows rapidly, resembling exponential growth.
However, as the population grows, the ratio $\frac{N}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{N}{K}$ is less than 1 , so $1 - \frac{N}{K} > 0$. Therefore the right-hand side of equation (1.1) is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $N = K$ then the right-hand side is equal to zero, and the population does not change.
Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{N}{K} > 1$, and $1 - \frac{N}{K} < 0$. Then the right-hand side of equation (1.1) is negative, and the population decreases. As long as $N > K$, the population decreases. It never actually

reaches K because $\frac{dN}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity.

We can solve this differential equation by the method of separation of variables.

$$\frac{dN}{dt} = \frac{rN(K-N)}{K}.$$

$$\frac{K}{N(K-N)}dN = rdt$$

and integrate

$$\int \frac{K}{N(K-N)}dN = \int rdt$$

Of course, $\int rdt = rt + C$, and for other part the method of partial fractions.

$$\frac{K}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N}$$

where A and B are coefficients yet to be determined. Thus we get,

$$K = A(K-N) + BN$$

For $N = 0$,

$$K = AK$$

$$\Rightarrow A = 1$$

For $N = K$

$$K = BK$$

$$\Rightarrow B = 1$$

from which it follows that $A = 1$ and $B = 1$. Thus,

$$\begin{aligned}
\int \frac{K}{N(K-N)} dN &= \int \frac{dN}{N} + \int \frac{dN}{K-N} \\
&= \ln N - \ln |K-N| \\
&= \ln \left| \frac{N}{K-N} \right|
\end{aligned}$$

Thus we get ,

$$\begin{aligned}
\ln \left| \frac{N}{K-N} \right| &= rt + C_1 \\
\Rightarrow \frac{N}{K-N} &= e^{rt+C_1} \\
\Rightarrow \frac{N}{K-N} &= C_2 e^{rt} \\
\Rightarrow N &= (K-N) C_2 e^{rt} \\
\Rightarrow N &= K C_2 e^{rt} - N C_2 e^{rt} \\
\Rightarrow N + N C_2 e^{rt} &= K C_2 e^{rt} \\
\Rightarrow N(1 + C_2 e^{rt}) &= K C_2 e^{rt} \\
\Rightarrow N = \left(\frac{K C_2 e^{rt}}{1 + C_2 e^{rt}} \right) &\left(\frac{\frac{1}{C_2} e^{-rt}}{\frac{1}{C_2} e^{-rt}} \right) \\
&= \frac{K}{(1 + C_2 e^{rt}) (\frac{1}{C_2} e^{-rt})} \\
&= \frac{K}{\frac{1}{C_2} e^{-rt} + 1} \\
&= \frac{K}{C e^{-rt} + 1}
\end{aligned}$$

Next, after simplification we get,

$$N = \frac{K}{1 + C e^{-rt}}$$

where C is a constant. At $t = 0$,

$$N_0 = \frac{K}{C+1}$$

This implies

$$C = \frac{K}{N_0} - 1$$

so ,

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}.$$

1.3.2 Prey-Predator Model

The Lotka-Volterra equations were formulated to capture the behavior of predator-prey relationships within ecosystems [3].

The Lotka-Volterra equations describe the dynamics of predator-prey interactions in ecological systems. These equations are a set of first-order non-linear ordinary differential equations (ODEs). Unlike stochastic models, which incorporate randomness, the solutions to the Lotka-Volterra equations are deterministic. This means that given same initial conditions,the outcomes will always be consistent. Additionally, the time in these equations is continuous, allowing for overlapping generations of predators and prey.

It's important to recognize that the creation of the Lotka-Volterra equations involved several assumptions, as is common in mathematical modeling. These assumptions shape the behavior of the predator and prey populations, influencing their interactions over time.

1.Abundant Food Supply: The prey population does not face scarcity of food.

2.Food-Prey Relationship: The amount of food available to the prey is linked directly to the size of the prey population.

3.Population Dynamics: The rate of population change is proportional to the population's

size.
4.Steady Environment: We assume a constant environment, and genetic adaptations are not considered negligible.
5.Persistent Predation: Predators continuously hunt and consume prey without interruption. Thus Lotka-Voltera equations can be written as:

$$\frac{dp}{dt} = \alpha p - \beta pq \tag{1.2}$$

$$\frac{dq}{dt} = \delta pq - \gamma q \tag{1.3}$$

where

p	=	number of prey
q	=	number of predators
$\frac{dp}{dt}$ and $\frac{dq}{dt}$	=	the instantaneous rates of the prey and predators, respectively.
t	=	time
$\alpha,\beta,\delta,\gamma$	=	positive real constants

In the prey equation (1.2) we can see the following :
Exponential Reproduction: The prey population is assumed to grow exponentially, denoted by the term αp.
Predation Rate: The equation reflects that the rate at which predators hunt and kill prey is directly proportional to the product of the prey and predator populations βpq. Essentially, it quantifies how frequently these two populations interact.
Population Impact: If either the prey or predator population is absent, predation cannot occur. In other words, the prey population's decline depends on the balance between birth and predation rates.

From the predator equation (1.3) we see that :
Predator Growth: The predator population's growth is directly linked to the frequency of interactions between predators and prey. This resembles the rate at which predators hunt down their prey, but with a distinct constant δpq. Unlike identical rates for predation and reproduction, here they differ.
Predator Decline: Since prey cannot actively kill predators, the decrease in the predator population results from natural causes or emigration. This decline follows an exponential decay pattern, represented by the term γq.
Predator Equation Summary: Essentially, the predator equation captures the balance between prey consumption and the natural mortality rate of the predator population.

1.3.3 Holling's Functional Responses

Holling Type- I Functional Response

In [6] it is formulated that,the general form of Holling type I functional response is given by:

$$g(N) = aN$$

where $g(N)$ is the feeding rate and a is positive constant. In the context of the classic predator-prey model proposed by Lotka and Volterra, we observe an interesting behavior: the feeding rate does not reach saturation as the prey density increases. This phenomenon diverges from the typical Holling type I functional response, where the feeding rate eventually levels off. Instead, we encounter a modified version expressed as $g(N)= min(aN,a)$, which saturates but does not exhibit the typical C^1 behavior.

Holling Type-II Functional Response

The general form of Holling type II functional response is given by :

$$g(N) = \frac{aN}{b+N}$$

This response models the fact that the consumption of prey is limited by satiation of predators, handling time (killing and eating) and time spent hunting prey. The curve of N against $g(N)$ can be plotted. Here the feeding rate saturates at the maximum feeding rate a. The feeding rate is half maximal at N = b.

Let N - available prey (no. of prey)

T- Total time (searching prey ;chasing/capturing prey; handling V number of prey)

T_h -handling time for each prey.

V - no. of victims (i.e no. of prey caught by the predator)

Total handling time is T_hV.

Time required for searching and capturing prey is $T - T_hV$.

$$V \propto N$$

$$V \propto T - T_hV$$

$$\therefore V = \alpha(T - T_hV)N$$

$$\text{i.e. } V = \frac{\alpha TN}{1 + \alpha T_hN}$$

$$i.e. V = \frac{aN}{b+N}$$

Here the rate of prey consumption increases at a decelerating rate with increasing prey density, eventually reaching a maximum, after which it saturates.

Holling Type-III Functional Response

The general form of Holling type III functional response is given by :

$$g(N) = \frac{aN^2}{b^2 + N^2}$$

The feeding rate initially increases with prey density but eventually levels off at the maximum feeding rate, denoted by a. However, there exists a point of inflection in the curve . This inflection point represents the scenario where, at low prey densities, the prey can successfully evade predators by seeking refuge. This type of functional response is commonly referred to as ‘sigmoidal’ or S-shaped curve.

1.3.4 Stability Analysis Using Routh-Hurwitz Criterion

- The criteria provides an analytical approach to evaluate the stability of systems of any order without the need to compute the roots of the characteristic equation. While the characteristic equation is essential for this assessment, the Routh–Hurwitz criteria suffices for linear system stability. It relies on the coefficients sequence within the characteristic equation.

Step 1: Calculating a Jacobian matrix.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial P_1}(P_1,P_2,\ldots,P_n) & \frac{\partial f_1}{\partial P_2}(P_1,P_2,\ldots,P_n) & \cdots & \frac{\partial f_1}{\partial P_n}(P_1,P_2,\ldots,P_n) \\ \frac{\partial f_2}{\partial P_1}(P_1,P_2,\ldots,P_n) & \frac{\partial f_2}{\partial P_2}(P_1,P_2,\ldots,P_n) & \cdots & \frac{\partial f_2}{\partial P_n}(P_1,P_2,\ldots,P_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial P_1}(P_1,P_2,\ldots,P_n) & \frac{\partial f_n}{\partial P_2}(P_1,P_2,\ldots,P_n) & \cdots & \frac{\partial f_n}{\partial P_n}(P_1,P_2,\ldots,P_n) \end{pmatrix}$$

where $\frac{\partial f_i}{\partial P_j}(P_1,P_2,\ldots,P_n)$ is the partial derivative of f_i with respect to its variable, $P_j(i,j = 1,2,\ldots,n)$.

Step 2: Find the Jacobian matrix.

The equilibrium value, $P_1^*,P_2^*,\ldots,P_n^*$, is used to calculate the Jacobian matrix. A local stability matrix, $\hat{J} = J|_{P_1=P_1^*,P_2=P_2^*,\ldots,P_n=P_n^*}$, is obtained. Then, using $\det(\hat{J}-\lambda I) = 0$, get the characteristic polynomial. Where I stands for the identity matrix, and rewrite as follows:

$$Q(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$$

with real coefficients a_i for $i = 1,2,\ldots,n$.

Step 3: Routh-Hurwiz criteria.

The following are the values for the n Hurwitz matrices:

$$H_1=(a_1)\,,H_2=\begin{pmatrix} a_1 & a_3 \\ a_0 & a_2 \end{pmatrix},\text{ and } H_n=\begin{pmatrix} a_1 & a_3 & a_5 & \dots & a_{2n-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n-2} \\ 0 & a_1 & a_3 & \dots & a_{2n-3} \\ 0 & a_0 & a_2 & \dots & a_{2n-4} \\ 0 & 0 & a_1 & \dots & a_{2n-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}$$

Note that if $j>n$, then $a_j=0$, $j=1,2,\dots,n$.

1.3.5 Type of Insects

Sterile Insects : Sterile insects are not self-replicating and therefore cannot become established in the environment. The sterile insect technique (SIT) is a powerful tool in biological pest control. Objective: Reduce populations of a target pest insect. Process involves ;Mass-produce insects of the same species, Sexually sterilize them (usually males) through techniques like irradiationRelease these sterile insects into the wild. Competition: The released sterile males compete with wild fertile males for females and the outcome will be mating between sterile males and wild females produces no viable offspring.

Fertile Insects : These are the regular, non-sterile insects that can reproduce and contribute to population growth. Fertile insects play essential roles in ecosystems, pollination, and food chains. Example : Aphids ,termites, etc.

Natural Enemies : Within the realm of ecological relationships, we encounter natural enemies organisms that either prey upon or compete with species considered harmful. These adversaries can be broadly divided into two groups: predators and parasites. Natural enemies, which include predators and parasites, play a vital role in ecological balance. Predators, being larger and more powerful, actively hunt down other species, while parasites exploit their hosts.
Examples: Lady beetles, beetles, etc

1.3.6 Degree of Effectiveness

It refers to the efficiency or impact of control measure in reducing pest populations. It represent the effectiveness of releasing sterile pest insects into the environment to reduce the reproductive success of pest population.
A higher degree of effectiveness ε implies that a greater proportion of the released sterile insect successfully compete, reduction in the overall pest population.

1.3.7 Control Measure

It is denoted by $u_1(t)$ and $u_2(t)$.It is used to optimize the control strategy achieve the desired objectives, such al minimizing pest density or maximizing plant density. (ie, achieve the desired outcome in managing pest popalation through control measures).

Literature Review

Scientists frequently employ mathematical models to understand the intricate dynamics between plants and pests. Mathematical models are commonly utilized by scientists to describe the interaction between plants and pests [1][4][12] . The later enables us to intervene the dynamic interaction among populations in the model. In pest control, many models have proven valuable in understanding the intervention mechanism . In this direction, determining the optimal control measure with respect to a certain performance index is often the research objective [1]. In this current work, [1] develop an optimal control model of plants-pests interaction with two control measures, namely, the release of predators as natural enemies of pests and the release of sterile pest insects. The sterile insect technique (SIT) is conducted by mass-rearing and periodically releasing sexually sterile pest insects using radiation into the wild targeted pest population to disrupt fecundity [1]. Our model thus consists of four interacting populations: the plant, the fertile insects, the sterile insects, and the predators. One additional issue we want to evaluate by the model is the release rates of predators and sterile insects. The more predators and sterile insects are released, the higher the cost of control will be. It is suggested by [5] that there is an optimal release rate in most cases that provided more effective control of pest insects. Thus, a fewest number of predators and sterile insects should be released as long as this improves control effectiveness[1].

In this research, as mention by [1],we develop a generic control model that describes the complex interac tion between plant and pest insects intervened by the release of sterile insects and natural enemies as control instruments. We focus in comparing the basic advantage favoring the control combinations to minimize the pest insect population jointly with the control cost. A salient feature introduced in this study is the evaluation of three different release strategies for sterile insects and natural enemies, namely, constant, proportional and saturating pro- portional release rates. The work extends to [2].

Interaction between pests and their natural enemies was also formulated by [8] in term of a prey-predator model, where stability properties, the existence of periodic solution were the main results. This model was extended by incorporating plant population and by implementing an indirect Z-control design to manage pest population [7].

Pest-natural enemy model with dependent instant killing and releasing rates was introduced by [10]. Beyond the stability analysis, this study particularly explores the relation between the number of natural enemies and their current density. It is found that the attain sability of bio control depends on the pest and predator initial densities and the predators released guided by the predator density is more sensitive to the pulse period and the number released predators [1]. As suggested by [12],because of the animals, pests, diseases, and weeds at least one-third to half of the world crop output is destroyed. Reducing the loss of food would lead to a major increase in the supply of food for use. A precise estimation of these damages is the first important step in reducing these losses. However, it is acknowledged that damage is rising due to different biotic and abiotic stresses in the face of the growing strength of agriculture and the environment on farmland. Thus, there is an urgent need to establish effective pest control techniques and agricultural pests enemy that is at the same time productive, healthy, and persistent. Using pest-resistant cultivators provides so many benefits and can form the center in which to build sustainable farming production. Pest control has appeared recently being one of the most important problems impacting ecologically sustainable due to the growing people communities and increasing food crisis. Hence, there is detailed research of effective pest control strategies by [12] .

Chapter 2

PEST CONTROL MODEL - I

2.1 Introduction

Over the past five decades, global population growth has surged at an unprecedented rate. Advances in agricultural technology have led to increased yields of major crops, but unfortunately, these developments have also contributed to environmental degradation. Despite these advancements, there remains a significant gap in our understanding of food losses and damages caused by biotic agents, particularly in developing countries[12]. The destruction of at least one-third to half of the world's crop output can be attributed to animals, pests, diseases, and weeds. Reducing food losses would significantly enhance the global food supply. However, accurately estimating these damages is a crucial initial step in mitigating losses. It is evident that damage is increasing due to various biotic and abiotic stresses, exacerbated by the intensification of agriculture and environmental pressures on farmland. Urgent efforts are needed to establish effective pest control methods that are both productive and sustainable. Utilizing pest-resistant cultivars offers numerous benefits and can serve as a cornerstone for building sustainable agricultural production. Pest control has emerged as a critical challenge for ecological sustainability, especially given the growing global population and food crises.

In pursuit of optimal pest management, researchers have explored a multifaceted approach combining infection control, chemical measures, and predation aspects. This balanced mix aims to mitigate pest populations effectively. Additionally, state-dependent dynamic impulsive mechanisms have been proposed, involving the release of natural enemies and targeted pesticide application. These strategies focus on impulsive interventions specific to pests. Various studies have employed prey-predator models and analyzed stage-structured populations. The overarching goal of this research is twofold: to enhance crop production and simultaneously reduce pest numbers. To achieve this, identifying suitable predators at the higher trophic level of the pest species becomes crucial.

The research aims to investigate the effectiveness of integrating natural enemies for pest management. Additionally, it highlights how monitoring pest control is significantly impacted by the removal of natural enemies due to predation by their own predators. Many species like spiders, birds, frogs, etc. are recognized as the natural enemy as they feed on agricultural pests. Biological monitoring is the beneficial of predators operation for controlling pests and their serious harm. Through the number of insects and mites, the natural enemies could be used as a biological control for pest management. The use of natural enemies is also useful for the biological management of the land area and wildland weeds.Therefore, the elimination of pests and the protection of the natural enemy from an agriculture sector and ecological viewpoint is very important[12].

2.2 Pest Control Model

In this model formulated by [12], The mathematical model is set up to protect crops from pests through the effective use of natural enemies .Here, we delve into a mathematical model that considers the interaction between prey (pests) and predators (natural enemies). To examine the effect of the natural enemy on control of the pest species, we first find a framework of prey–predator in three dimensions.

$$\frac{dP_1}{dt} = r_1 P_1 \left(1 - \frac{P_1}{K_1}\right) - \alpha_1 P_1 P_2, \tag{2.1}$$

$$\frac{dP_2}{dt} = r_2 P_2 \left(1 - \frac{P_2}{K_2}\right) - \frac{m P_3 P_2^2}{\alpha_2^2 + P_2^2} + \alpha_3 P_1 P_2, \tag{2.2}$$

$$\frac{dP_3}{dt} = \frac{\gamma P_3 P_2^2}{\alpha_2^2 + P_2^2} - \mu P_3 \tag{2.3}$$

with initial data $P_1(0) > 0, P_2(0) > 0$ and $P_3(0) > 0$.

The following assumptions were made in forming the above equations .

1. The plant grows logistically with intrinsic growth rate r_1 in an environment with carrying capacity K_1

2. The term $\frac{mP_3P_2^2}{\alpha_2^2+P_2^2}$ represents the predation loss in the pest equation.

3. Additionally, we assume that natural enemies can be removed from the ecosystem due to predation by their own predators, apart from natural death.

4. Functional Responses of Natural Enemies: Natural enemies exhibit different functional responses when preying on pest populations. In our model, we adopt the Holling type III functional response, which captures the relationship between prey

consumption and prey density.This approach contributes to sustainable agriculture and ecological balance.

2.3 Boundedness of the System

Theorem 2.3.0.1. *The solutions of the system (2.1) - (2.3) in the region R_+^3 are bounded [12].*

Proof. Let the total Population be $N(P_1, P_2, P_3) = P_1 + P_2 + P_3$.

Then the derivative of $N(P_1, P_2, P_3)$ with respect to time t is ;

$$\begin{aligned}
\frac{dN}{dt} &= \frac{dP_1}{dt} + \frac{dP_2}{dt} + \frac{dP_3}{dt} \\
&= r_1 P\left(1 - \frac{P_1}{K_1}\right) - \alpha_1 P_1 P_2 + r_2 P_2\left(1 - \frac{P_2}{K_2}\right) - \frac{mP_3P_2^2}{\alpha_2^2 + P_2^2} + \alpha_3 P_1 P_2 + \frac{\gamma P_3 P_2^2}{\alpha_2^2 + P_2^2} - \mu P_3 \\
&\leqslant r_1 P_1\left(1 - \frac{P_1}{K_1}\right) + r_2 P_2\left(1 - \frac{P_2}{K_2}\right) - \mu P_3, \\
\frac{dN}{dt} + \tau N &\leqslant r_1 P_1\left(1 - \frac{P_1}{K_1}\right) + r_2 P_2\left(1 - \frac{P_2}{K_2}\right) - \mu P_3 + \tau(P_1 + P_2 + P_3) \\
&= (r_1 + \tau)P_1 + (r_2 + \tau)P_2 - \frac{r_1}{K_1}P_1^2 - \frac{r_2}{K_2}P_2^2 - (\mu - \tau)P_3
\end{aligned}$$

$$\Rightarrow \frac{dN}{dt} + \tau N \leqslant \frac{K_1(r_1+\tau)^2}{4r_1} + \frac{K_2(r_2+\tau)^2}{4r_2} = L.$$

Here, $0 < \tau < \mu$ and $L = \frac{K_1(r_1+\tau)^2}{4r_1} + \frac{K_2(r_2+\tau)^2}{4r_2}$. Thus we get,

$$\frac{dN}{dt} + \tau N \leq L$$

Compare with the first order differential equation , $\frac{dy}{dx}+Py=Q$

$$\begin{aligned} P=\tau \quad Q=L \\ \therefore IF=e^{\int Pdt} \\ =e^{\int \tau dt} \\ =e^{\tau t} \end{aligned}$$

$\therefore N(IF)=\int Q.\text{IF}\,dt.$

$Ne^{\tau t}\leq \int L\cdot e^{\tau t}dt+C$

$$\begin{aligned} &=\frac{Le^{\tau t}}{\tau}-\int 0+C \\ &\leqslant L\frac{e^{\tau t}}{\tau}+C \\ \Rightarrow Ne^{\tau t} &\leqslant e^{\tau t}\left[\frac{L}{\tau}+\frac{C}{e^{\tau t}}\right] \\ N(t) &\leqslant \frac{L}{a}+Ce^{-\tau t} \end{aligned}$$

At $t=0$

$$\begin{aligned} N_0 &= \frac{L}{\tau}+C \\ \Rightarrow C &= N_0-\frac{L}{\tau} \\ \therefore N(t) &\leqslant \frac{L}{\tau}+\left(N_0-\frac{L}{\tau}\right)e^{-\tau t} \end{aligned}$$

This differential inequality has the solution

$$0<N(t)\leqslant e^{-\tau t}\left(N_0-\frac{L}{\tau}\right)+\frac{L}{\tau},$$

when $t\to\infty$, yields $0<N(t)<\frac{L}{\tau}$.

This completes the proof. □

2.4 Local Stability Analysis

2.4.1 Equilibrium Points of the System

Equilibrium points of a system of differential equation is a constant solution to the system and is obtained by equating the equation (2.1) - (2.3) to zero.

$$r_1 P_1 \left(1 - \frac{P_1}{K_1}\right) - \alpha_1 P_1 P_2 = 0 \tag{2.4}$$

$$r_2 P_2 \left(1 - \frac{P_2}{K_2}\right) - \frac{m P_3 P_2^2}{\alpha_2^2 + P_2^2} + \alpha_3 P_1 P_2 = 0 \tag{2.5}$$

$$\frac{\gamma P_3 P_2^2}{\alpha_2^2 + P_2^2} - \mu P_3 = 0 \tag{2.6}$$

Equilibrium point E_0 :

Clearly $P_1 = 0$, $P_2 = 0$ and $P_3 = 0$ are equilibrium points of the differential equation (2.4), (2.5), (2.6) respectively.

Therefore , $E_0\,(P_1 = 0, P_2 = 0, P_3 = 0)$ is the trivial equilibrium point of the system of differential equation.

Equilibrium point E_1 :

Clearly $P_1 = 0$ and $P_3 = 0$ are equilibrium points of the differential equation (2.4) and (2.6) respectively. Substituting $P_1 = 0$ and $P_3 = 0$ in (2.5) we get ,

$$P_2 \left(r_2 \left(1 - \frac{P_2}{K_2}\right) - \frac{m P_3 P_2}{\alpha_2^2 + P_2^2} + \alpha_3 P_1 \right) = 0$$

$$\Rightarrow r_2 \left(1 - \frac{P_2}{K_2}\right) = 0$$

$$\Rightarrow \frac{K_2 - P_2}{K_2} = 0$$

$$\Rightarrow P_2 = K_2$$

Therefore , $E_1\,(P_1 = 0, P_2 = K_2, P_3 = 0)$ is the equilibrium point of the system of differential equation.

Equilibrium point E_2 :

Clearly, $P_2 = 0$ and $P_3 = 0$ are equilibrium points of the differential equation (2.5) and (2.6) respectively. Substituting $P_2 = 0$ and $P_3 = 0$ in (2.4) we get ,

$$
\begin{aligned}
& r_1 P_1 \left(1 - \frac{P_1}{K_1}\right) - \alpha_1 P_1 P_2 = 0 \\
& \Rightarrow P_1 \left(r_1 \left(1 - \frac{P_1}{K_1}\right) - \alpha_1 P_2\right) = 0 \\
& \Rightarrow r_1 \left(1 - \frac{P_1}{K_1}\right) = 0 \\
& \Rightarrow \frac{K_1 - P_1}{K_1} = 0 \\
& \Rightarrow P_1 = K_1
\end{aligned}
$$

Therefore , $E_2\,(P_1 = K_1, P_2 = 0, P_3 = 0)$ is the equilibrium point of the system of differential equation.

Equilibrium point E_3 **and** E_4 :

Clearly, $P_1 = 0$ are equilibrium points of the differential equation (2.4).Substituting $P_1 = 0$ in (2.6) we get ,

$$
\begin{aligned}
& P_3 \left(\frac{\gamma P_2^2}{\alpha_2^2 + P_2^2} - \mu\right) = 0 \\
& \Rightarrow \frac{\gamma P_2^2}{\alpha_2^2 + P_2^2} = \mu \\
& \Rightarrow \gamma P_2^2 = \mu(\alpha_2^2 + P_2^2) \\
& \Rightarrow \gamma P_2^2 = \mu \alpha_2^2 + \mu P_2^2 \\
& \Rightarrow (\gamma - \mu) P_2^2 = \mu \alpha_2^2
\end{aligned}
$$

$$\Rightarrow P_2^2 = \frac{\mu\alpha_2^2}{\gamma-\mu}$$
$$\Rightarrow P_2 = \pm\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$$
$$\therefore P_2 = +\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}} \text{ and } P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$$

Next, substituting $P_2 = +\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$ in equation (2.5),

From equation (2.5) we have,

$$P_2\left(r_2\left(1-\frac{P_2}{K_2}\right)-\frac{mP_3P_2}{\alpha_2^2+P_2^2}+\alpha_3P_1\right)=0$$
$$\Rightarrow r_2\left(1-\frac{P_2}{K_2}\right)-\frac{mP_3P_2}{\alpha_2^2+P_2^2}=0$$
$$\Rightarrow \left(r_2(1-\frac{P_2}{K_2})\right)(\alpha_2^2+P_2^2)=mP_3P_2$$
$$\Rightarrow \frac{r_2\left(\frac{K_2-P_2}{K_2}\right)(\alpha_2^2+P_2^2)}{mP_2}=P_3$$
$$\Rightarrow \frac{r_2(K_2-P_2)(\alpha_2^2+P_2^2)}{K_2mP_2}=P_3$$
$$\Rightarrow \frac{r_2(K_2-\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}})(\alpha_2^2+\frac{\mu\alpha_2^2}{\gamma-\mu})}{K_2m\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}}=P_3$$

$$\Rightarrow \frac{\gamma\alpha_2^2r_2(K_2\sqrt{\gamma-\mu}-\sqrt{\mu}\alpha_2)(\sqrt{\gamma-\mu})}{K_2m(\gamma-\mu)\sqrt{\mu}\alpha_2}=P_3$$
$$\Rightarrow \frac{\gamma\alpha_2r_2(K_2\sqrt{\gamma-\mu}-\sqrt{\mu}\alpha_2)}{K_2m(\gamma-\mu)\sqrt{\mu}}=P_3$$

Similarly, substituting $P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$ in equation (2.5),

From equation (2.5) we get,

$$P_3 = \frac{\gamma\alpha_2r_2(K_2\sqrt{\gamma-\mu}+\sqrt{\mu}\alpha_2)}{K_2m(\gamma-\mu)\sqrt{\mu}}$$

Therefore , $E_3\left(P_1=0, P_2=-\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3=\frac{\gamma\alpha_2 r_2(K_2\sqrt{\gamma-\mu}+\sqrt{\mu}\alpha_2)}{K_2 m(\gamma-\mu)\sqrt{\mu}}\right)$ and

$E_4\left(P_1=0, P_2=\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3=\frac{\gamma\alpha_2 r_2(K_2\sqrt{\gamma-\mu}-\sqrt{\mu}\alpha_2)}{K_2 m(\gamma-\mu)\sqrt{\mu}}\right)$ are the equilibrium points of the system of differential equation.

Equilibrium point E_5 :

Clearly, $P_3=0$ are equilibrium points of the differential equation (2.6). Substituting $P_3=0$ in equation (2.4) we get ,

$$P_1\left(r_1\left(1-\frac{P_1}{K_1}\right)-\alpha_1 P_2\right)=0$$

$$\Rightarrow r_1\left(1-\frac{P_1}{K_1}\right)=\alpha_1 P_2 \tag{2.7}$$

$$\Rightarrow \frac{r_1}{K_1}\left(1-\frac{P_1}{K_1}\right)=P_2 \tag{2.8}$$

Similarly from equation (2.5) and substituting $P_3=0$ we have ,

$$P_2\left(r_2\left(1-\frac{P_2}{K_2}\right)-\frac{mP_3P_2}{\alpha_2^2+P_2^2}+\alpha_3 P_1\right)=0$$

$$\Rightarrow r_2\left(1-\frac{P_2}{K_2}\right)=-\alpha_3 P_1 \tag{2.9}$$

$$\Rightarrow 1-\frac{P_2}{K_2}=-\frac{\alpha_3 P_1}{r_2}$$

$$\Rightarrow P_2=\left(\frac{\alpha_3 P_1}{r_2}+1\right)K_2 \tag{2.10}$$

Equating equation (2.8) and (2.10) we get ,

$$\frac{r_1}{K_1}\left(1-\frac{P_1}{K_1}\right)=\left(\frac{\alpha_3 P_1}{r_2}+1\right)K_2$$

$$\Rightarrow \frac{r_1}{\alpha_1} - \frac{P_1 r_1}{\alpha_1 K_1} = \frac{\alpha_3 P_1 K_2}{r_2} + K_2$$
$$\Rightarrow \frac{r_1}{\alpha_1} - K_2 = P_1 \left(\frac{\alpha_3 K_2}{r_2} + \frac{r_1}{\alpha_1 K_1} \right)$$
$$\Rightarrow P_1 = \frac{(r_1 - \alpha_1 K_2) r_1 K_1}{\alpha_1 \alpha_3 K_1 K_2 + r_1 r_2}$$

To find P_2 , from equation (2.9) we have,

$$P_1 = \frac{r_2}{\alpha_3} \left(\frac{P_2}{K_2} - 1 \right)$$

$$\Rightarrow \frac{r_2}{\alpha_3 K_2} (P_2 - K_2) = P_1 \tag{2.11}$$

Also from equation (2.7)

$$\left(1 - \frac{P_1}{K_1} \right) = \frac{\alpha_1 P_2}{r_1}$$

$$\Rightarrow P_1 = \left(1 - \frac{\alpha_1 P_2}{r_1} \right) K_1 \tag{2.12}$$

Now equating equation (2.11) and (2.12) we get ,

$$\frac{r_2}{\alpha_3 K_2} (P_2 - K_2) = \left(1 - \frac{\alpha_1 P_2}{r_1} \right) K_1$$
$$\Rightarrow K_1 - \frac{\alpha_1 P_2 K_1}{r_1} = \frac{r_2 P_2}{\alpha_3 K_2} - \frac{r_2}{\alpha_3}$$

$$\Rightarrow \frac{\alpha_3 K_1 + r_2}{\alpha_3} = P_2 \left(\frac{r_2}{\alpha_3 K_2} + \frac{\alpha_1 K_1}{r_1} \right)$$
$$\Rightarrow P_2 = \frac{(\alpha_3 K_1 + r_2) K_2 r_1}{r_1 r_2 + K_1 K_2 \alpha_1 \alpha_3}$$

Therefore , $E_5 \left(P_1 = \frac{(r_1 - \alpha_1 K_2) r_1 K_1}{\alpha_1 \alpha_3 K_1 K_2 + r_1 r_2}, P_2 = \frac{(\alpha_3 K_1 + r_2) K_2 r_1}{r_1 r_2 + K_1 K_2 \alpha_1 \alpha_3}, P_3 = 0 \right)$ is the equilibrium point of the system of differential equation.

Equilibrium point E_6 and E_7

We know that ,

$$P_2 = \pm\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$$
$$\therefore P_2 = +\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}} \text{ and } P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$$

Also, from equation (2.12) we have,

$$P_1 = \left(1-\frac{\alpha_1 P_2}{r_1}\right)K_1$$
$$\Rightarrow P_1 = K_1\left(1-\frac{\alpha_1\alpha_2\sqrt{\mu}}{r_1(\sqrt{\gamma-\mu})}\right)$$

Next from equation (2.5) we have

$$r_2\left(1-\frac{P_2}{K_2}\right)-\frac{mP_3P_2}{\alpha_2^2+P_2^2}+\alpha_3P_1 = 0$$
$$\Rightarrow \frac{mP_3P_2}{\alpha_2^2+P_2^2} = r_2\left(1-\frac{P_2}{K_2}\right)+\alpha_3P_1$$
$$\Rightarrow \frac{mP_3P_2}{\alpha_2^2+P_2^2} = \frac{r_2(K_2-P_2)+K_2\alpha_3P_1}{K_2}$$
$$\Rightarrow P_3 = \frac{(r_2(K_2-P_2)+K_2\alpha_3P_1)(\alpha_2^2+P_2^2)}{mK_2P_2}$$
$$\Rightarrow P_3 = \frac{(\alpha_2^2+P_2^2)(r_2K_2-r_2P_2+K_2\alpha_3P_1)}{mK_2P_2}$$
$$\Rightarrow P_3 = \frac{\alpha_2^2r_2K_2-\alpha_2^2r_2P_2+\alpha_2^2K_2\alpha_3P_1+P_2^2r_2K_2-P_2^2r_2P_2+P_2^2K_2\alpha_3P_1}{mK_2P_2}$$

Substituting the value of P_1 and P_2 we get ,

$$\Rightarrow P_3 = \frac{\gamma\alpha_2\left(\sqrt{\mu}r_1r_2\alpha_2+K_2\left(\sqrt{\mu}K_1\alpha_1\alpha_2\alpha_3+\sqrt{\gamma-\mu}r_1\left(r_2+K_1\alpha_3\right)\right)\right)}{m\sqrt{\mu}(\gamma-\mu)K_2r_1} = C_1$$

when$P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$

and

$$P_3 = \frac{\gamma\alpha_2\left(-\sqrt{\mu}r_1r_2\alpha_2 + K_2\left(\sqrt{\mu}K_1\alpha_1\alpha_2\alpha_3 + \sqrt{\gamma-\mu}r_1\left(r_2+K_1\alpha_3\right)\right)\right)}{m\sqrt{\mu}(\gamma-\mu)K_2r_1} = C_2$$

when $P_2 = +\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}$

Therefore , $E_6\left(P_1 = K_1\left(1+\frac{\alpha_1\alpha_2\sqrt{\mu}}{r_1(\sqrt{\gamma-\mu})}\right), P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = C_1\right)$ and

$E_7\left(P_1 = K_1\left(1-\frac{\alpha_1\alpha_2\sqrt{\mu}}{r_1(\sqrt{\gamma-\mu})}\right), P_2 = \frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = C_2\right)$ are the equilibrium points of the system of differential equation.

The model (2.1) ,(2.2) and (2.3) has the following equilibrium points.

i) The trivial equilibrium $E_0\left(P_1 = 0, P_2 = 0, P_3 = 0\right)$, which always exists.

ii) $E_1\left(P_1 = 0, P_2 = K_2, P_3 = 0\right)$.

iii) $E_2\left(P_1 = K_1, P_2 = 0, P_3 = 0\right)$.

iv) $E_3\left(P_1 = 0, P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = \frac{\gamma r_2\alpha_2\left(\sqrt{\gamma-\mu}K_2+\sqrt{\mu}\alpha_2\right)}{m\sqrt{\mu}(\gamma-\mu)K_2}\right)$.

v) $E_4\left(P_1 = 0, P_2 = \frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = \frac{\gamma r_2\alpha_2\left(\sqrt{\gamma-\mu}K_2-\sqrt{\mu}\alpha_2\right)}{m\sqrt{\mu}(\gamma-\mu)K_2}\right)$.

vi) $E_5\left(P_1 = \frac{K_1r_2(r_1-K_2\alpha_1)}{r_1r_2+K_1K_2\alpha_1\alpha_3}, P_2 = \frac{K_2r_1(r_2+K_1\alpha_3)}{r_1r_2+K_1K_2\alpha_1\alpha_3}, P_3 = 0\right)$.

vii) $E_6\left(P_1 = K_1\left(1+\frac{\sqrt{\mu}\alpha_1\alpha_2}{\sqrt{\gamma-\mu}r_1}\right), P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = C_1\right)$.

viii) $E_7\left(P_1 = K_1\left(1-\frac{\sqrt{\mu}\alpha_1\alpha_2}{\sqrt{\gamma-\mu}r_1}\right), P_2 = \frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}}, P_3 = C_2\right)$.

2.4.2 Local Stability

In the study of differential equations, understanding the stability of equilibrium points is crucial. Here are the key concepts related to stability assessment:

Asymptotic Stability: An equilibrium point is asymptotically stable if all the eigenvalues of its Jacobian matrix have negative real parts.
Unstable Equilibrium: If at least one eigenvalue of the Jacobian matrix has a positive real part, the equilibrium point is considered unstable. Unstable equilibria lead to unpredictable behavior in the system.

Routh–Hurwitz Stability Criterion: This criterion relies on the characteristic polynomial of the Jacobian matrix. By analyzing the polynomial's roots, we can determine stability properties. If all roots have negative real parts, the system is stable. To assess the stability of equilibrium points locally, we compute the Jacobian matrix $J(P_1, P_2, P_3)$ at any given equilibrium point (P_1, P_2, P_3). The Jacobian matrix captures the linearization of the system near the equilibrium. By analyzing its eigenvalues, we gain insights into stability behavior.

$$J(P_1, P_2, P_3) = \begin{pmatrix} -\frac{P_1 r_1}{K_1} + \left(1 - \frac{P_1}{K_1}\right) r_1 - P_2\alpha_1 & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & M_1 & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & M_2 & -\mu + \frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}$$

where $M_1 = -\frac{P_2 r_2}{K_2} + \left(1 - \frac{P_2}{K_2}\right) r_2 + \frac{2mP_2^3 P_3}{\left(P_2^2+\alpha_2^2\right)^2} - \frac{2mP_2 P_3}{P_2^2+\alpha_2^2} + P_1\alpha_3$ and
$M_2 = -\frac{2\gamma P_2^3 P_3}{\left(P_2^2+\alpha_2^2\right)^2} + \frac{2\gamma P_2 P_3}{P_2^2+\alpha_2^2}$

(i)The Jacobian matrix at equilibrium point $E_0\,(P_1=0,P_2=0,P_3=0)$ is

$$\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -\mu \end{pmatrix}$$

and the eigen values at the E_0 is r_1, r_2 and $-\mu$. Therefore the equilibrium point is saddle.

(ii) The Jacobian matrix for the equilibrium point $E_1\,(P_1=0,P_2=K_2,P_3=0)$ is

$$\begin{pmatrix} r_1-K_2\alpha_1 & 0 & 0 \\ K_2\alpha_3 & -r_2 & -\frac{mK_2}{K_2^2+\alpha_2^2} \\ 0 & 0 & -\mu+\frac{\gamma K_2^2}{K_2^2+\alpha_2^2} \end{pmatrix}$$

The eigenvalues of $J\,(E_1)$ are $-r_2, r_1-K_2\alpha_1$ and $\frac{\gamma K_2^2-\mu K_2^2-\mu\alpha_2^2}{K_2^2+\alpha_2^2}$. Therefore the equilibrium point $E_1\,(P_1=0,P_2=K_2,P_3=0)$, is locally asymptotically stable if $r_1-K_2\alpha_1<0$ and $\frac{\gamma K_2^2-\mu K_2^2-\mu\alpha_2^2}{K_2^2+\alpha_2^2}<0$.

(iii) The Jacobian matrix for the equilibrium point $E_2\,(P_1=K_1,P_2=0,P_3=0)$ is

$$\begin{pmatrix} -r_1 & -K_1\alpha_1 & 0 \\ 0 & r_2+K_1\alpha_3 & 0 \\ 0 & 0 & -\mu \end{pmatrix}$$

The eigenvalues of $J\,(E_2)$ are $-\mu, -r_1$ and $r_2+K_1\alpha_3$. Since two eigenvalues are negative and other is always positive, therefore the system around the (E_2) point is saddle point.

(iv) The Jacobian matrix for the equilibrium point $E\,(P_1=0, P_2\neq 0, P_3\neq 0)$ is

$$\begin{pmatrix} r_1-P_2\alpha_1 & 0 & 0 \\ P_2\alpha_3 & r_2-\frac{2P_2r_2}{K_2}+\frac{2mP_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}-\frac{2mP_3P_2}{P_2^2+\alpha_2^2} & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & -\frac{2\gamma P_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}+\frac{2\gamma P_2P_3}{P_2^2+\alpha_2^2} & -\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}$$

The characteristic polynomial of Jacobian matrix is

$$x^3+a_1x^2+a_2x+a_3=0$$

where, $a_1=\left(-r_1+P_2\alpha_1-r_2+\frac{2P_2r_2}{K_2}-\frac{2mP_2^3P_3}{\left(P_2^2-\alpha_2^2\right)^2}+\frac{2mP_3P_2}{P_2^2+\alpha_2^2}+\mu-\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)$,

$$a_2=(r_1-P_2\alpha_1)\left(r_2-\frac{2P_2r_2}{K_2}+\frac{2mP_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}-\frac{2mP_3P_2}{P_2^2+\alpha_2^2}\right)+\left(r_2-\frac{2P_2r_2}{K_2}+\frac{2mP_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}-\frac{2mP_3P_2}{P_2^2+\alpha_2^2}\right)\left(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)$$
$$+\left(-\frac{2\gamma P_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}+\frac{2\gamma P_2P_3}{P_2^2+\alpha_2^2}\right)\left(\frac{mP_2}{P_2^2+\alpha_2^2}\right)+(r_1-P_2\alpha_1)\left(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right),$$

$$a_3=(r_1-P_2\alpha_1)\left(\left(r_2-\frac{2P_2r_2}{K_2}+\frac{2mP_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}-\frac{2mP_3P_2}{P_2^2+\alpha_2^2}\right)\left(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)+\left(-\frac{2\gamma P_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}+\frac{2\gamma P_2P_3}{P_2^2+\alpha_2^2}\right)\left(\frac{mP_2}{P_2^2+\alpha_2^2}\right)\right)$$

For the local asymptotically stable of the system, the following Routh-Hurwitz criterion must be satisfied: (1) $a_1>0, a_2>0, a_3>0$, (2) $a_1a_2-a_3>0$.

(v) The Jacobian matrix for the equilibrium point $E_5\,(P_1\neq 0, P_2\neq 0, P_3=0)$ is

$$\begin{pmatrix} -\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1 & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & -\frac{P_2r_2}{K_2}+\left(1-\frac{P_2}{K_2}\right)r_2+P_1\alpha_3 & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & 0 & -\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}$$

The characteristic polynomial of the Jacobian matrix is

$$x^3+a_1x^2+a_2x+a_3=0$$

where $a_1=\left(\frac{P_1r_1}{K_1}-\left(1-\frac{P_1}{K_1}\right)r_1+P_2\alpha_1+\frac{P_2r_2}{K_2}-\left(1-\frac{P_2}{K_2}\right)r_2-P_1\alpha_3+\mu+\frac{\gamma P_2^2}{P_2^2-\alpha_2^2}\right)$

$$a_2=\left(\frac{\mu P_1r_1}{K_1}+\frac{\mu P_2r_2}{K_2}+\frac{P_1P_2r_1r_2}{K_1K_2}+\frac{m\gamma P_2^3P_3}{\left(P_2^2+\alpha_2^2\right)^2}-\frac{m\mu P_2P_3}{P_2^2+\alpha_2^2}-\frac{\gamma P_1P_2^2r_1}{K_1\left(P_2^2+\alpha_2^2\right)}-\frac{mP_1P_2P_3r_1}{K_1\left(P_2^2+\alpha_2^2\right)}-\frac{\gamma P_2^3r_2}{K_2\left(P_2^2+\alpha_2^2\right)}+P_1P_2\alpha_1\right.$$

$a_3=\left(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)\left(\left(-\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1\right)\left(-\frac{P_2r_2}{K_2}+\left(1-\frac{P_2}{K_2}\right)r_2+P_1\alpha_3\right)+\right.$
$\left.(-P_1\alpha_1)(P_2\alpha_3)\right)$.

For the local asymptotically stable of the system, the following Routh-Hurwitz criterion must be satisfied: (1) $a_1>0, a_2>0, a_3>0$, (2) $a_1a_2-a_3>0$.

(vi) The Jacobian matrix for the equilibrium point $E\,(P_1\neq 0, P_2\neq 0, P_3\neq 0)$ is

$$\begin{pmatrix} -\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1 & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & M_1 & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & M_2 & -\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}$$

The characteristic equation of system around the equilibrium is

$$x^3+a_1x^2+a_2x+a_3=0$$

where $a_1=\left(\frac{P_1r_1}{K_1}-\left(1-\frac{P_1}{K_1}\right)r_1+P_2\alpha_1-M_1+\mu-\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)$
$a_2=\left(-\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1\right)\left((M_1)(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2})+(M_2)(-\frac{mP_2}{P_2^2+\alpha_2^2})\right)+$
$P_1\alpha_1(P_2\alpha_3)(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}))$
$a_3=\left(-\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1\right)\left(M_1(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2})+(M_2)(\frac{mP_2}{P_2^2+\alpha_2^2})\right)+$
$P_1\alpha_1(P_2\alpha_3)\left(-\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2}\right)$.

For the local asymptotically stable of the system, the following Routh-Hurwitz criterion must be satisfied: (i) $a_1>0, a_2>0, a_3>0$, (ii) $a_1a_2-a_3>0$.

2.5 Conclusion

In our research, we developed a three-dimensional model that considers the interactions among prey (crops), pests (predators of crops), and natural enemies of pests (predators of pests). Our goal was to devise an effective strategy for pest population control by combining pesticides and natural enemies.

System Boundaries and Equilibrium Points: The system exhibits bounded behavior, which is essential for stability. We established criteria for the existence and stability of equilibrium points. These points represent steady states where the population dynamics balance.Proper monitoring and understanding interference effects are critical for sustainable pest management. Holistic Perspectives: Researchers will provide comprehensive insights into farming system productivity, ecological balance, and economic sustainability. Flexible management practices and development strategies will drive truly sustainable agriculture.

Chapter 3

PEST CONTROL MODEL - II

3.1 Introduction

Pests are harmful organisms that affect the health and productivity of plants. They can cause damage by feeding on plant tissues, transmitting diseases, or competing for resources. To control pest in plants, there are various methods that can be used, such as cultural, physical, biological, and chemical techniques. Cultural methods involve modifying the environment or the plant to make it less attractive or suitable for pests. Physical methods involve removing or excluding pests from plants by hand, water, traps, or barriers. Biological methods involve using natural enemies, such as predators, parasites, or pathogens, to reduce pest populations.

Chemical methods involve applying pesticides, such as insecticides, fungicides, or herbicides, to kill or repel pests[1]. However, chemical methods can have negative impacts on the environment and human health, so they should be used with caution and as a last resort. The most effective and sustainable way to control pest in plants is to use integrated pest management (IPM), which combines different methods in a way that minimizes the use of pesticides and the impact on the environment. IPM also considers the economic, social, and ecological aspects of pest control. Scientists frequently employ mathematical models to characterize the interplay between plants and pests. These models serve as tools for evaluating the efficacy of control strategies. By doing so, we can actively intervene in the dynamic interactions within populations. In the field of pest control, various models have provided valuable insights into the mechanisms underlying effective interventions.

Our system of equation consists of four interconnected components: plants, fertile insects, sterile insects, and predators (Natural enemies). Our objective is to evaluate how different release rates for both predators and sterile insects impact the system. As the number of released organisms increases, so does the cost of control. Existing research indicates that an optimal release rate exists in most situations, leading to more efficient pest insect management. Therefore, our strategy involves releasing the minimum number of predators and sterile insects while maintaining enhanced control effectiveness. In this research, we present a comprehensive control model that captures the intricate dynamics between plants and pest insects. Our focus lies in evaluating various control strategies, specifically the release of sterile insects and natural enemies, to mitigate pest populations while considering control costs[1].

Key aspects of our study; Release Strategies: We investigate three distinct release strategies for sterile insects and natural enemies: constant, proportional, and saturating proportional release rates. Natural Enemies: Unlike previous models, we incorporate the class of natural enemies into our framework. This inclusion allows for a more realistic assessment of pest birth rates.

3.2 Pest Control Model

The model formulated by [1], The interactions of plant, pests and natural enemies are distributed into three classes of populations but four state variables. We denote by $P(t)$ the density of plants at time t, by $F(t)$ the population of fertile pest insects at time t, by $S(t)$ the population of (male) sterile pest insects at time t and by $E(t)$ the population of natural enemies at time t.

3.2.1 Assumptions

To avoid the complexity of the model, we make certain assumptions.
1. The plant population exhibits logistic growth, characterized by an intrinsic growth rate r_p, within an environment with a carrying capacity of k.

2. Only pest consumes the plant with consumption rate a_1 for fertile insects and a_2 for sterile insects.The plant-pest interaction is following a predator-prey Holling type II function with handling time c_1 and c_2, and constant of half-saturation m. Thus, response functions of plants consumption by fertile and sterile insects are, respectively, given by $g_F(P)$ and $g_S(P)$, where

$$g_F(P) = \frac{a_1 P}{m + c_1 P} \tag{3.1}$$

$$g_S(P) = \frac{a_2 P}{m + c_2 P} \tag{3.2}$$

3. Predators feed on both reproductive and non-reproductive pests, with distinct consumption rates:b_F for fertile pests and b_S for sterile pests. Their feeding behavior adheres to a Holling type II response curve.

4. The efficiency of plant consumption by fertile insects is denoted as e_1, while the efficiency of plant consumption by sterile insects is represented by e_2. Since the population of sterile insects is small, we can infer that $e_2$is also low. We assume that e_2 lies within the range of [0, 1]. Similarly, the conversion efficiency of insect consumption by natural enemies is denoted as e_3. Again, we assume that e_3 falls within the interval [0, 1]. This implies that when a natural enemy consumes an insect, it can produce at most one predator.

5. In this model, there is no migration activity for the population. Recruitment occurs exclusively through the reproductive process or manual release.

6. The introduction of sterile pest insects leads to a reduction in the birth rate of pests. Initially, the intrinsic birth rate is denoted as r_h. However, due to the release of sterile pest insects, this birth rate decreases to $r_hF/(F+S)$. Additionally, based on our assumptions, we consider that the growth of fertile insects is influenced by the availability of plants as their food resource (i.e.pest only survive on plant) . Consequently, the pest birth rate r can be expressed as:

$$r = \frac{e_1 a_1 P}{m + c_1 P} \frac{r_h F}{1 + F + S} \tag{3.3}$$

7. Biological control agents exhibit comparable survivability to their wild counterparts. However, there are three primary factors contributing to pest mortality:Natural causes represented by d_F and d_F . Self-interaction effects denoted by α_F and α_S . Fertile-sterile interactions captured by β. For the natural enemies, we use the symbols d_E and α_E to represent their corresponding mortality rates.

8. Sterile insects are strategically released based on the abundance of fertile insects, as governed by the rate function $R_1(F)$. while , the introduction of natural enemies is

contingent upon both the fertile and sterile insect populations, as described by the rate function $R_2(F,S)$

3.2.2 Mathematical Model

The intricate relationships between plants, pest insects, and their natural enemies find representation in a system of nonlinear ordinary differential equations. These equations serve as the governing dynamics, describing the transitions within this ecological framework.

The growth of plant P serves a dual purpose: it provides essential commodities and serves as a food source for pests. The governing equation for its dynamics is as follows:

$$\frac{dP}{dt} = r_p P\left(1 - \frac{P}{k}\right) - \frac{a_1 P}{m + c_1 P}F - \frac{a_2 P}{m + c_2 P}S \tag{3.4}$$

Here, the first term on the right-hand side of (3.4) represents logistic growth, characterized by the intrinsic growth rate r_p and the carrying capacity k. The subsequent two terms account for the consumption of plant material by both fertile F and sterile S pests. These intake rates adhere to the Holling type II functional form.

The population growth rate of fertile pest insects, denoted by F, is influenced by several factors. Here's the formulation of the dynamics:

$$\frac{dF}{dt} = \frac{e_1 a_1 P}{m + c_1 P}\frac{r_h F}{1 + F + S}F - \frac{b_F F}{1 + F + S}E - (d_F + \alpha_F F + \beta S)F \tag{3.5}$$

Birth Rate: The increase in population due to births is captured by the first term on the right-hand side of the equation (3.5). Specifically, it depends on the availability of plant resources P and the presence of sterile insects S. Sterile insects in the denominator

have a negative impact on growth, effectively lowering the birth rate. However, this reduction is counterbalanced by the positive effect of plant ingestion.
Predation Effect: The second term represents the population reduction caused by predation by natural enemies.The rate of predation is denoted by b_F . The intake rate follows a Holling type II function, reflecting how predators consume pest insects.
Mortality Factors: The last terms account for the death of fertile pest insects. These include natural mortality d_F, self-regulation due to population density $\alpha_F F$, and the impact of sterile insects βS.

The population of sterile pest insects, denoted by S, experiences growth due to plant consumption and declines due to predation and natural death. Additionally, the population increases through the manual release of sterile insects into the wild. The dynamics of this system can be described by the following differential equation:

$$\frac{dS}{dt} = \frac{e_2 a_2 P}{m + c_2 P} S - (d_S + \alpha_S S + \beta F) S - \frac{b_S S}{1 + F + S} E + \varepsilon_1 u_1 R_1(F). \tag{3.6}$$

Where, in equation (3.6) , u_1=$u_1(t)$ represents the control measure for applying Sterile Insect Technique (SIT). The release rate of sterile pest insects, denoted by $R_1(F)$, depends on the number of fertile counterparts. The effectiveness of this control action is characterized by the parameter ε_1 .

The population of natural enemies can be intentionally augmented to control pests. This involves releasing them at a rate denoted by $R_2(F,S)$, which depends on the abundance of fertile and sterile pest insects. The control action is represented by u_2=$u_2(t)$, with an effectiveness factor of ε_2. The dynamics of natural enemies can be described as follows:

$$\frac{dE}{dt} = \frac{e_3 (b_F F + b_S S)}{1 + F + S} E - (d_E + \alpha_E E) E + \varepsilon_2 u_2 R_2(F,S) \tag{3.7}$$

In this equation, The first term on the right-hand side represents the growth of natural enemies due to food availability. The second term accounts for natural enemy mortality. The control action contributes through the last term, adjusting the population based on the release rate and its effectiveness.

We refer to equations (3.4)-(3.7) as the dynamical system, which comprises a set of equations describing the behavior of state variables. We assume that the system satisfies the following conditions;
Initial Conditions: The system starts with initial values,

$$P(0) = P_0, F(0) = F_0, S(0) = S_0, E(0) = E_0 \tag{3.8}$$

These initial values are all non-negative.

Terminal Time Conditions: At a fixed finite horizon $T > 0$ is a fixed finite horizon of control, we have terminal values;

$$P(T) = P_T, F(T) = F_T, S(T) = S_T, E(T) = E_T \tag{3.9}$$

These terminal values are free parameters.

Bounded Control Policies: We enforce the following bounded control policies:

$$0 \le u_1(t) \le \bar{u}_1 \tag{3.10}$$

$$0 \le u_2(t) \le \bar{u}_2 \tag{3.11}$$

for all $t \in [0,T]$ .The upper bounds $\bar{u}_1$ in (3.10) and $\bar{u}_2$ in (3.11) are determined based on the release rates $R_1(F)$ and $R_2(F,S)$.

Admissible Control Functions:The set of all admissible control functions is denoted by :

$$\mathbb{U} = \{u(t) \mid u(t) \in L^{\infty}(0,T), 0 \leq u(t) \leq \bar{u}\},$$

where $u(t) = (u_1(t), u_2(t))^{\top}$ and L^{∞} be the set of all Lebesgue integrable functions.

3.2.3 Release Rates

The effectiveness of pest control through augmentation is influenced by the number of released biological control agents. Striking the right balance is crucial because increasing the agent population can escalate control costs without consistently ensuring effective pest suppression in the wild environment . To explore the impact of natural enemy and sterile insect release rates on pest suppression mechanisms, we examine three distinct time-independent release strategies: constant, proportional, and saturating proportional release rates.

The constant control rates for the release of sterile insects and natural enemies, we set $R_1(F) = 1$ and $R_2(F,S) = 1$. In this scenario, The control variable $u_1(t)$ represents the number of sterile insects released at time t. The control variable $u_2(t)$ represents the number of natural enemies released at time t. The parameters $\bar{u}_1$ and $\bar{u}_2$ denote the maximum availability of these biological agents.

In the context of proportional control rates, we consider release rates that are directly proportional to the number of corresponding pest insects.The details for both sterile insects and natural enemies.

Sterile Insects:We set $R_1(F) = F$. Consequently, we define $u_1(t)$ as the proportion of sterile insects to be released relative to the number of fertile insects in the environment. The maximum availability of sterile insects is denoted by $\bar{u}_1$, which is set to 1.

Natural Enemies: Since natural enemies prey on both fertile and sterile insects, we define $R_2(F,S) = F + S$ The control variable $u_2(t)$ represents the share of natural enemies to be released relative to the total pest population. Again, the maximum availability of natural enemies is $\bar{u}_2$, set to 1.

The concept of saturating proportional rates involves maintaining proportional release rates for small pest populations but transitioning to saturation as the populations grow. The details for both sterile insects and natural enemies.
Sterile Insects: We set $R_1(F) = F/(1+F)$. The release rate is proportional to the pest population size, but it saturates as the population increases. The parameter $\bar{u}_1$ represents the maximum availability of sterile insects, similar to the constant release rate case.
Natural Enemies: For natural enemies, we define $R_2(F,S) = (F+S)/(1+F+S)$ .Again, the release rate is proportional to the total pest population, with saturation effects. The control variable $u_2(t)$ represents the share of natural enemies to be released. The upper bound $\bar{u}_2$ remains consistent with the constant release rate.
We summarize different release rate strategies for pest control in Table 3.1 :

Release Rate	Sterile Insect	Natural Enemy	Control Upper Bound
Constant	$R_1(F) = 1$	$R_2(F,S) = 1$	$\bar{u}_i \geq 1$
Proportional	$R_1(F) = F$	$R_2(F,S) = F + S$	$\tilde{u}_i = 1$
Saturating proportional	$R_1(F) = \frac{F}{1+F}$	$R_2(F,S) = \frac{F+S}{1+F+S}$	$\bar{u}_i \geq 1$

Table 3.1

Theorem 3.2.3.1. *Given model (3.4)-(3.7) with initial conditions (3.8) . The solutions to this model are positive and bounded. [1]*

Proof. Let define the total 'population' N as

$N(t) = P(t) + F(t) + S(t) + E(t)$,

Thus,

$\frac{dN}{dt} = \frac{dP}{dt} + \frac{dF}{dt} + \frac{dS}{dt} + \frac{dE}{dt}$.

Terms in the right-hand side are then substituted by those of model (3.4)-(3.7) we have ,

$$\frac{dN}{dt} = \left(r_p P\left(1-\frac{P}{k}\right) - \frac{a_1 P}{m+c_1 P}F - \frac{a_2 P}{m+c_2 P}S\right) + \frac{e_1 a_1 P}{m+c_1 P}\frac{r_h F}{1+F+S}F - \frac{b_F F}{1+F+S}E$$
$$-(d_F + \alpha_F F + \beta S)F + \left(\frac{e_2 a_2 P}{m+c_2 P}S - (d_S + \alpha_S S + \beta F)S - \frac{b_S S}{1+F+S}E + \varepsilon_1 u_1 R_1(F)\right)$$
$$+\left(\frac{e_3(b_F F + b_S S)}{1+F+S}E - (d_E + \alpha_E E)E + \varepsilon_2 u_2 R_2(F,S)\right)$$

Thus we may write

$$\frac{dN}{dt} \le r_p P\left(1-\frac{P}{k}\right) + \bar{r}F - (d_F + \alpha_F F)F - d_S S - d_E E + \bar{R},$$

where $\bar{R} = \sup_{t\in[0,T]}\{\varepsilon_1 u_1 R_1 + \varepsilon_2 u_2 R_2\}$ and $\bar{r} = \max\left\{\frac{e_1 a_1 P}{m+c_1 P}\frac{r_h F}{1+F+S}\right\}$ is the maximum saturation. To strengthen the expression, let us define the following quantities:

$$d = \min\{d_F, d_S, d_E\} = a,$$
$$\theta_1 = \frac{r_p}{4k}\left(k + \frac{dk}{r_p}\right)^2,$$
$$\theta_2 = \frac{\bar{r}^2}{4\alpha_F}.$$

Then we obtain

$$\frac{dN}{dt} \leq -aN - \frac{r_p}{k}\left(P - \frac{1}{2}\left(k + \frac{ak}{r_p}\right)\right)^2 + \theta_1 - \alpha_F\left(F - \frac{\bar{r}}{2\alpha_F}\right)^2 + \theta_2 + \bar{R}.$$

Since $\theta = \theta_1 + \theta_2 + \bar{R}$, we have

$$\frac{dN}{dt} + aN \leq \theta$$

Compare with the first order differential equation , $\frac{dy}{dx} + Py = Q$

$$\begin{aligned} P = a \quad Q &= \theta \\ \therefore IF &= e^{\int P dt} \\ &= e^{\int a dt} \\ &= e^{at} \end{aligned}$$

$\therefore N(IF) = \int Q.\text{IF}\, dt.$

$Ne^{at} \leq \int \theta \cdot e^{at} dt + C$

$$\begin{aligned} &= \frac{\theta e^{at}}{a} - \int 0 + C \\ &\leqslant \theta \frac{e^{at}}{a} + C \\ \Rightarrow Ne^{at} &\leqslant e^{at}\left[\frac{\theta}{a} + \frac{C}{e^{at}}\right] \\ N(t) &\leqslant \frac{\theta}{a} + Ce^{-at} \end{aligned}$$

At $t = 0$

$$\begin{aligned} N_0 &= \frac{\theta}{a} + C \\ \Rightarrow C &= N_0 - \frac{\theta}{a} \\ \therefore N(t) &\leqslant \frac{\theta}{a} + \left(N_0 - \frac{\theta}{a}\right) e^{-at} \end{aligned}$$

This differential inequality has the solution

$$0 \leq N(t) \leq \frac{\theta}{a} + \left(N_0 - \frac{\theta}{a}\right) e^{-at}$$

This solution is bounded by the steady-state value $\bar{N} = \frac{\theta}{a}$ as time t becomes infinite, meaning that the model is mathematically and ecologically well-posed with bounded state variables. The invariant region $\mathbb{S}$ is then given by

$$\mathbb{S} = \left\{ (P,F,S,E)^\top \in \mathbb{R}_+^4 \,\middle|\, 0 \leq P+F+S+E \leq \frac{\theta}{a} \right\}$$

This completes the proof. $\square$

3.3 Conclusion

We have introduced a straightforward analytical model for pest control, described by a system of ordinary nonlinear differential equations. This model captures the dynamic interactions between plant and pest populations. Within this framework, we incorporate two control strategies: the release of sterile pest insects (SIT) and the deployment of natural enemies as predators. Our model encompasses four distinct population classes: plants, fertile insects, sterile insects, and natural enemies. Importantly, it allows flexibility in adjusting the release rates of both sterile insects and natural enemies.

We see that by introducing sterile insects and natural enemies , it helps to reduce the number of fertile pest which inturn result in growth of the plant.

Chapter 4

CONCLUSION AND FUTURE SCOPE

4.1 Conclusion

The overall conclusion drawn from the research and modeling presented in the document is that a comprehensive approach combining natural enemies, targeted pesticide application, and monitoring techniques can effectively manage pest populations, increase crop production, and maintain ecological balance. By integrating natural enemies into pest control strategies , the study demonstrates a more sustainable and efficient method for pest management.

The importance of utilizing natural enemies as biological control agents to regulate pest populations and reduce the reliance on chemical pesticides. The significance of the use of natural enemy minimize impacts on populations and ecosystem health and enhance crop production. The role of mathematical modeling in understanding the dynamics of prey-predator systems and optimizing pest control strategies.

The overall conclusion for the topic of pest control using a combination of natural enemies and sterile insect releases, as discussed in the document, can be summarized as follows:
Effectiveness of Integrated Pest Management: The research highlights the effectiveness of integrated pest management strategies that combine the release of natural enemies and sterile insects to control pest populations in agricultural settings. By leveraging the complementary effects of natural enemies and sterile insects, the study demonstrates improved pest suppression and cost-effectiveness compared to individual control methods.
Optimal Control Strategies: The optimization model developed in the study provides insights into optimal control strategies for managing pest populations while minimizing control costs. By determining the optimal release rates of sterile insects and natural enemies, the model offers a systematic approach to achieving pest control objectives efficiently.

Stability and Feasibility: The analysis of the system dynamics within the invariant region ensures the stability and feasibility of the pest control model. By confining the state variables within bounded regions, the model maintains realistic and sustainable pest management practices.
Cost-Effectiveness and Sustainability: The study emphasizes the importance of considering both the population dynamics of pests and the economic costs of control measures in designing sustainable pest management strategies. The findings suggest that a balanced approach to releasing sterile insects and natural enemies can lead to cost-effective pest control outcomes.
Practical Implications: The research outcomes have practical implications for agricultural practices, pest control programs, and decision-making processes in pest management. The identified optimal control strategies can guide practitioners and policymakers in implementing effective and sustainable pest control measures.

The research contributes to the advancement of integrated pest management approaches by offering a quantitative framework for optimizing control strategies using natural enemies and sterile insect releases. The study underscores the importance of interdisciplinary research in addressing pest control challenges and promoting sustainable agricultural practices.

4.2 Future Scope

In our exploration of prey–predator mathematical models, we aim to enhance ecological systems that describe the dynamics of crop and pest interactions in agriculture. Our focus extends to increasing crop density while simultaneously reducing pest populations. Here are the key points and future directions: High-Resolution Monitoring: Modern agriculture demands precise and effective monitoring techniques. We recognize the need for high-resolution data collection to inform pest management strategies. Integrated Pest Control: Effective pest control is essential for crop health and productivity. Our research emphasizes a balanced approach that combines natural enemies and other control measures.

Intelligent Airborne Platforms: We explore mathematical modeling and analysis for an intelligent airborne platform. This platform aims to monitor crops efficiently, detect pest damage, and optimize interventions. Challenges and Opportunities: Detecting pest damage remains a challenge for many countries. By formulating real-world problems mathematically, we can devise innovative solutions. Rigorous scientific definitions, concepts, and assumptions underpin our reasonings.

The global community faces critical environmental issues, including floods, climate change, rising sea levels, and extreme weather events. To tackle these challenges effectively, future research requires collaboration across disciplines. Here are key considerations and opportunities: Multidisciplinary Approach: Researchers, environmental modelers, and computational experts must work together. Building reliable mathematical models that align with real-world observations is essential. These models serve as valuable tools for decision-makers and farmers. Optimal Control Techniques: Understanding system parameters is crucial for applying optimal control strategies. By fine-tuning parameters, we can enhance the performance of future control systems.

Emerging Challenges: Realistic systems should incorporate diverse requirements, reflecting the complexity of agricultural ecosystems. Investigating mutual interference in predator–prey systems is an open challenge. Proper monitoring and understanding interference effects are critical for sustainable pest management. Holistic Perspectives: Researchers will provide comprehensive insights into farming system productivity, ecological balance, and economic sustainability. Flexible management practices and development strategies will drive truly sustainable agriculture.

Future Research Directions:

The study opens up avenues for further research in optimizing pest control strategies, refining control models, and exploring additional factors that influence pest dynamics and control outcomes. Future studies could focus on validating the model predictions through field experiments and expanding the analysis to different pest species and agricultural systems.

Bibliography

[1] Toni Bakhtiar et al. “Mathematical model of pest control using different release rates of sterile insects and natural enemies”. In: *Mathematics* 10.6 (2022), p. 883.

[2] R Bhattacharyya and B Mukhopadhyay. “Mathematical study of a pest control model incorporating sterile insect technique”. In: *Natural Resource Modeling* 27.1 (2014), pp. 61–79.

[3] Tyler Blaszak and W Hu. “Lotka-volterra models of predator-prey relationships”. In: *web. mst. edu, Retrieved* (2019), pp. 09–30.

[4] RM Colombo, Elena Rossi, et al. “A modeling framework for biological pest control”. In: *Mathematical Biosciences and Engineering* 17.2 (2020), pp. 1413–1427.

[5] David W Crowder. “Impact of release rates on the effectiveness of augmentative biological control agents”. In: *Journal of Insect Science* 7.1 (2007), p. 15.

[6] JHP Dawes and MO3046076 Souza. “A derivation of Holling’s type I, II and III functional responses in predator–prey systems”. In: *Journal of theoretical biology* 327 (2013), pp. 11–22.

[7] Dibyendu Sekhar Mandal et al. “A mathematical study of a crop-pest–natural enemy model with Z-type control”. In: *Mathematics and Computers in Simulation* 187 (2021), pp. 468–488.

[8] Dibyendu Sekhar Mandal et al. “Study of a predator–prey model with pest management perspective”. In: *Journal of Biological Systems* 27.03 (2019), pp. 309–336.

[9] University of North Carolina Wilmington. *The Logistic Equation*. [Internet]. Available from: [1](https://math.libretexts.org/@go/page/91088) [cited 2024 Apr 23]. 2022.

[10] Xia Wang, Yuan Tian, Sanyi Tang, et al. “A Holling type II pest and natural enemy model with density dependent IPM strategy”. In: *Mathematical Problems in Engineering* 2017 (2017).

[11] Wikipedia contributors. *Biological pest control — Wikipedia, The Free Encyclopedia*. [Online; accessed 23-April-2024]. 2024. URL: https://en.wikipedia.org/w/index.php?title=Biological_pest_control&oldid=1215695508.

[12] Sudhakar Yadav. “Mathematical Modeling and Optimal Control to Biological Models”. PhD thesis. 2022.

[13] Jing Yang, Xiaorong Hou, and Yajun Li. “A generalization of Routh–Hurwitz stability criterion for fractional-order systems with order $\alpha(1, 2)$”. In: *Fractal and Fractional* 6.10 (2022), p. 557.