First Order Delay Differential Equations

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, " First Order Delay Differential Equations" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Mridini Gawas and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " First Order Delay Differential Equations." This report is divided into four chapters.

FIRST CHAPTER :

The first chapter introduces the concept of delay differential equation and the analytical solution of DDE's have been discussed.

SECOND CHAPTER :

The second chapter deals with the simple cases of DDE, existence and uniqueness of solutions and some general equations have been analyzed.

THIRD CHAPTER :

In the third chapter we have discused the linear system with delay.

FOURTH CHAPTER :

At the end, the fourth chapter deals with a discussion of the delayed logistic equation.

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Notations and Abbreviations

DDEs:	Delay Differential Equations
LDDE:	Linear Delay Differential Equation
DE:	Differential Equation
ODE:	Ordinary Differential Equation
FDE:	Functional Differential Equation
PDE:	Partial Differential Equation
BVP:	Boundary Value Problem
Non-LDDEs:	Nonlinear Delay Differential Equations
SDDEs:	Stochastic Delay Differential Equations
NDDEs:	Neutral Delay Differential Equations

Chapter 1

INTRODUCTION

This chapter introduces the concept of differential equation with delay, called as delay differential equations(DDEs). These differential equation depend on past history, and are therefore used in many models because they are more realistic than models independent of past history.

Time delays are present in so many natural and man-made processes (biological, medical, chemical, physical, engineering, economic, etc.) that ignoring them is tantamount to ignoring reality. For example, consider reforestation. After re-planting a cut forest, it will take a minimum of 20 years to reach any type of maturity. For some species (redwoods for example), it will take much longer. Therefore, it is clear that any mathematical model for forest harvesting and regeneration must include time delays. Another example is that animals need time to digest food before they can engage in other activities and responses. Therefore, any model for species dynamics that does not include time delays is at best an approximation.

For example, in many applications, the system is assumed to be governed by the principle of cause and effect, i.e., the future status of the system is not dependent on the past but is determined by the present. However, it is important to remember that this is just a first approximation of the real situation. A more accurate model will need to include some of the system's past history. When a model does not include a dependence on the past history, it is usually composed of ODEs or PDEs. Models that include past history typically include DDEs or FDEs.

1.1 Definition

A delay differential equation is a differential equation where the time derivatives at the current time depend on the solution and its derivative at previous times.

For single delay, it is represented as

$$x'(t) = f(t, x(t), x(t-\tau)) \tag{1.1}$$

At time t , the evolution of system depends on the current time t, current state $x(t)$ of system and at some time $\tau > 0$ in the past and is called as delay.

- If τ is constant then above DDE is a constant DDE.
- If $\tau = \tau(t)$ depends on time t then above DDE is a time dependent DDE.
- If $\tau = \tau(x(t))$ depends on state then above DDE is state dependent.

For multiple delays, we have

$$x'(t) = f(t, x(t), x(t-\tau_1(t, x(t))), x(t-\tau_2(t, x(t))), \ldots) \tag{1.2}$$

For $t \geq 0$ and $\tau_i > 0$, are the delays, τ_i , i = 1, 2, 3, ...
In DDEs the derivative at any time depend on the solution at previous times, more generally that is $\tau_i = \tau_i(t, x(t))$.

Some examples,

$$x'(t) = -2x(t-1) \tag{1.3}$$

$$x'(t) = x(t) - x\left(\frac{t}{2}\right) + x'(t-1) \tag{1.4}$$

$$x'(t) = x(t)x(t-1) + t^2 x(t+2) \tag{1.5}$$

$$x''(t) = -x'(t) - x'(t-1) - 3\sin x(t) + \cos t \tag{1.6}$$

DDE model depends on the initial function to determine a unique solution, because $x'(t)$ depends on the solution at prior times. Then it is necessary to supply an initial auxilliary function called the "history" function, the auxilliary function in many models is constant, τ : $\max \tau_i$.

In ordinary differential equation, the derivative of the unknown function is related to the function's value at the same instant. Solutions are determined by initial conditions at a single point in time. For a first-order ordinary differential equation (ODE), consider a initial value problem of the form :

$$x'(t) = f(t, x(t)), \quad \text{where } t \geq t_0 \tag{1.7}$$

with initial value at initial time point

$$x(t_0) = x_0 \tag{1.8}$$

They are used to model systems where the present state directly determines the future state.This implies that ODEs cannot directly account for incubation,life history, duration of events and many more.

In delay differential equation, the derivative of the unknown function at a certain time depends on the function's values at earlier times. Solutions require knowledge of the function over an interval of time, known as the initial history function. For a first-order delay differential equation (DDE), consider a initial value problem of the form :

$$x'(t) = f(t, x(t), x(t-\tau)), \quad \text{where } t \geq t_0, \tag{1.9}$$

we must provide initial data $x(t) = \phi(t)$ on the interval $[t_0 - \tau, t_0]$ for all $t \in [t_0 - \tau, t_0]$. The function $\phi(t)$ is known as the history function or initial function. They model systems where the current rate of change is influenced by past states, such as biological systems with gestation delays or engineering systems with time lags in feedback loops.

1.2 Classification of Delay Differential Equations (DDEs)

Delay differential equations can be classified as :-

- Linear delay differential equations (LDDEs).
- Nonlinear delay differential equations (Non-LDDEs).
- Stochastic delay differential equations (SDDEs).
- Neutral delay differential equations (NDDEs).
- Autonomous delay differential equations (never changing under the change of time).

- Non-autonomous delay differential equations.

1.2.1 Linear Delay Differential Equations (LDDEs):

LDDEs are differential equations where the dependent variable and its derivatives appear linearly, and the equation involves delays in the form of past values of the dependent variable. An example is:

$$x'(t) = ax(t) + bx(t-\tau)$$

where a and b are constants, and τ is the delay.

1.2.2 Nonlinear Delay Differential Equations (Non-LDDEs):

Non-LDDEs are differential equations where the dependent variable or its derivatives appear nonlinearly. They can involve delays and nonlinearity in various forms, making their analysis more complex. An example is:

$$x'(t) = ax(t) + bx(t-\tau)^2$$

1.2.3 Stochastic Delay Differential Equations (SDDEs):

SDDEs are differential equations that incorporate random or stochastic components, making their solutions probabilistic. They often arise in modeling systems subject to random fluctuations or noise. An example is:

$$dx(t) = [ax(t) + bx(t-\tau)]dt + \sigma dW(t)$$

where $dW(t)$ represents a Wiener process.

1.2.4 Neutral Delay Differential Equations (NDDEs):

NDDEs are differential equations where delays appear both in the state variables and their derivatives. These equations often arise in systems with delayed feedback or in control systems. An example is:

$$x'(t) = ax(t) + bx'(t-\tau)$$

1.2.5 Autonomous Delay Differential Equations:

Autonomous DDEs are differential equations where the coefficients and delay do not explicitly depend on time. These equations describe systems whose behavior remains unchanged under time translations. An example is:

$$x'(t) = f(x(t), x(t-\tau))$$

1.2.6 Non-autonomous Delay Differential Equations:

Non-autonomous DDEs are differential equations where the coefficients or the delay explicitly depend on time. These equations describe systems whose behavior changes over time due to external influences or time-varying parameters. An example is:

$$x'(t) = f(t, x(t), x(t-\tau))$$

1.3 Analytical solutions of DDEs

Real life problems with delay are generally too complex for analytical solutions. However, for some delay differential equation such as linear first order delay differential equations with single constant delay and constant coefficients ' the method of steps ' is used to find the analytical solutions.

1.3.1 The Method of Steps

We begin with constant coefficient delay differential equation, defined for $t > t_0$ and initial function on interval $[t_0 - \tau, t_0]$ where τ is the delay. We are looking for a continuous extension into the future. We first consider the interval $[t_0, t_0 + \tau]$ on which the DDE reduces to an ODE. We find a solution valid on this interval and then use this solution as the initial function for the interval $[t_0 + \tau, t_0 + 2\tau]$. Then we find a solution on $[t_0 + \tau, t_0 + 2\tau]$, and this way the solution is extended forward from interval to interval. Continuing in this way yields a solution of ODEs valid on $[t_0 - \tau, \infty)$ which becomes smoother in time t as t increases. At each step in the process, we are solving an ODE for which, under the hypothesis of the uniform Lipschitz continuity of the right hand side of equations, a unique solution is generated.
This method is explain with the help of the following example:
Consider DDE of the form $x'(t) = x(t-1)$ and history function $x(t) = 1$, $\forall t \in [-1, 0]$.
Solution as follows:
The basic idea behind the method of steps is to transform the DDE model to a sequence of a finite number of ODEs through dividing the domain of the DDE into sub-domains, in each of which the DDE is transformed into ODE. Then to solve these ODEs starting from the first sub-domain using the given history function. The solution in the first domain is used as a history for the next sub-domain, and this process is repeated until the domain

of the DDE is covered.

The above equation on interval $[0,1]$ can be written as the non-autonomous ODE as $x'(t) = f(x, x(t))$ with $f(t, x(t)) = \phi_0(t-1)$ where $\phi_0(t) = 1$, $t \in [-1, 0]$.

Now by making use of Integral form of the solutions,

we have $\forall t \in [0,1]$.

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds \tag{1.10}$$

$$= x(0) + \int_0^t \phi_0(s-1)\, ds \tag{1.11}$$

$$= x(0) + \int_0^t 1\, ds \tag{1.12}$$

$$= x(0) + t \tag{1.13}$$

$$= 1 + t \tag{1.14}$$

as $x(t) = 1, \forall t \in [-1, 0] \quad \text{so } x(0) = 1$

Proceeding as before , we can write given DDE on the interval [1,2] as the non-autonomous ODE as

$$x'(t) = f(x, x(t)) \quad \text{with} \quad f(t, x(t)) = \phi_1(t-1)$$

where $\phi_1(t)$ is defined on $[0,1]$ as $x(t) = 1 + t$.

$\therefore$ the given DDE can be written as

$$x'(t) = \phi_1(t-1), \quad t \in [1,2]$$

where $\phi_1(t) = 1 + t$ for $t \in [0, 1]$.

By making use of the integral form of the solution for $t \in [1, 2]$, we have:

$$x(t) = x(1) + \int_1^t f(s, x(s))\, ds \tag{1.15}$$
$$= x(1) + \int_0^t \phi_1(s-1)\, ds \tag{1.16}$$
$$= x(1) + \int_0^t \phi_1(s)\, ds \tag{1.17}$$

From the solution $x(t) = 1 + t$ as found earlier, $x(1) = 2$.

Therefore, for $t \in [1, 2]$, we have:

$$x(t) = 2 + \int_0^t (1+s)\, ds \tag{1.18}$$
$$= 2 + t + \frac{t^2}{2} \tag{1.19}$$

Continuing as previously, for $t \in [2, 3]$, we have:

$$x(t) = x(2) + \int_2^t \phi_2(s-1)\, ds \tag{1.20}$$

Here the history function is

$\phi_2(t) = 2 + t + \frac{t^2}{2}$ for $t \in [1, 2]$ and $x(2) = 6$.

So, for $t \in [2, 3]$ we have

$x(t) = \frac{10}{3} + 2t + \frac{t^2}{2} + \frac{t^3}{6}$

We can continue further. This will give us the general form of the solution.

Chapter 2

SIMPLE CASES

2.1 Simplest DDE

2.1.1 Example

The simplest example of a DDE is given by

$$x'(t) = -x(t-\tau) \tag{2.1}$$

for t ≥ 0, where $\tau > 0$ is called the delay.

Suppose the initial condition for (2.1) is given by

$$x(t) = 1 \tag{2.2}$$

for t $\in [-\tau, 0]$. Following the procedure called the method of steps, the solution $x(t)$ for $t \in [(n-1)\tau, n\tau], n \in \mathbb{N}$, can be determined in the following way.

For $t \in [0, \tau]$, it follows that $t - \tau \in [-\tau, 0]$. Therefore,

$$x'(t) = -x(t-\tau) = -1. \tag{2.3}$$

From this, we can conclude that

$$x(t) = x(0) + \int_0^t (-1)\, ds = 1 - t, \quad t \in [0, \tau]. \tag{2.4}$$

Similarly, we can show that

$$x'(t) = -x(t-\tau) = -[1-(t-\tau)], \quad t \in [\tau, 2\tau]. \tag{2.5}$$

Therefore,

$$\begin{aligned} x(t) &= x(\tau) + \int_\tau^t -[1-(s-\tau)]\, ds && (2.6)\\ &= 1 - \tau + \left[-s + \frac{1}{2}(s-\tau)^2\right]\Bigg|_{s=\tau}^{s=t} && (2.7)\\ &= 1 - t + \frac{1}{2}(t-\tau)^2, \quad t \in [\tau, 2\tau]. && (2.8) \end{aligned}$$

It can be generalised in the form

$$x(t) = 1 + \sum_{k=1}^{n} \frac{(-1)^k [t-(k-1)\tau]^k}{k!}, \quad t \in [(n-1)\tau, n\tau], \quad n \in \mathbb{N}. \tag{2.9}$$

2.2 Existence and Uniqueness

Let $\tau \geq 0$ be a constant in $J = [\xi, \xi + a]$, where $\xi \geq 0$, and $a > 0$. The equation

$$x'(t) = f(t, x(t-\tau)) \quad \text{for} \quad t \in J \tag{2.10}$$

is called a delay differential equation, where $\tau > 0$ is called the delay. An initial condition for (2.10) is given by

$$x(t) = \varphi(t) \quad \text{for} \quad t \in J^- = [\xi - \tau, \xi], \tag{2.11}$$

where φ is a given continuous function.

Theorem 2.1: We consider the initial value problem (2.10), (2.11), where f is continuous in the strip $S = J \times \mathbb{R}$, φ is continuous in J^-, and $\tau > 0$ is a constant in J. Then, there exists exactly one solution.

Proof: Let

$$x_n'(t) = f(t, x_n(t-\tau)) \quad \text{for} \quad t \in [\xi + (n-1)\tau, \xi + n\tau], \tag{2.12}$$

where $n \in \mathbb{N}$.

For $t \in [\xi, \xi + \tau]$, it follows that $t - \tau \in [\xi - \tau, \xi]$. Therefore,

$$x_1'(t) = f(t, x_1(t-\tau)) = f(t, \varphi(t-\tau)), \tag{2.13}$$

and

$$x_1(\xi) = \varphi(\xi). \tag{2.14}$$

From this, we can conclude that

$$x_1(t) = x_1(\xi) + \int_\xi^t f(s, x_1(s-\tau))\,ds = \varphi(\xi) + \int_\xi^t f(s, \varphi(s-\tau))\,ds. \tag{2.15}$$

Hence, x_1 is uniquely defined for $t \in [\xi, \xi + \tau]$. Similarly, we can show that

$$x_2'(t) = f(t, x_2(t-\tau)) = f(t, x_1(t-\tau)), \quad t \in [\xi + \tau, \xi + 2\tau], \tag{2.16}$$

and

$$x_2(\xi+\tau) = x_1(\xi+\tau). \tag{2.17}$$

Therefore,

$$x_2(t) = x_2(\xi+\tau) + \int_{\xi+\tau}^{t} f(s, x_2(s-\tau))\,ds = x_1(\xi+\tau) + \int_{\xi+\tau}^{t} f(s, x_1(s-\tau))\,ds. \tag{2.18}$$

So x_2 is uniquely defined for $t \in [\xi+\tau, \xi+2\tau]$. We can conclude that x_n is defined by x_{n-1} for all $n \in \mathbb{N}$. Therefore, x_n is uniquely defined for $t \in [\xi+(n-1)\tau, \xi+n\tau]$, $n \in \mathbb{N}$. From this, we can conclude that x, defined by

$$x(t) = \begin{cases} \varphi(t) & \text{for } t \in J^- = [\xi-\tau, \xi], \\ \varphi(\xi) + \int_{\xi}^{t} f(s, x(s-\tau))\,ds & \text{for } t \in J = [\xi, \xi+a] \end{cases} \tag{2.19}$$

is well-defined for all values $t \in J^- \cup J = [\xi-\tau, \xi+a]$. Furthermore, $x(t)$ is a solution of the initial value problem (2.10), (2.11). From the definition of $x(t)$, we can conclude that $x(t)$ is the only solution.

2.3 General Equation

The general equation of the simplest DDE is given by

$$x'(t) = -\alpha x(t-\tau), \tag{2.20}$$

where α is a constant, and $\tau > 0$ is the delay. The case $\alpha > 0$ corresponds to negative feedback, and the case $\alpha < 0$ corresponds to positive feedback.

By scaling this DDE, the number of parameters can be reduced, which makes solving easier. We will determine a DDE with $U(\mu) = x(t)$, where the delay is the constant 1. For the scaling, we set $\mu := \eta t$, $\eta > 0$. Then,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dU}{d\mu} \cdot \frac{d\mu}{dt} \\
\because \frac{d\mu}{dt} &= \eta \\
\frac{dx}{dt} &= \frac{dU}{d\mu} \cdot \eta \\
\therefore \frac{dU}{d\mu} &= \frac{1}{\eta} \frac{dx}{dt} \\
\frac{dU}{d\mu} &= \frac{1}{\eta}(-\alpha)x(t-\tau) \\
\frac{dU}{d\mu} &= (-\alpha)\eta^{-1}x(t-\tau) \\
\frac{dU}{d\mu} &= (-\alpha)\eta^{-1}U(\eta t - \eta\tau)
\end{aligned}$$

$$\frac{dU}{d\mu} = (-\alpha)\eta^{-1}U(\mu - \eta\tau) \tag{2.21}$$

If we set $\eta := 1/\tau$ and $\beta := \alpha\tau$, then

$$\frac{dU}{d\mu} = -\beta U(\mu - 1). \tag{2.22}$$

We introduce the following linear operator, defined on the differentiable functions:

$$L(U) = \frac{dU}{d\mu} + \beta U(\mu - 1). \tag{2.23}$$

We seek (complex) values of λ such that $U(\mu) = e^{\lambda\mu}$ is a solution of (2.22). Filling this expression in for $L(U)$, we get

$$L(e^{\lambda\mu}) = \lambda e^{\lambda\mu} + \beta e^{\lambda(\mu-1)} = e^{\lambda\mu}(\lambda + \beta e^{-\lambda}). \tag{2.24}$$

We want to solve the characteristic equation

$$h(\lambda) \equiv \lambda + \beta e^{-\lambda} = 0. \tag{2.25}$$

Then, $L(e^{\lambda\mu})$ is the zero function, hence $U(\mu) = e^{\lambda\mu}$ is a solution of (2.22).

To solve (2.25), we set $\lambda := x + iy$. Considering real and imaginary parts, we get the system

$$\begin{aligned} x &= -\beta e^{-x}\cos(y), \\ y &= \beta e^{-x}\sin(y). \end{aligned} \tag{2.26}$$

We call $\lambda \in \mathbb{C}$ a root of (2.22) of order l, where $l \geq 1$, if

$$h(\lambda) = h'(\lambda) = h''(\lambda) = \ldots = h^{(l-1)}(\lambda) = 0, \quad h^{(l)}(\lambda) \neq 0.$$

Lemma : The following hold.

1. If $\beta < 0$, then there is exactly one real root and it is positive.
2. If $0 < \beta < 1/e$, then there are exactly two real roots $x_1 < x_2$, both negative. $x_1 \to -\infty$ and $x_2 \to 0$ as $\beta \to 0$.

1. If $\beta = \frac{1}{e}$, then there is a single real root of order two, namely $\lambda = -1$.
2. If $\beta > \frac{1}{e}$, then there are no real roots.

Proof.

1. For (1): h is an increasing function that crosses the x-axis, hence there is exactly one real root. Because $\beta < 0$, it follows that $\lambda = -\beta e^{-\lambda} > 0$.

2. For (2): h has a minimum below the x-axis, hence there are exactly two real roots. Because $\beta > 0$, it follows that $\lambda = -\beta e^{-\lambda} < 0$. The last assertion follows from the following:
   $\beta = -\lambda e^{\lambda} \to 0$ if and only if $\lambda \to 0$ or $\lambda \to -\infty$.

3. For (3): h has a minimum on the x-axis, hence there is exactly one real root. Filling $\beta = \frac{1}{e}$ in for h gives the root $\lambda = -1$. Furthermore, $h'(-1) = 0$ and $h''(-1) = 1$, hence the order of $\lambda = -1$ is two.

4. For (4): h has a minimum above the x-axis.

Corollary: The following hold for (2.20).

1. If $\alpha < 0$, then $x = 0$ is unstable.

2. If $0 < \alpha\tau < \pi/2$, then $x = 0$ is asymptotically stable.

3. If $\alpha\tau = \pi/2$, then $x = \sin(\pi\mu/2)$ and $x = \cos(\pi\mu/2)$ are solutions.

4. If $\alpha\tau > \pi/2$, then $x = 0$ is unstable.

Theorem: For every real α and $\tau > 0$, the following are equivalent.

1. Every solution of (2.20) is oscillatory.

2. $\alpha\tau > 1/e$.

Proof: For every real λ, $U(\mu) = e^{\lambda\mu}$ is either a monotonic or constant function.

Therefore, $x(t) = U(\mu)$ is not oscillatory for real λ.

So $x(t)$ is oscillatory if and only if λ is a complex (not real) root.

By Lemma , λ is a complex (not real) root if and only if $\beta = \alpha\tau > 1/e$.

This proves the theorem.

2.4 Scalar DDE

2.4.1 General Equation

The homogeneous equation of the scalar DDE is given by:

$$x'(t) = ax(t) + bx(t-\tau), \tag{2.27}$$

where a and b are constants, and $\tau > 0$ is the delay. The nonhomogeneous equation will be discussed in the next chapter.

We seek a nontrivial solution of (2.27) of the form:

$$x(t) = e^{\lambda t}c, \quad c \neq 0, \tag{2.28}$$

where λ is complex and c is a constant.

The characteristic equation we want to solve is given by:

$$h(\lambda) \equiv \lambda - a - be^{-\lambda\tau} = 0. \tag{2.29}$$

This expression can be simplified by multiplying the right-hand side by τ and setting:

$$z := \lambda\tau, \quad \alpha := a\tau, \quad \beta := b\tau.$$

We get:

$$h(z) \equiv z - \alpha - \beta e^{-z} = 0. \tag{2.30}$$

Chapter 3

LINEAR SYSTEM

3.1 Preliminaries:

A DDE with a single delay is of the form

$$x'(t) = f(t, x(t), x(t-\tau)) \quad \text{for} \quad t \geq \xi, \tag{3.1}$$

where f is a given continuous function, $\xi \geq 0$ is a constant, and $\tau > 0$ is the delay. An initial condition for (3.1) is given by

$$x(t) = \phi(t) \quad \text{for} \quad t \in [\xi - \tau, \xi], \tag{3.2}$$

where ϕ is a given continuous function.

We are interested in determining the state of the system(3.1),(3.2) at time $t \geq \xi \geq 0$. The state of a system at time $t \geq 0$ includes all information needed to determine the state of the system at future times $s \geq t$. Therefore, the state of the system (3.1), (3.2) at time

$t \geq \xi$ includes $x(\eta)$ for all $\eta \in [t-\tau, t]$. Hence, we conclude that this state, which we denote by x_t, is given by

$$x_t(\theta) := x(t+\theta) \text{ for } -\tau \leq \theta \leq 0. \tag{3.3}$$

In this chapter, we will look at linear systems with delay. We will discuss linear DDEs of the form

$$x'(t) = L(x_t) \quad \text{ for } \quad t \geq \xi, \tag{3.2}$$

where $\xi \geq 0$ is a constant, x_t is defined as (3.3), and L is the map $C \to \mathbb{C}^n$ where $C = C([-\tau, 0], \mathbb{C}^n)$. An initial condition for (3.4) is of the form

$$x_\xi = \phi, \tag{3.3}$$

where x_ξ is defined as (3.3), and ϕ is a given continuous function in C.

The map L is linear if it satisfies

$$L(a\phi + b\psi) = aL(\phi) + bL(\psi), \quad \phi, \psi \in C, \quad a, b \in \mathbb{C}. \tag{3.4}$$

In the next section, an example of the following linear DDE will be discussed:

$$x'(t) = Ax(t) + Bx(t-\tau), \tag{3.5}$$

where A, B are $n \times n$ matrices, and $\tau > 0$ is the delay. This equation can be rewritten in the form of (3.4) by defining the map L as

$$L(y) = Ay(0) + By(-\tau), \tag{3.6}$$

because then, using (3.3),

$$L(x_t) = Ax_t(0) + Bx_t(-\tau) = Ax(t) + Bx(t-\tau), \tag{3.7}$$

which is precisely (3.6).

We introduce the Laplace transform:

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt, \tag{3.8}$$

where $f(t)$ is a function on $[0,\infty)$. The domain of $F(s)$ consists of all the values for which the integral in (3.7) exists. The Laplace transform of f is denoted by both F and $L\{f\}$. We let $f(t)$ be an exponentially bounded function of order α; that is,

$$|f(t)| \le Me^{\alpha t} \quad \text{for all} \quad t \ge T$$

for some positive constants M, T. Then, if $f(t)$ is also piecewise continuous on $[0,\infty)$, then $L\{f\}(s)$ exists for $s > \alpha$.

The key property of the Laplace transform that we exploit is that the transform of a convolution is the product of the transforms. If $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$, then their convolution, denoted by $f * g$, is defined by

$$(f*g)(t) := \int_0^t f(s)g(t-s)\, ds = \int_0^t f(t-s)g(s)\, ds. \tag{3.8}$$

The Laplace transform satisfies

$$L\{f*g\}(s) = F(s)G(s),$$

where $F = L\{f\}$ and $G = L\{g\}$.

3.2 Linear DDE

3.2.1 General Equation

The general equation of the homogeneous linear DDE is given by

$$x'(t) = Ax(t) + Bx(t-\tau) \quad \text{for } t \geq 0, \tag{3.9}$$

where A and B are $n \times n$ matrices

and $\tau > 0$ is the delay.

The initial condition for (3.9) is given by

$$\varphi(t) := x(t) \quad \text{for } t \in [-\tau, 0]. \tag{3.10}$$

Example:

We are given a homogeneous linear DDE:

$$x'(t) = Ax(t) + Bx(t-\tau) \quad \text{for } t \geq 0, \tag{3.11}$$

where $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

and $\tau > 0$ is the delay. The initial condition for (3.11) is given by

$$\varphi(t) := x(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for } t \in [-\tau, 0]. \tag{3.12}$$

Solution:

We need to determine the eigenvalues and the corresponding eigenvectors of matrix A.

The characteristic equation

$$h(\lambda) \equiv \det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = 0 \tag{3.13}$$

Therefore, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$.

To find eigenvectors

Consider for $\lambda_1 = 1$, we have

$$(A - \lambda_1 I)u = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} u = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Therefore, eigenvector corresponding to λ_1 is $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now consider for $\lambda_2 = 2$, we have

$$(A - \lambda_2 I)v = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Therefore, eigenvector corresponding to λ_2 is $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Therefore, the solution of $x'(t) = Ax(t)$ is

$$x_h(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{3.14}$$

where c_1, c_2 are constants.

For $t \in [0, \tau]$,

$$x'(t) = Ax(t) + B\varphi(t-\tau) \tag{3.15}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.16}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.17}$$

A particular solution of (3.17) is of the form

$$x_p(t) = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \tag{3.18}$$

where d_1, d_2 are constants.

Substitute $x_p(t)$ for $x(t)$ in (3.17) gives

$$x_p'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} d_2 + 1 \\ -d_1 + 3d_2 + 1 \end{pmatrix}. \tag{3.19}$$

Therefore, $d_1 = -2$ and $d_2 = -1$.

The particular solution of (3.17) is

$$x_p(t) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.20}$$

Hence, the general solution of (3.17) is of the form

$$x(t) = x_h(t) + x_p(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.21}$$

where c_1, c_2 are constants. Now to find the values of c_1 and c_2

Since $x(0) = \varphi(0)$ and $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The eqn (3.17) will be

$$x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 - 2 \\ c_1 + 2c_2 - 1 \end{pmatrix}. \tag{3.22}$$

$$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 - 2 \\ c_1 + 2c_2 - 1 \end{pmatrix} \tag{3.23}$$

Therefore, $c_1 = 4$ and $c_2 = -1$.

and the general solution of (3.11), (3.12), for $t \in [0, \tau]$, is

$x(t) = x_h(t) + x_p(t) = 4e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

In a similar way, general solutions of the initial value problem (3.11), (3.12) defined on other intervals of the form $[(n-1)\tau, n\tau]$, $n \in \mathbb{N}$, can be determined.

3.2.2 Laplace Transform

The general equation of the non-homogeneous linear DDE is given by

$$x'(t) = Ax(t) + Bx(t - \tau) + f(t) \quad \text{for } t \geq 0, \tag{3.24}$$

where $x(t)$ is an $n \times 1$ column vector function representing the state of the system,

A and B are $n \times n$ matrices,

f is a given continuous function.

and $\tau > 0$ is the delay.

The initial condition for (3.24) is given by

$$x(t) = \varphi(t) \quad \text{for } t \in [-\tau, 0]. \tag{3.25}$$

Applying the Laplace transform to the left hand side of (3.24), we get

$$\begin{aligned}
\mathscr{L}\{x'\}(s) &= \int_0^\infty e^{-st}x'(t)dt && (3.26)\\
&= [e^{-st}x(t)]\Big|_{t=\infty}^{t=0} + s\int_0^\infty e^{-st}x(t)dt && (3.27)\\
&= sX(s) - \varphi(0) && (3.28)
\end{aligned}$$

Applying the Laplace transform to the right-hand side of (3.24),we get

$$\begin{aligned}
sX(s) - \varphi(0) &= \int_0^\infty e^{-st}[Ax(t) + Bx(t-\tau) + f(t)]dt && (3.29)\\
&= A\int_0^\infty e^{-st}x(t)dt + B\int_0^\infty e^{-st}x(t-\tau)dt + \int_0^\infty e^{-st}f(t)dt && (3.30)\\
&= AX(s) + B\left[\int_0^\tau e^{-st}\varphi(t-\tau)dt + \int_\tau^\infty e^{-st}x(t-\tau)dt\right] + F(s) && (3.31)\\
&= AX(s) + B\left[\int_0^\tau e^{-st}\varphi(t-\tau)dt + \int_0^\infty e^{-s(t+\tau)}x(t)dt\right] + F(s) && (3.32)\\
&= [A + e^{-s\tau}B]X(s) + B\Phi(s) + F(s) && (3.33)
\end{aligned}$$

where $\Phi = \mathscr{L}\{\phi(\cdot - \tau)\}$

Rearranging the terms, we have

X(s) = K(s)$[\phi(0) + B\Phi(s) + F(s)]$,

where

$K(s) = (sI - A - e^{-s\tau}B)^{-1}$ is a matrix-valued function.

In order to make use of the convolution result, we need to know the inverse transform k of K. In view of the calculations above, we see that k is a solution of (3.24), (3.25), with

$f = 0$, for the initial data

$$\xi(\theta) = \begin{cases} I & \text{for } \theta = 0, \\ 0 & \text{for } \theta \in [-\tau, 0). \end{cases} \tag{3.34}$$

In spite of the discontinuity of ξ at zero, the method of steps readily establishes that the solution k exists for $t \geq 0$. The matrix function k is called the fundamental matrix solution of (3.24),(3.25).

Lemma: We may express the solution of the initial value problem (3.24), (3.25) as

$$x(t) := x(t;\phi,f) = x(t;\phi,0) + x(t;0,f). \tag{3.35}$$

Proof. For $t \in [0,\tau]$, the following hold:

$$x'(t;\phi,f) = Ax(t;\phi,f) + B\phi(t-\tau) + f(t); \tag{3.36}$$

$$x'(t;\phi,0) = Ax(t;\phi,0) + B\phi(t-\tau); \tag{3.37}$$

$$x'(t;0,f) = Ax(t;0,f) + f(t). \tag{3.38}$$

So, for $t \in [0,\tau]$,
Adding (3.37) and (3.38)

$$x'(t;\phi,0) + x'(t;0,f) = A[x(t;\phi,0) + x(t;0,f)] + B\phi(t-\tau) + f(t). \tag{3.39}$$

By integrating on the both sides we get,

$$x(t) := x(t;\phi,f) = x(t;\phi,0) + x(t;0,f) \tag{3.40}$$

for t $\in [0,\tau]$. We can conclude from (3.40) that (3.35) holds for all $t \geq 0$, by uniqueness of the solutions $x(t;\phi,f)$, $x(t;\phi,0)$, and $x(t;0,f)$ for all $t \geq 0$.

Chapter 4

APPLICATIONS

The logistic equation is given by

$$x'(t) = ax\left(1 - \frac{x}{K}\right), \tag{4.1}$$

where a is the growth rate and K is the carrying capacity of the ecosystem.

Now consider the delayed logistic equation:

The model we have employed to demonstrate population dynamics is as follows:

- The initial normalized population is chosen to be small (typically 0.01) as it cannot be zero. A zero initial population signifies a non-existent species.
- The equation is normalized.
- The growth rate is denoted by a, which is finite, positive, and time-independent.
- The delay τ is also finite and positive.

With these assumptions, the single delay logistic equation is:

$$x'(t) = ax(t)\,[1 - bx(t-\tau)] \quad \text{for} \quad t \geq 0, \tag{4.2}$$

where a, $b > 0$ are constants with $b \equiv 1/K$, and $\tau > 0$ is the delay. As initial condition for (14), consider

$$x(t) = 0.01 \quad \text{for} \quad t \in [-\tau, 0]. \tag{4.3}$$

We denote the solution of the initial value problem (4.2), (4.3) by $x(t)$.

To determinine the equilibrium points of (4.2), which we denote by x^*. Because x^* is constant, it follows that $x^* = x(t) = x(t-\tau)$. Substituting this result in (4.2), we get

$$ax^*\,(1 - bx^*) = 0. \tag{4.4}$$

Therefore, the equilibrium points of (4.2) are given by $x^* = 0$ and $x^* = 1/b$.

Now linearize the equation(14)
Let $p(t)$ be a small variation in the population such that higher powers of p may be neglected. Substituting $x(t) \equiv x^* + p(t)$ in (4.2) gives the DDE:

$$p'(t) = a[x^* + p(t)]\,[1 - bx^* - bp(t-\tau)] \quad \text{for} \quad t \geq 0. \tag{4.5}$$

with initial condition for (4.5) :

$$p(t) = 0.01 \quad \text{for} \quad t \in [-\tau, 0]. \tag{4.6}$$

Substituting $x^* = 0$ in (4.5) gives

$$p'(t) = ap(t)\,[1 - bp(t-\tau)]\,. \tag{4.7}$$

Neglecting higher powers of p, we get the DDE:

$$p'(t) = ap(t). \tag{4.8}$$

Therefore,

$$p(t) = ce^{at}, \tag{4.9}$$

where c is a constant.

Taking the initial condition (4.6) , the solution of the initial value problem (4.5), (4.6), for $x^* = 0$, is given by

$$p_1(t) = 0.01e^{at}. \tag{4.10}$$

Since $a > 0$, it follows that $x^* = 0$ is unstable.

Substituting $x^* = 1/b$ in (4.6) gives

$$p'(t) = a\left[\frac{1}{b} + p(t)\right][-bp(t-\tau)]. \tag{4.11}$$

Neglecting higher powers of p, we get the DDE:

$$p'(t) = -ap(t-\tau). \tag{4.12}$$

The equilibrium point of (4.12) is given by $p^* = 0$. Because $p^* \neq x^*$ and we need to have $p^* = x^*$, we will add a certain constant to (4.12) such that $p^* = x^*$ holds. The constant has to be equal to a/b, because then, we have the following DDE:

$$p'(t) = -a\left[p(t-\tau) - \frac{1}{b}\right], \tag{4.13}$$

where the equilibrium point is given by $p^* = x^* = 1/b$. Instead of (4.6), we consider (4.10) as initial condition for (4.13).

To get the idea see Figure 4.1. We denote the solution of the initial value problem (4.13), (4.10) by $p_2(t)$.

Using the package DDE23 in Matlab, plots of the numerical solution $x(t)$ for different values of τ, a, and b can be made. Because changing the value of b does not affect the shapes of the plots, we will only consider one value for b. See Figure 4.1, 4.2 ans 4.3 for the plots of $x(t)$, $p_1(t)$, and $p_2(t)$ for some values of τ, a, and $b = 0.5$. Because determining $p_2(t)$ analytically is quite difficult, the numerical solution $p_2(t)$ is included in these plots.

From the plots in Figure 4.1, 4.2 and 4.3, we can conclude that p1(t) and p2(t) approximate x(t) very well, except for the last case. For this case, $a\tau = (1.5)^2 = 2.25 > \frac{\pi}{2}$. By Corollary , it follows that x^*= 1/b is unstable. From the plot of this case, we can conclude that x(t) contains complex (not real) values, and p1(t) and p2(t) only contain real values. Therefore, x(t) cannot be approximated by p1(t) and p2(t). For the other cases, we can conclude from the plots, and Corollary , that x^* = 1/b is stable.

Furthermore, we can conclude from these plots that the cases where has the same value, the graphs look similar. For example, consider the cases τ = 0.5, a = 1, and τ = 1, a = 0.5. Then, the plots of these cases look the same, but the time intervals are different. The same holds for the other cases with the same value of aτ.

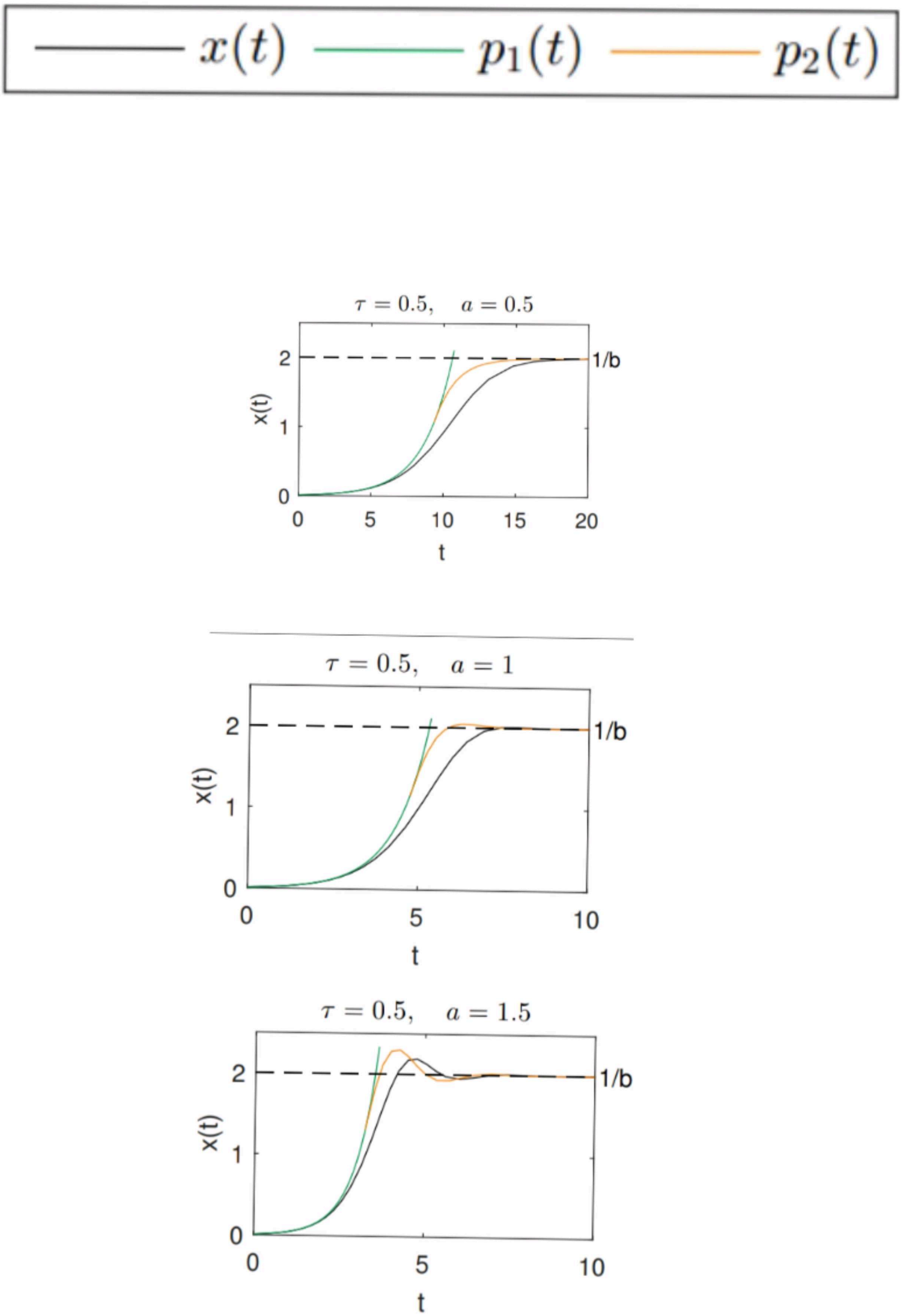

Figure 4.1: Plots of $x(t)$, $p_1(t)$, and $p_2(t)$, for $b = 0.5$, $\tau = 0.5$ and different values of a.

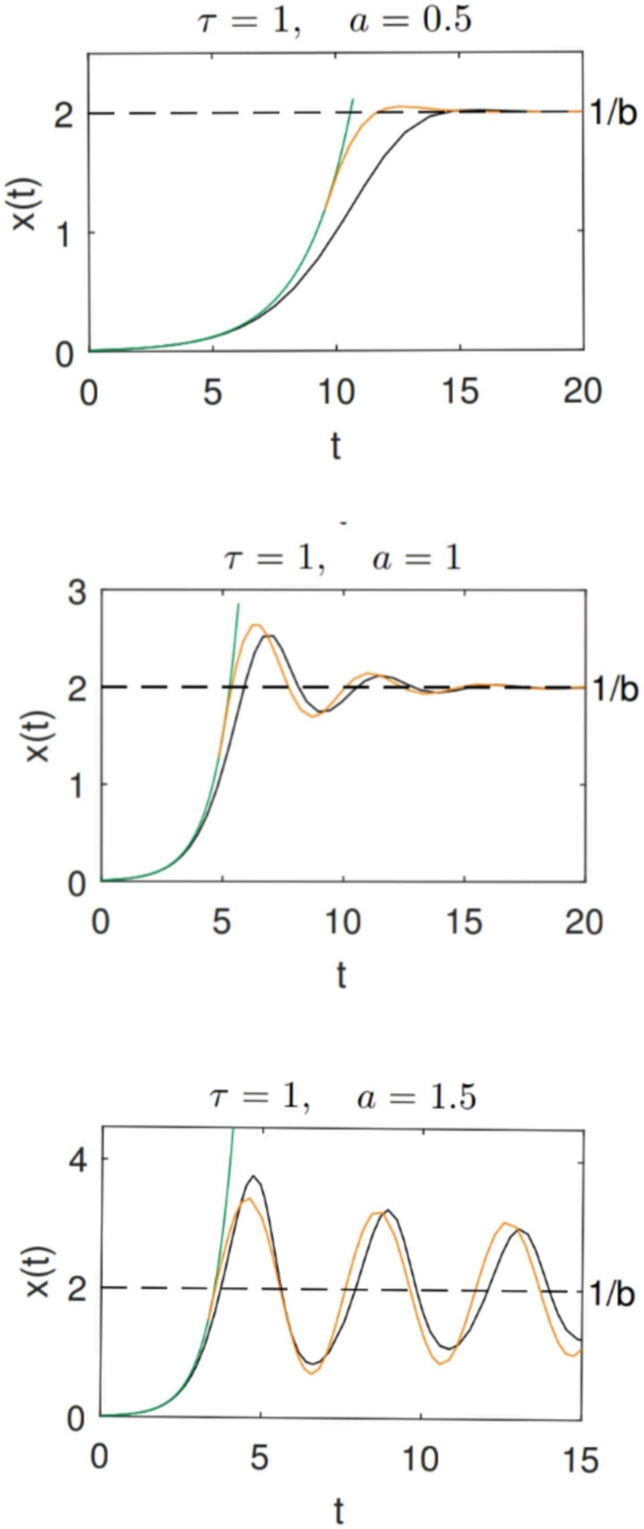

Figure 4.2: Plots of $x(t)$, $p_1(t)$, and $p_2(t)$, for $b = 0.5$, $\tau = 1$ and different values of a.

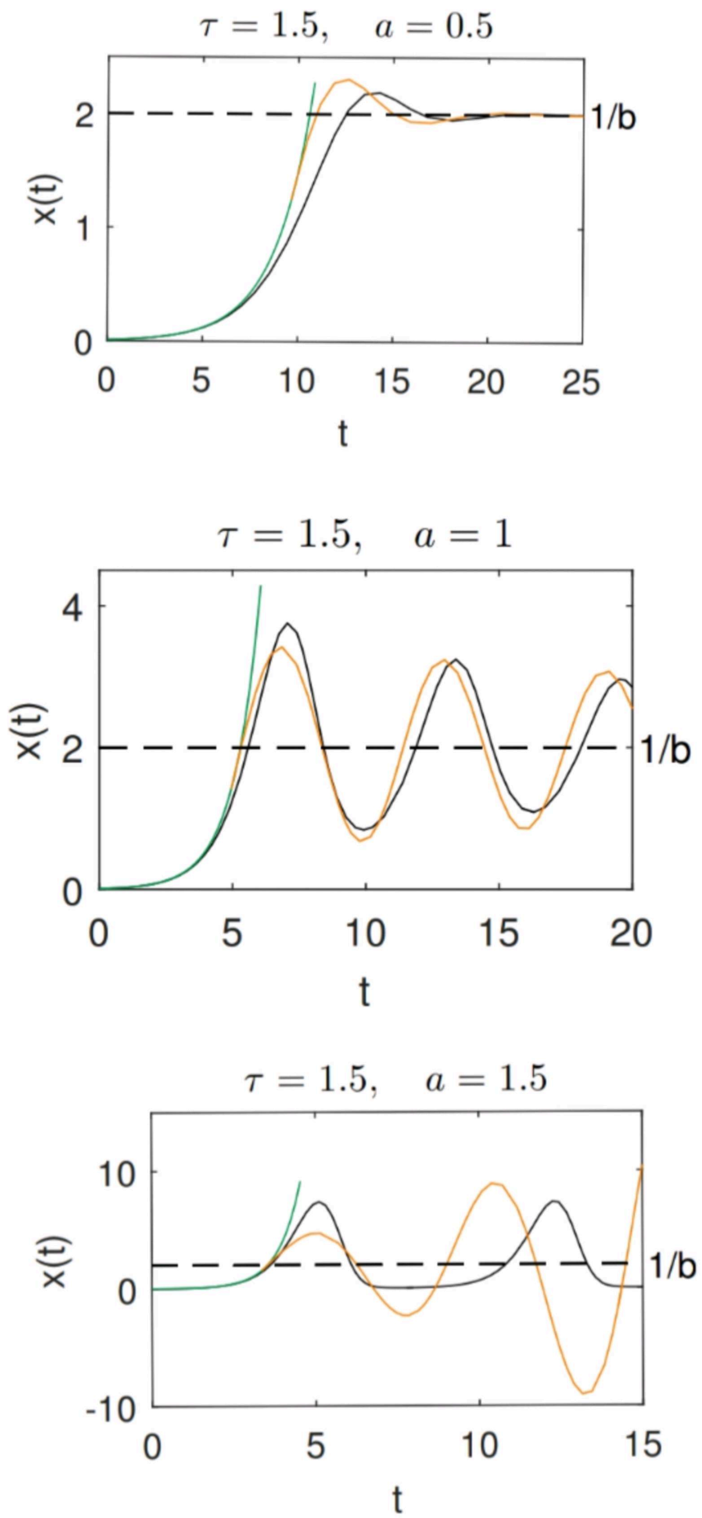

Figure 4.3: Plots of $x(t)$, $p_1(t)$, and $p_2(t)$, for $b = 0.5$, $\tau = 1.5$ and different values of a.

Chapter 5

CONCLUSION

The analysis of linear delay differential equations (DDEs) has provided insights into their solutions and stability properties. Through the determination of eigenvalues and eigenvectors, we derived the general solution for a linear DDE with a constant delay. The Laplace transform was employed to analyze non-homogeneous linear DDEs, leading to the derivation of fundamental matrix solutions. Furthermore, the applications of DDEs were explored, particularly in modeling population dynamics with the logistic equation and its delayed variant. Stability analysis revealed the influence of parameters such as the growth rate and delay on equilibrium points and system behavior.

Continuation: Moving forward, further investigations into the behavior of DDEs under varying parameters and initial conditions can deepen our understanding of complex systems. Numerical methods, such as MATLAB's DDE23 package, facilitate the exploration of DDE dynamics and the comparison of analytical and numerical solutions. Additionally, the study of DDEs in interdisciplinary contexts, such as biology and ecology, can provide valuable insights into real-world phenomena and contribute to the development of predictive models for dynamic systems.

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