Generalizations Of Mean Value Theorems

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Miss BHAVANA VINOD KHEDEKAR

22P0410011

ABC ID : 167-556-954-549

201801607

Under the Supervision of

Dr. M. Kunhanandan

School of Physical & Applied Sciences

Mathematics Discipline

GOA UNIVERSITY

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Examined by: Seal of the School

DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Generalizations Of Mean Value Theorem" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. M. Kunhanandan and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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Signature: BKhedekar

Student Name:Bhavana Vinod Khedekar

Seat no: 22P0410011

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This is to certify that the dissertation report "Generalizations of Mean Value Theorem" is a bonafide work carried out by Miss Bhavana Vinod Khedekar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

Signature : ______________

Supervisor : Dr. M. Kunhanandan

Date: 10/05/2024

Signature of HoD of the Dept

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the dissertation report is: " Generalization Of Mean Value Theorem" This report is divided into six chapters. Each chapter describes a different variant or generalizations of the Mean Value Theorem.

Chapter 1 is an introduction to this topic, it has some basic results related to mean value theorem, it's geometric interpretation and it's application. In **Chapter 2** we see that The mean value theorem is motivation for many functional equations, some of which are discussed in this chapter. Further, we briefly describe the mean value theorem for divided differences and give some applications in defining the functional means. We also examine the limiting behaviour of mean values.This chapter also deals with Pompeiu's mean value theorem and stamate type of functional equations.

In **Chapter 3**, we examine and extend Lagrange's mean value theorem to functions in two variables and Cauchy's mean value theorem for the functions in two variables. **Chapter 4** examines the mean value theorem and it's generalizations for functions with symmetric derivative. Here, we introduce the notion of symmetric differentiation and then derive the mean value theorem for symmetrically differentiable functions.

Chapter 5 deals with the integral mean value theorem and its generalizations. Some applications of it are given, such as finding the integral representations of the arithmetic , geometric, logarithmic, and identric means. In **Chapter 6** we examine Flett's theorem in $\mathbb{R}$ its geometric interpretation as well as extension for Flett's theorem in $\mathbb{C}$. Lastly the **Chapter 7** deals with the conclusion of this project.

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ABSTRACT

Mean Value theorem is a very important result in analysis. It originated from Rolle's theorem, which was proved by the French mathematician Michel Rolle (1652-1719) for polynomials in 1691. Rolle's theorem got its recognition when Joseph Lagrange (1736-1813) presented this mean value theorem in this book Theorie des functional analytiques in 1791. It recieved further recognition when agustin Luis Cauchy (1789-1857) proved his mean value theorem in his book Equationnes differentielles ordinaires. Many functional equations are studied motivated by various mean value theorems.

The main goal of my thesis is to study Mean value theorem and some related topics such as, Mean value theorem and associated functional equations, Two dimensional mean value theorem, Mean value theorem for some generalized derivatives, Integral mean value theorems and related topics and extend variants of Mean value Theorems in real variable case to holomorphic functions.

Keywords: Mean Value Theorem, functional Equations, holomorphic functions, Divided difference, Symmetric derivative, functional mean, Flett's theorem, Complex Rolle's theorem.

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Chapter 1

INTRODUCTION

Mean Value Theorem is a very important result in analysis. It was originated from Rolle's Theorem, which was proved by the French mathematician Michel Rolle (1652-1719) for a polynomials in 1691. Rolle's theorem got its recognition when Joseph Lagrange (1736-1813) presented this mean value theorem in this book Theorie des functional analytiques in 1791. It recieved further recognition when Agustin Luis Cauchy (1789-1857) proved his mean value theorem in his book Equationnes differentielles ordinaires. Many functional equations are studied motivated by various mean value theorems.

1.1 Basic results related to Mean Value Theorem

proposition 1.1.0.1. *If a differentiable function $f : \mathbb{R} \to \mathbb{R}$, attains an extreme value at a point c in an open interval (a,b), then $f'(c) = 0$.*

Proof. Let us assume that f has a maximum at $x = c$, then there exists some positive δ such that

$f(x) < f(c)$ for all $0 < |x-c| < \delta$ which implies $f(x) - f(c) < 0$ for all $0 < |x-c| < \delta$

case 1: When $x-c > 0$. Then $\frac{f(x)-f(c)}{x-c} < 0$ for all $0 < |x-c| < \delta$

which implies $\frac{f(x)-f(c)}{x-c} < 0$ if $c < x < c+\delta$

Thus, $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$,

implies $Rf'(c) \leq 0$.

Case 2: When $x-c < 0$. Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $0 < |x-c| < \delta$

which implies $\frac{f(x)-f(c)}{x-c} > 0$ if $c-\delta < x < c$

Thus $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$,

implies $Lf'(c) \geq 0$.

But we also given that $f'(c)$ exists. So

$f'(c) = Lf'(c) = Rf'(c) = 0$

Similarly, we can show that $f'(c) = 0$, if f has minimum at c. $\square$

proposition 1.1.0.2. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ attains its extreme values on any closed and bounded interval [a,b].*

Proof. Let us prove this result by using the method of contradiction. We will prove that f attains its maximum on the closed interval [a,b]. The proof that f attains its minimum on [a,b] can be proved on the similar lines.

By hypothesis, f is continuous on [a,b], so f is bounded on [a,b] such that there exists m, M such that we have $m \leq f(x) \leq M$ using the boundedness theorem.

Assuming that M is the least upper bound, we need to prove that there exists c in [a,b] such that $f(c) = M$.

Now, assume that there is no such c in [a,b], then we have $f(x) < M$ for all x in [a,b].

Define a function $h(x) = \frac{1}{[M-f(x)]}$ on [a,b]. Now, we know that $h(x) > 0$ because $f(x) < M$ for all x in [a,b] and h is also continuous on [a,b]. So using the boundedness theorem, we have h(x) is bounded on [a,b]. This implies that there exists $k > o$ such that $h(x) \leq k$, for all x in [a,b].

$\Rightarrow \frac{1}{[M-f(x)]} \leq k$

$\Rightarrow M - f(x) \geq \frac{1}{k}$

Adding $f(x) - \frac{1}{k}$ on both sides, we have

$\Rightarrow M - \frac{1}{k} \geq f(x)$

$\Rightarrow f(x) \leq M - \frac{1}{k}$

This contradicts the fact that M is the least upper bound of f(x). Hence, our assumption that there exists no such c in [a,b] such that $f(c) = M$ is wrong.

Therefore f attains its maximum on [a,b].

We can prove that f attains its minimum on [a,b] on similar lines.

□

Theorem 1.1.0.3. *(Rolle's Theorem): If f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), and $f(x_1) = f(x_2)$, then there exists a point $\eta \in (x_1, x_2)$ such that $f'(\eta) = 0$.*

Proof. Since f is continuous and $[x_1, x_2]$ is a closed bounded interval, then by proposition (1.1.0.2) f attains its maximum and minimum value on this interval. If both of these occur at the end points x_1, x_2, then maximum and minimum value are equal and the function is constant, hence $f'(\eta) = 0$ for all η in (x_1, x_2). If this is not the case then one of the extreme values occurs at a point $\eta \in (x_1, x_2)$ and by proposition (1.1.0.1), we have $f'(\eta) = 0$. □

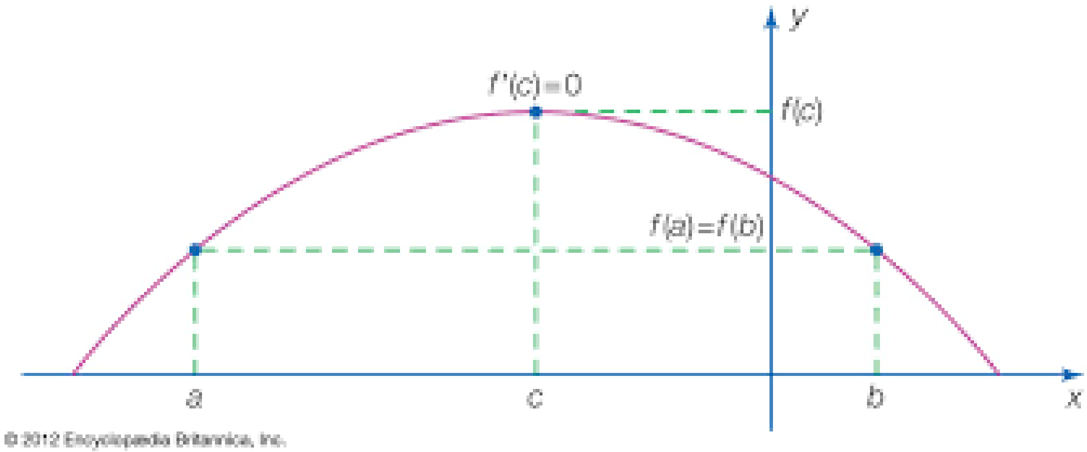

Figure 1.1: Geometric Interpretation Of Rolle's Theorem

Geometrically this means that If there is a horizontal secant line to the graph of f, then there exists a horizontal tangent to the graph that is supported at a point between the two points of intersection of the secant line.

Theorem 1.1.0.4. *(Lagrange's Mean Value Theorem): For every real valued function f differentiable on an interval I and for all pairs $x_1 \neq x_2$ in I, there exists a point η depending on x_1 and x_2 such that*

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(\eta(x_1,x_2)) \tag{1.1}$$

Proof. The proof follows the idea that the mean value theorem is just a modified version of Rolle's theorem. We consider the function

$$h(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1)$$

This is the equation of the line intersecting the graph of f at $(x_1, f(x_1))$ and $(x_2, f(x_2))$. If we define

$$g(x) = f(x) - h(x)$$

then g is the result of rotating f and shifting it down to the x-axis. Since f and h are continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2) so is g and $g(x_1) = g(x_2) = 0$ and thus g satisfies the hypothesis of Rolle's theorem. Now we may apply Rolle's theorem which results in the existence of an $\eta \in (x_1,x_2)$, where

$$0 = g'(\eta) = f'(\eta) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\eta)$$

The proof of the theorem is now complete.

$\square$

The mean value theorem(MVT) has the geometric interpretation as shown in the figure 1.1. The tangent line to the graph of function f at $\eta(x_1,x_2)$ is parallel to the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$

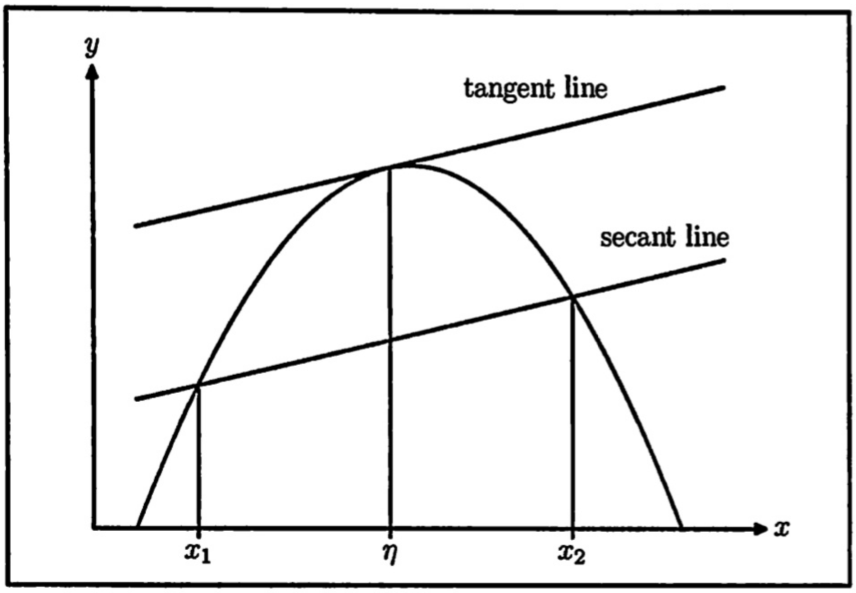

Figure 1.2: A Geometrical Interpretation of the Mean Value Theorem

1.2 Applications of the MVT

In this section we will see how Mean value theorem can be used in proving various other results.

Lemma 1.2.0.1. *If $f'(x) = 0$ for all x in (a,b), then f is a constant on [a,b].*

Proof. Let x_1, x_2 be any two points in (a,b),
and suppose $f(x_1) \neq f(x_2)$, then by the mean value theorem there is a $c \in (a,b)$

such that
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - X_1} \neq 0$$

contradicting the hypothesis that $f'(x) = 0$ for all $x \in (a,b)$

□

Lemma 1.2.0.2. *If $f'(x) = g'(x)$ for all x in (a,b), then f and g differ by a constant on [a,b].*

Proof. let $h(x) = f(x) - g(x)$, then $h'(x) = 0$, so by lemma 1.1 $h(x) = c$, where c is a constant. thus, f and g differ by a constant.

□

Lemma 1.2.0.3. *If $f'(x) > 0$ (< 0) for all x in (a,b),then f is a strictly increasing (decreasing) function on [a,b].*

Proof. Let $x_1 < x_2$ be in [a,b], then by the mean value theorem there is a c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

and since $x_2 - x_1 > 0$ we have $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$,and f is increasing.

□

Example: 1.2.0.4. *The mean value theorem can be used to prove **Bernoulli's inequality**: If $x > -1$, then*

$$(1+x)^n \geq 1 + nx$$

for all $n \in \mathbb{R}$

Proof. First, we suppose $x \geq 0$ and let $f(t) = (1+t)^n$, for $t \in [0,x]$. Thus f satisfies the hypothesis of the mean value theorem and we have an $\eta \in (0,x)$ with

$$f(x) - f(0) = (x-0)f'(\eta)$$

Thus we have

$$(1+x)^n - 1 = xn(1+\eta)^{n-1} \geq nx$$

and hence $(1+x)^n \geq 1+\eta x$. The case when $-1 < x < 0$ can be handled similarly by considering $f(t) = (1+t)^n$ for $t \in [x,0]$

□

Theorem 1.2.0.5. *(Cauchy's Mean Value Theorem):Let f and g be two functions defined on [a,b] such that*

1. *f and g are both continuous on* $[a,b]$
2. *f and g are both derivable (a,b)*
3. *g'(x) does not vanish at any point of (a,b)*

Then there exists a number c between a and b such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Proof. We prove this theorem by using Rolle's theorem.

Now let's take the auxillary function F(x) such that,

$$F(x) = f(x) + P(g(x)) \tag{1.2}$$

In the above equation, P is chosen such that $F(x)$ always satisfies the Rolle's Theorem in $[a,b]$, Now by definition of Rolle's theorem,

$F(a) = F(b)$

$\implies f(a) + P(g(a)) = f(b) + P(g(b))$

$\implies f(b) - f(a) = P(g(b) - g(a))$

$\implies P = \frac{[f(b)-f(a)]}{[g(b)-g(a)]}$

In equation (1.2)

$F'(x) = f'(x) - \frac{[f(b)-f(a)]}{[g(b)-g(a)]} g'(x)$

AS, $F(x)$ satisfies Rolle's Theorem,

$F'(C) = 0$ where, $c \in (a,b)$

$\implies f'(c) - \frac{[f(b)-f(a)]}{[g(b)-g(a)]} g'(c) = 0$

$\implies \frac{f'(c)}{g'(c)} = \frac{[f(b)-f(a)]}{[g(b)-g(a)]}$

Thus, Cauchy's Theorem is proved.

$\square$

Chapter 2

MVT AND RELATED FUNCTIONAL EQUATIONS

In this section we illustrate a functional equation that arises from the mean value theorem and then we present a systematic study of this functional equation and its various generalizations. These functional equations characterize polynomials of various degrees.

2.1 Functional Equations

In Mathematics, a functional equation is an equation in which one or several functions appear as unknowns. So, differential equations and integral equation are functional equations.

Definition 2.1.0.1. For distinct real numbers $x_1, x_2, ..., x_n$, the divided difference of the function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f[x_1] = f(x_1)$$

and

$$f[x_1, x_2, .., x_n] = \frac{f[x_1, x_2, .., x_{n-1}] - f[x_2, x_3, .., x_n]}{x_1 - x_n}$$

For all $n \geq 2$ It is easy to see that

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

and

$$f[x_1, x_2, x_3] = \frac{(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}$$

In view of divided difference, equation (2.1) in the mean value theorem takes the form

$$f[x_1, x_2] = f'(\eta(x_1, x_2)).$$

Theorem 2.1.0.2. *The function $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f[x, y] = h(x + y), \quad x \neq y, \tag{2.1}$$

if and only if,

$$f(x) = ax^2 + bx + c \;\; and \;\; h(x) = ax + b \tag{2.2}$$

where a,b,c are arbitrary constants.

Proof. Equation (2.1), using the definition of the divided difference of f, can be rewritten as

$$f(x) - f(y) = (x-y)h(x+y) \quad \text{for} \quad x \neq y \tag{2.3}$$

which is true for $x = y$. If f satisfies equation (2.3), so does f+b, where b is an arbitrary constant. Therefore we may assume without loss of generality $f(0) = 0$. Putting $y = 0$ in in equation (2.3), we see that

$$f(x) = xh(x) \tag{2.4}$$

Hence by equation (2.4), equation (2.3) transforms into

$$xh(x) - yh(y) = (x-y)h(x+y) \tag{2.5}$$

Again if h satisfies equation (2.5) so also $h+c$, where c is an arbitrary constant. So we may suppose $h(0) = 0$. Therefore, letting $x = -y$ in equation (2.5), we obtain

$$-yh(-y) = yh(y) \tag{2.6}$$

that is h is an odd function. Taking this into consideration and replacing y by $-y$ in equation (2.5),we get

$$xh(x) - yh(y) = (x+y)h(x-y) \tag{2.7}$$

comparing equation (2.7) with equation (2.5),we have

$$(x-y)h(x+y) = (x+y)h(x-y) \tag{2.8}$$

and substituting

$$u = x+y \quad \text{and} \quad v = x-y \tag{2.9}$$

in equation (2.8), we obtain

$$vh(u) = uh(v) \tag{2.10}$$

for all $u, v \in \mathbb{R}$. Thus

$$h(u) = au \tag{2.11}$$

If we do not assume $h(0) = 0$, then we have in general

$$h(u) = au+b \tag{2.12}$$

By equation (2.4) this gives $f(x) = x(ax+b)$ and, if we do not assume $f(0) = 0$, then

$$f(x) = ax^2+bx+c$$

So we have indeed proved that all solutions of equation (2.1) are of the form

$$f(x) = ax^2+bx+c$$

$$h(x) = ax+b$$

where a,b,c are arbitrary constants, as asserted. The converse of this theorem is straightforward and the proof is now complete.

□

Theorem 2.1.0.3. *If the quadratic polynomial $f(x) = ax^2 + bx + c$ with $a \neq 0$, is a solution of the functional equation*

$$f(x+y) - f(x) = hf'(x+\theta h) \quad (0 < \theta < 1) \tag{2.13}$$

assumed for all $x \in \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$, then $\theta = \frac{1}{2}$. Conversely, if a function f satisfies the above functional differential equation with $\theta = \frac{1}{2}$, then the only solution is a polynomial of degree at most two.

Proof. Suppose the polynomial

$$f(x) = ax^2 + bx + c \tag{2.14}$$

is a solution of equation (2.13). Then inserting equation (2.14) into (2.13), we have

$$a(x+h)^2 + b(x+h) + c - ax^2 - bx - c = h(2a(x+\theta h) + b) \tag{2.15}$$

that is

$$ah^2(1-2\theta) = 0$$

Since a and h are nonzero, we have

$$\theta = \frac{1}{2}$$

This proves the if part of the theorem. Next, we prove the converse of the theorem. Letting $\theta = \frac{1}{2}$ and $h = y - x$ in equation (2.13), we see that

$$f(x) - f(y) = (x - y)f'(\frac{x+y}{2}), \quad x \neq y$$

Thus, by Corollary, f is a polynomial of degree at most two and the proof of theorem is now complete.

□

2.2 Integral Representation Of Divided Difference

In this section, we prove the mean value theorem for divided difference and then present some applications towards the study of means. We begin this section with an integral representation of divided differences. In this section $f^{(n)}$ will denote the n^{th} derivative of a function f while f' will represent the first derivative of f.

Theorem 2.2.0.1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has a continuous n^{th} derivative in the interval*

$$min\{x_0, x_1, .., x_n\} \leq x \leq max\{x_0, x_1, .., x_n\}.$$

If all the points $x_0, x_1, .., x_n$ are all distinct, then

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} f^{(n)}\big(x_0 + \sum_{k=1}^{n} t_k(x_k - x_{k-1})\big) dt_n \tag{2.16}$$

$$= f[x_0, x_1, ..., x_n]$$

where $n \geq 1$.

Proof. We prove this theorem by induction. If $n = 1$, the representation given in (2.18) reduces to

$$f[x_0,x_1] = \int_0^1 f'(t_1(x_1 - x_0) + x_0)dt_1$$

First we show that the integral on the right side of the equation is equal to divided difference of f based on the two distinct points x_0 and x_1. consider the integral

$$\int_0^1 f'(t_1(x_1 - x_0) + x_0)dt_1$$

Since $x_1 \neq x_0$, introducing a new variable z for $t_1(x_1 - x_0) + x_0$, we get

$$dz = (x_1 - x_0)dt_1$$

that is

$$dt_1 = \frac{dz}{x_1 - x_0}$$

Since $t_1 = 0$, the new variable $z = x_0$ and similarly if $t_1 = 1$, then $z = x_1$. Hence, we have

$$\begin{aligned}\int_0^1 f'(t_1(x_1 - x_0) + x_0)dt_1 &= \int_{x_0}^{x_1} f'(z)\frac{dz}{x_1 - x_0} \\ &= \frac{\int_{x_0}^{x_1} f'(z)dz}{x_1 - x_0} \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}\end{aligned}$$

Next, assuming that the integral representation in (2.18) holds for $n-1$, that is

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-2}} (x_0 + \sum_{k=1}^{n-1} t_k(x_k - x_{k-1}))dt_{n-1} = f[x_0,x_1,..,x_{n-1}] \qquad (2.17)$$

We will show that (2.18) holds for integer n. Let

$$w = t_n(x_n - x_{n-1} + .. + t_1(x_1 - x_0)) + x_0$$

be the new variable. Hence

$$dt_n = \frac{dt_w}{x_n - x_{n-1}}$$

for $x_n \neq x_{n-1}$. If $t_n = 0$, then $w = w_0$,where

$$w_0 = t_{n-1}(x_{n-1} - x_{n-2}) + .. + t_1(x_1 - x_0) + x_0$$

Similarly, if $t_n = t_{n-1}$, then $w = w_1$, where

$$w_1 = t_{n-1}(x_n - x_{n-2}) + .. + t_1(x_1 - x_0) + x_0$$

Now applying the induction hypothesis, we have

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 .. \int_0^{t_{n-1}} f^(n)(x_0 + \sum_{k=1}^{n} t_k(x_k - x_{k-1})dt_n$$

$$= \int_0^1 dt_1 \int_0^{t_1} dt_2 .. \int_0^{t_{n-2}} \frac{f^{(n-1)}(w_1) - f^{(n-1)}(w_0))}{x_n - x_{n-1}} dt_{n-1}$$

$$= \frac{f[x_0, x_1, .., x_{n-2}, x_n] - f[x_0, x_1, .., x_{n-2}, x_{n-1}]}{x_n - x_{n-1}}$$

$$f[x_0, x_1, .., x_n]$$

This completes the proof of the theorem.

□

From the above integral representation, we see that the integrand is a continuous function of the variables $x_0, x_1, ..., x_n$ and therefore the left side, $f[x_0, x_1, .., x_n]$, is also a continuous function of these variables. If $f(x)$ has a continuous n^{th} derivative, then the above integral representation defines uniquely the continuous extension of $f[x_0, x_1, .., x_n]$.

Theorem 2.2.0.2. *Let $f : [a,b] \to \mathbb{R}$ be a real valued function with continuous n^{th} derivative and $x_0, x_1, .., x_n$ in [a,b]. Then there exists a point η in the interval $[min\{x_0, x_1, .., x_n\}, max\{x_0, x_1, .., x_n\}]$ such that*

$$f[x_0, x_1, .., x_n] = \frac{f^{(n)}(\eta)}{n!}. \tag{2.18}$$

Proof. Since $f^{(n)}(x)$ is continuous on $[a,b]$, the function $f^{(n)}(x)$ has a maximum and a minimum on $[a,b]$. Let

$$m = minf^{(n)}(x) \quad and \quad M = maxf^{(n)}(x)$$

Then from the integral representation of $f[x_0, x_1, .., x_n]$, we have

$$m \prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k \leq f[x_0, x_1, .., x_n] \leq M \prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k$$

where $t_0 = 1$. Using the fact that

$$\prod_{k=1}^{n} \int_0^{t_{k-1}} dt_K = \int_0^1 dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n = \frac{1}{n!}$$

we obtain from the above inequalities

$$m \leq f[x_0, x_1, .., x_n](n!) \leq M$$

Since $f^{(n)}(x)$ is continuous, by applying the intermediate value theorem to it, we have

$$f[x_0, x_1, .., x_n](n!) = f^{(n)}(\eta)$$

for some $\eta \in [minx_0, x_1, ., x_n, maxx_0, x_1, .., x_n] = \frac{f^{(n)}(\eta)}{n!}$ and the proof of the theorem is now complete.

$\square$

We observe that Some of the nodes in the divided differences can coalesce if f is suitably differentiable.
For this consider,

$$\begin{aligned}
f[b,b,a,a] &= \frac{f[b,b,a] - f[b,a,a]}{b-a} \\
&= \frac{1}{b-a}[f[b,b,a] - f[b,a,a]] \\
&= \frac{1}{b-a}[\frac{f[b,b] - f[b,a]}{b-a} - \frac{f[b,a] - f[a,a]}{b-a}] \\
&= \frac{1}{(b-a)^2}[f[b,b] - 2f[b,a] + f[a,a]] \\
&= \frac{f'(b) - 2f[b,a] + f'(a)}{(b-a)^2}
\end{aligned}$$

Here f[b,b,a,a] is denoted by $f[b^{[2]}, a^{[2]}]$. Similarly, in general

$$f[b^{[n]}, a^{[n]}] = f[b, b, b, ..b, a, a.., a]$$

In above divided difference a and b appear exactly n times each. Suppose f is (2n-1) times continuously differentiable in the interval $[a,b]$. Further, we assume that $f^{(2n-1)}(x)$ is strictly monotone in $[a,b]$. Then by the mean value theorem for divided differences, there exists one point $\eta \in [a,b]$ such that

$$f[b^{[n]},a^{[n]}] = \frac{f^{(2n-1)}(\eta)}{(2n-1)!}$$

Strict monotonicity of $f^{(2n-1)}(x)$ forces η to be a mean value, that is, $a < \eta < b$. Further, since $f^{(2n-1)}(x)$ is strictly monotone, such a η is also unique, and this defines a functional mean $M_f^n(a,b)$ in a and b. Hence

$$M_f^n(a,b) = (f^{(2n-1)})^{-1}\{(2n-1)!f[b^{[n]},a^{[n]}]\}.$$

Note that in the above formula $(f^{(2n-1)})^{-1}$ denotes the inverse function of $f^{(2n-1)}$. If $n = 1$, then $m_f^n(a,b)$ reduces

$$M_f^n(a,b) = (f')^{-1}(\frac{f(b)-f(a)}{b-a})$$

Example: 2.2.0.3. *If $f(x) = x^m$, where m is a positive integer greater than or equal to n, then $M_f^n(a,b) = \frac{a+b}{2}$.*

To see this, If $f(x) = x^m$, then it can be shown by the definition of divided difference on function f and using induction on n that

$$f[x_0,x_1,..,x_{m-1}] = x_0 + x_1 + ... + x_{m-1} \tag{2.19}$$

Letting $m = 2n$ and

$x_0 = x_1 = \ldots = x_{n-1} = a$ *and* $x_n = x_{n+1} = \ldots = x_{2n-1} = b$

in (8), we get

$$f[b^{[n]}, a^{[n]}] = n(a+b)$$

Therefore,

$$\begin{aligned} M_f^n(a,b) &= (f^{(2n-1)})^{-1}\{(2n-1)!f[b^{[n]}, a^{[n]}]\} \\ &= \frac{1}{2n!}\{(2n-1)!n(a+b)\} \\ &= \frac{a+b}{2} \end{aligned}$$

Example: 2.2.0.4. *If $f(x) = \frac{1}{x}$, then $M_f^n(a,b) = \sqrt{ab}$*

To see this, If $f(x) = \frac{1}{x}$,then by using definition of divided difference on f and by using induction on n, we can show that,

$$f[x_0, x_1, .., x_n] = \frac{(-1)^{n-1}}{x_0 x_1 .. x_{n-1}} \tag{2.20}$$

Hence

$$f[b^{[n]}, a^{[n]}] = \frac{(-1)^{2n-1}}{a^n b^n}$$

Since $f^{(2n-1)}(x) = \frac{(-1)^{2n-1}(2n-1)!}{x^{2n}}$ *thus we have,*

$$\begin{aligned} M_f^n(a,b) &= (f^{(2n-1)})^{-1}\{(2n-1)!f[b^{[n]},a^{[n]}]\} \\ &= (a^n b^n)^{\frac{1}{2n}} \\ &= \sqrt{ab} \end{aligned}$$

Conclusion: Let $f(x) = x^p$, where $p \in \mathbb{R}$. The first example shows that, p is a positive integer greater than or equal to n, then the functional mean $M_f^n(a,b)$ is the arithmetic mean of a and b. The second example illustrates that if $p = -1$, then the functional mean $M_f^n(a,b)$ is the geometric mean of a and b.

Theorem 2.2.0.5. *Suppose that* $f(x) = x^l$, *for some nonnegative integer l,then*

$$f[x_1,..,x_n] = \begin{cases} 0 & \textit{for } l < n-1 \\ 1 & \textit{for l=n-1} \\ x_1+...+x_n & \textit{for l=n} \end{cases}$$

for all positive integers n.

Proof. Let $f(x) = x^l$, where l is a natural number. We would like to evaluate $f[x_1,x_2,..,x_n]$. First consider

$$
\begin{aligned}
f[x_1,x_2] = \frac{f(x_1)-f(x_2)}{x_1-x_2} &= \frac{x_1^l - x_2^l}{x_1-x_2} \\
&= \frac{(x_1-x_2)\sum_{k=0}^{l-1} x_1^k x_2^{l-1-k}}{x_1-x_2} \\
&= \sum_{k=0}^{l-1} x_1^k x_2^{l-1-k} = \sum_{p_1+P_2=l-1} x_1^{p_1} x_2^{p_2}
\end{aligned}
$$

where p_1 and p_2 are nonnegative integers.

Next we consider

$$
\begin{aligned}
f[x_1,x_2,x_3] &= \frac{f[x_1,x_3]-f[x_2,x_3]}{x_1-x_2} \\
&= \frac{\sum_{p_1+p_2=l-1} x_1^{p_1} x_3^{p_3} - \sum_{p_2+p_3=l-1} x_2^{p_2} x_3^{p_3}}{x_1-x_2} \\
&= \frac{1}{x_1-x_2}[(x_1-x_2)x_3^{l-2} + (x_1^2-x_2^2)x_3^{l-3} + \dots + (X_1^{l-2}-x_2^{l-2})x_3] \\
&= x_3^{l-2} + (x_1+x_2)x_3^{l-3} + (x_1^2+x_1x_2+x_2^2)x_3^{l-4} + .. + \sum_{p_1+p_2=l-2} x_1^{p_1} x_2^{p_2} x_3 \\
&= \sum_{p_1+p_2+p_3=l-2} x_1^{p_1} x_2^{p_2} x_3^{p_3}
\end{aligned}
$$

where p_1, p_2 and p_3 are non-negative integers. Similarly it can be shown that

$$
f[x_1,x_2,..,x_k] = \sum_{p_1+p_2+..+p_k=l-k+1} x_1^{p_1} x_2^{p_2} .. x_k^{p_k},
$$

where $p_1, p_2, .., p_k$ are nonnegative integers. Hence

$$
f[x_1,x_2,..,x_l] = \sum_{p_1+p_2+..+p_l=1} x_1^{p_1} x_2^{P_2} \dots x_l^{p_l} = \sum_{j=1}^{l-1} x_j
$$

$$f[x_1,x_2,..,x_{l-1}] = \sum_{p_1+..+p_{l-1}=0} x_1^{p_1} x_2^{p_2} ... x_{l-1}^{p_{l-1}} = 1$$

and

$$f[x_1,x_2,..,x_{l-1}] = 0$$

This completes the proof. □

The following theorem gives an alternate representation for the n-point divided difference $f[x_1,x_2,..,x_n]$ of f.

Theorem 2.2.0.6. *The n-points divided difference of f can be expressed as*

$$f[x_1,x_2,..,x_n] = \sum_{j=1}^{n} \frac{f(x_j)}{\prod_{k\neq j,k=1}^{n}(x_j - x_k)} \tag{2.21}$$

for all positive integers n.

Proof. The proof will be based on induction on n. The epression is trivially true for $n = 1$ and is easy to establish for $n = 2$. Assuming it is true for n, we establish it for (n+1), using the recursive form of the definition of divided difference as a starting point. We have, by the definition

$$f[x_1,x_2,..,x_{n+1}] = \frac{1}{x_1 - x_{n+1}}(f[x_1,x_2,..,x_n] - f[x_2,x_3,..,x_{n+1}]).$$

By the induction hypothesis, the right side of the above expression yields

$$\frac{1}{x_1 - x_{n+1}}[\sum_{j=1}^{n} f(x_j) \prod_{k=1,k\neq j}^{n} \frac{1}{x_j - x_k} - \sum_{j=2}^{n+1} f(x_j) \prod_{k=2,k\neq j}^{n+1} \frac{1}{x_j - x_k}]$$

Separating the terms based on the left and right endpoints from the above expansion, we get

$$\frac{f(x_1)}{x_1 - x_{n+1}} \prod_{k=2}^{n} \frac{1}{x_1 - x_k} + \frac{1}{x_1 - x_{n+1}} \sum_{j=2}^{n} f(x_j) \prod_{k=1,k\neq j}^{n} \frac{1}{x_j - x_k}$$

$$-\frac{f(x_{n+1})}{x_1 - x_{n+1}} \prod_{k=2}^{n} \frac{1}{x_{n+1} - x_k} - \frac{1}{x_1 - x_{n+1}} \sum_{j=2}^{n} f(x_j) \prod_{k=2,k\neq j}^{n+1} \frac{1}{x_j - x_k}$$

Rearranging, we get

$$\frac{f(x_1)}{x_1 - x_{n+1}} \prod_{k=2}^{n} \frac{1}{x_1 - x_k} + \frac{f(x_{n+1})}{x_{n+1} - x_k} \prod_{k=2}^{n} \frac{1}{x_{n+1-x_k}}$$

$$+\sum_{j=2}^{n} \frac{f(x_j)}{x_1 - x_{n+1}}[\prod_{k=1,k\neq j}^{n} \frac{1}{x_j - x_k} - \prod_{k=2,k\neq j}^{n+1} \frac{1}{x_j - x_k}]$$

Combining and factoring, we have

$$f(x_1)\prod_{k=2}^{n+1}\frac{1}{x_1-x_k}+f(x_{n+1})\prod_{k=1}^{n}\frac{1}{x_{n+1}-x_k}$$

$$+\sum_{j=2}^{n}\frac{f(x_j)}{x_1-X_{j+1}}[\frac{1}{x_j-X_1}-\frac{1}{x_j-x_{n+1}}]\prod_{k=2,k\neq j}^{n}\frac{1}{x_j-x_k}$$

The third term can be condensed, yielding

$$f(x_1)\prod_{k=2}^{n+1}\frac{1}{x_1-x_k}+f(x_{n+1})\prod_{k=1}^{n}\frac{1}{x_{n+1}-x_k}$$

$$+\sum_{j=2}^{n}\frac{f(x_j)}{(x_j-x_1)(x_j-x_{n+1})}\prod_{k=2,k\neq j}^{n}\frac{1}{x_j-x_k}$$

This gives

$$f(x_1)\prod_{k=2}^{n+1}\frac{1}{x_1-x_k}+\sum_{j=2}^{n}f(x_j)\prod_{k=1,k\neq j}^{n+1}\frac{1}{x_j-x_j}+f(x_{n+1})\prod_{k=1}^{n}\frac{1}{x_{n+1}-x_k}$$

and we have established the relationship

$$f[x_1,x_2,..,x_{n+1}]=\sum_{j=1}^{n+1}f(x_j)\prod_{k=1,k\neq j}^{n+1}\frac{1}{x_j-x_k}$$

$\square$

2.3 Limiting Behavior of mean values

If x is a number in the interval (a,b) then by applying Langrange's mean value theorem to the interval $[a,x]$, it is possible to choose a number η_x in (a,x) as a function of x such that

$$f[a,x] = f'(\eta_x)$$

In this section, we examine the behavior of the mean value η_x as x approaches the left end point a of the interval [a,x].

Example: 2.3.0.1. *Consider the function $f(t) = t^2$ on the interval $[1,2]$. Applying Lagrange's mean value theorem to f on the interval $[1,x]$, where $x \in (1,2)$, we obtain*

$$\frac{f(x) - f(1)}{x-1} = f'(\eta_x)$$

for some η_x in $(1,x)$. Since $f(t) = t^2$, the mean value η_x is given by

$$\eta_x = \frac{1}{2}(x+1)$$

Now evaluating the limit of $\frac{\eta_x - 1}{x-1}$ as x approaches 1 from the right, we get

$$\lim_{x \to 1^+} \frac{\eta_x - 1}{x-1} = \lim_{x \to 1^+} \frac{\frac{1}{2}(x+1) - 1}{x-1} = \frac{1}{2}$$

Similarly,

Example: 2.3.0.2. *if we consider another function $f(t) = e^t$ on the interval [0,2], then again we have*

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\eta_x - 0}{x-0} &= \lim_{x\to 0^+} [\frac{1}{x} ln(\frac{e^x - 1}{x})] \\
&= \lim_{x\to 0^+} [\frac{1}{x} ln(1 + \frac{x}{2} + \frac{x^2}{3!} + ...)] \\
&= \frac{1}{2}
\end{aligned}$$

These two examples indicate that as x approaches the left end point of the interval from the right, the mean value η_x approaches the midpoint between x and the left end point of the interval. This is true for many functions by the following theorem.

Theorem 2.3.0.3. *Suppose the function f is continuously differentiable on $[a,b]$ and twice differentiable at a with $f"(a) \neq 0$.If η_x denotes the mean value in $f[a,x] = f'(\eta_x)$, then*

$$\lim_{x\to a+} \frac{\eta_x - a}{x-a} = \frac{1}{2}$$

Proof. To establish this theorem ,we evaluate

$$\lim_{x\to a+} \frac{f(x) - f(a) - (x-a)f'(a)}{(x-a)^2}$$

in two different ways. First, using the mean value theorem, we get

$$\begin{aligned}
\lim_{x\to a+}\frac{f(x)-f(a)-(x-a)f'(a)}{x-a)^2} &= \lim_{x\to a+}\frac{(x-a)f'(\eta_x)-(x-a)f'(a)}{(x-a)^2}\\
&= \lim_{x\to a+}\frac{f'(\eta_x)-f'(a)}{(x-a)}\\
&= lim_{x\to a+}\frac{f'(\eta_x)-f'(a)}{\eta_x-a}\lim_{x\to a+}\frac{\eta_x-a}{x-a}.\\
&= f\text{''}(a)\lim_{x\to a+}\frac{\eta_x-a}{x-a}
\end{aligned}$$

Then by applying L'Hospital rule, we get

$$\lim_{x\to a+}\frac{f(x)-f(a)-(x-a)f'(a)}{(x-a)^2} = \lim_{x\to a+}\frac{f'(x)-f'(a)}{2(x-a)} = \frac{1}{2}f\text{''}(a)$$

Since $f\text{''}(a)\neq 0$, we obtain

$$\lim_{x\to a+}\frac{\eta_x-a}{x-a} = \frac{1}{2}$$

□

2.4 Pompeiu's Mean Value Theorems

Pompeiu derived a variant of Lagrange's mean value theorem, known as Pompeiu's mean value theorem.

Theorem 2.4.0.1. *For every real valued function f differentiable on an interval [a,b] not containing 0 and for all pairs $x_1 \neq x_2$ in [a,b], there exists a point ε in (x_1, x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\varepsilon) - \varepsilon f'(\varepsilon) \tag{2.22}$$

Proof. Define a real valued function F on the interval $[\frac{1}{b}, \frac{1}{a}]$ by

$$F(x) = t f(\frac{1}{t}) \tag{2.23}$$

Since f is differentiable on [a,b] and 0 is not in [a,b], we see that F is differentiable on $[\frac{1}{b}, \frac{1}{a}]$ and

$$F'(t) = f(\frac{1}{t}) - \frac{1}{t} f'(\frac{1}{t}) \tag{2.24}$$

Applying the mean value theorem to F on the interval $[x, y] \in [\frac{1}{b}, \frac{1}{a}]$, we get

$$\frac{F(x) - F(y)}{x - y} = F'(\eta) \tag{2.25}$$

for some $\eta \in (x, y)$.Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$, and $\varepsilon = \frac{1}{\eta}$. Then since $\eta \in (x, y)$, we have

$$x_1 < \varepsilon < x_2$$

now using (2),(3),(4), we have

$$\frac{x f(\frac{1}{x}) - y f(\frac{1}{y}))}{x - y} = f(\frac{1}{\eta}) - \frac{1}{\eta} f'(\frac{1}{\eta}) \tag{2.26}$$

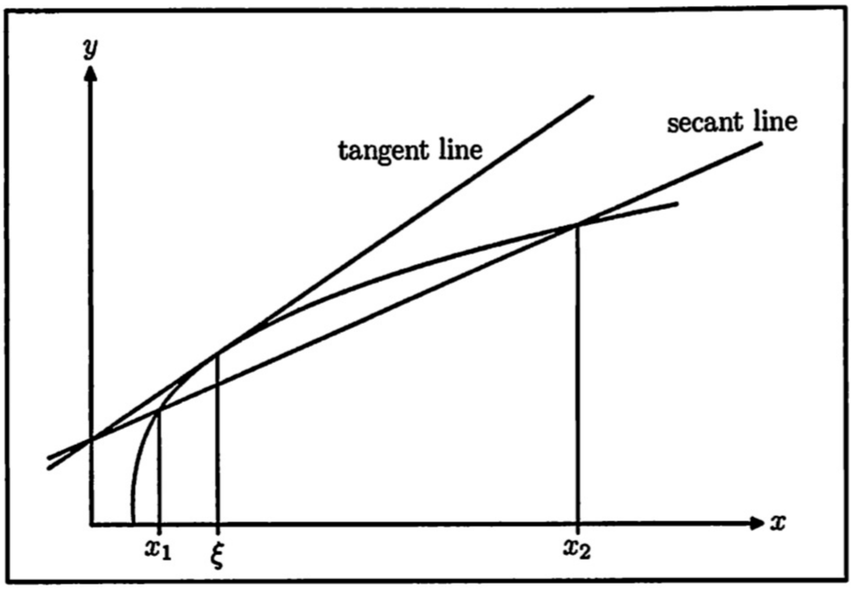

Figure 2.1: An Interpretation of Pompeiu's Mean Value Theorem

that is

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\varepsilon) - \varepsilon f'(\varepsilon)$$

this completes the proof of the theorem. □

Geometrical interpretation of above theorem. The equation of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_1) - f(x_2)}{x_2 - x_1}(x - x_1)$$

This line intersects the y-axis at the point (0,y), where y is

$$
\begin{aligned}
y &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(0 - x_1) \\
&= \frac{x_2 f(x_1) - x_1 f(x_2) - x_1 f(x_2) + x_1 f(x_1)}{x_2 - x_1} \\
&= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2}
\end{aligned}
$$

The equation of the tangent line at the point $(\varepsilon, f(\varepsilon))$ is

$$y = (x - \varepsilon) f'(\varepsilon) + f(\varepsilon)$$

This tangent line intersects the y-axis at the point (0,y),where

$$y = -\varepsilon f'(\varepsilon) + f(\varepsilon)$$

If this tangent line intersects the y-axis at the same point as the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then we have

$$\frac{x_1 f(x_2) - x_2 f(x_2)}{x_1 - x_2} = f(\varepsilon) - \varepsilon f'(\varepsilon)$$

Hence the geometric meaning of this is that the tangent at the point $(\varepsilon, f(\varepsilon))$ intersects on the y-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

2.5 Stamate Type Equations

The algebraic expression (2.22) yields a functional equation.The right hand side of (2.22) depends on ε and not directly on x_1 and x_2. Thus we have the following functional equation

$$\frac{xf(y)-yf(x)}{x-y} = h(\varepsilon(x,y)) \quad \forall x,y \in \mathbb{R},\ x \neq y \tag{2.27}$$

Similar to the divided difference, a variant of it was defined as

$$f\{x_1\} = f(x_1)$$

and

$$f\{x_1,x_2,..,x_n\} = \frac{x_n f\{x_1,x_2,..,x_{n-1}\} - x_1 f\{x_2,x_3,..,x_n\}}{x_1 - x_n}$$

An easy computation shows that $f\{x_1,x_2\} = \frac{x_2 f(x_1)-x_1 f(x_2)}{x_2-x_1}$

and

$$f\{x_1,x_2,..,x_n\} = \sum_{i=1}^{n} \Big(\prod_{j\neq i}^{n} \frac{x_j}{x_i - x_j}\Big) f(x_i)$$

Chapter 3

TWO-DIMENSIONAL MEAN VALUE THEOREM

Lagrange's mean value theorem for functions in one variable can be extended to functions in two variables or more. The main goal of this chapter is to present results regarding the mean value theorem for functions in two variables and then discuss Cauchy's mean value theorem for functions in two variables.

3.1 MVT's for Functions in Two Variables

The following result appeared in the book by Courant (1964).

Theorem 3.1.0.1. *For every function $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous partial derivatives f_x and f_y and for all distinct pairs (x,y) and (u,v) in $\mathbb{R}^2$, there exists an intermediate point*

(η, ε) *on the line segment joining the points* (x,y) *and* (u,v) *such that*

$$f(u,v) - f(x,y) = (u-x)f_x(\eta,\varepsilon) + (v-y)f_y(\eta,\varepsilon) \tag{3.1}$$

Proof. Let (x,y) and (u,v) be any two points in the plane $\mathbb{R}^2$.
Let $h = u - x$ and $k = v - y$. Let $\mathscr{L}$ be line-segment obtained by joining the points (x,y) and (u,v). The co-ordinates of any point on this line-segment are given by $(x + ht, y + kt)$ for some $t \in [0,1]$. We define a function $F : [0,1] \to \mathbb{R}$ by

$$F(t) = f(x+ht, y+kt) \tag{3.2}$$

keeping x, y, u, v fixed for the moment. The derivative of this function is given by

$$F'(t) = hf_x(x+ht, y+kt) + kf_y(x+ht, y+kt), \tag{3.3}$$

where f_x and f_y are partial derivatives of f with respect to x and y, respectively. Applying the mean value theorem to F yields,

$$F(1) - F(0) = F'(t_0) \tag{3.4}$$

Where $t_0 \in (0,1)$. The definition of F in (3.2) yields,

$$f(u,v) - f(x,y) = F'(t_0))$$

using (3.3) in the above equation, we get

$$f(u,v)-f(x,y)=hf_x(x+ht_0,y+kt_0)+kf_y(x+ht_0,y+kt_0),$$

which is

$$f(u,v)-f(x,y)=(u-x)f_x(\eta,\varepsilon)+(v-y)f_y(\eta,\varepsilon),$$

Where (η,ε) is the point on line-segment $\mathscr{L}$ whose co-ordinates are given by $(x+ht_0,y+kt_0)$. This completes the proof of the theorem.

□

The geometric interpretation of this theorem is that the difference between the values of the function at the points (u,v) and (x,y) is equal to the differential at an intermediate point (η,ε) on the line-segment joining the two points.

Example: 3.1.0.2. *A function $f:\mathbb{R}^2\to\mathbb{R}$ whose partial derivatives f_x and f_y exists and have the value 0 at every point on the plane is constant.*
To see this,
Let (x,y) and (u,v) be any two arbitrary points on the plane and apply the above theorem to f. Then we have

$$f(u,v)-f(x,y)=(u-x)f_x(\eta,\varepsilon)+(v-y)f_y(\eta,\varepsilon)$$

for some (η, ε) *on the line-segment joining the points* (x,y) *and* (u,v)*. Since the partial derivatives are zero at every point, we see that*

$$f(x,y) = f(u,v)$$

for all $x, y, u, v \in \mathbb{R}$*. Therefore f is a constant function on* $\mathbb{R}^2$*.*

3.2 Cauchy's MVT for Functionals in Two Variables

Theorem 3.2.0.1. *For all real valued functions g and f in* $\mathbb{R}^2$ *with continuous partial derivatives* f_x, f_y, g_x, g_y *and for all distinct pairs* (x,y) *and* (u,v) *in* $\mathbb{R}^2$*, there exists an intermediate point* (η, ε) *on the line segment joining the points* (x,y) *and* (u,v) *such that*

$$[f(u,v) - f(x,y)][(u-x)g_x(\eta,\varepsilon) + (v-y)g_y(\eta,\varepsilon)]$$

$$= [g(u,v) - g(x,y)][(u-x)f_x(\eta,\varepsilon) + (v-y)f_y(\eta,\varepsilon)].$$

Proof. The proof is analogous to the one for the one variable case. We define an auxilliary function

$$\Psi(s,t) = [f(u,v) - f(s,t)][g(u,v) - g(x,y)] - [f(u,v) - f(x,y)][g(u,v) - g(s,t)]$$

Then $\Psi(u,v) = \Psi(x,y) = 0$, Ψ is differentiable wherever f and g are, hence by the mean value theorem for functions of two variables there is a point (η,ε) on the line segment joining (x,y) and (u,v), such that

$$(u-x)\Psi_s(\eta,\varepsilon)+(v-y)\Psi_t(\eta,\varepsilon)=0$$

carrying out the partial derivatives on Ψ yields

$$(u-x)\{g_x(\eta,\varepsilon)[f(u,v)-f(x,y)]-f_x(\eta,\varepsilon)[g(u,v)-g(x,y)]\}$$

$$+(v-y)\{f_y(\eta,\varepsilon)[g(u,v)-g(x,y)]-g_y(\eta,\varepsilon)[f(u,v)-f(x,y)]\}=0$$

which proves the results.

$\square$

Chapter 4

MEAN VALUE THEOREM FOR SOME GENERALISED DERIVATIVES

In this chapter, we shall study some mean value theorems and their generalizations when a function has generalized derivative such as symmetric derivative.

4.1 Symmetric Differentiaton of Real Function

The derivative of a real function f, that is a function from the real line $\mathbb{R}$ into itself, at a point x is given by the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

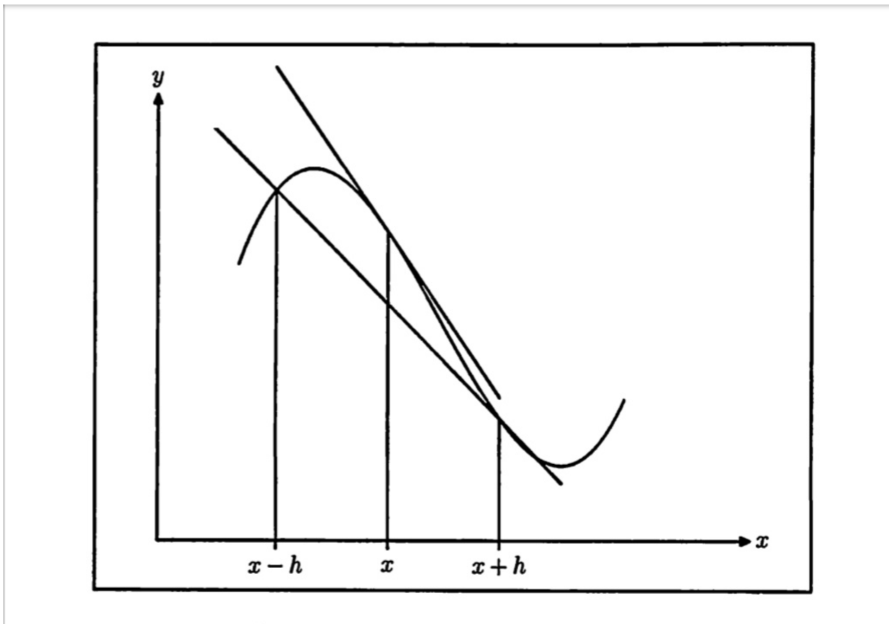

Figure 4.1: A Geometric Interpretation of the symmetric Derivative

Definition 4.1.0.1. A real function f on an interval (a,b) is said to be **symmetrically differentiable** at a point x in (a,b) if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. we shall denote this as $f^s(x)$. If a function is symmetrically differentiable at every point of an interval, then we say that it is symmetrically differentiable.

Theorem 4.1.0.2. *Every differentiable function is symmetrically differentiable.*

Proof. Let x be an arbitrary point. We would like to show that the symmetric derivative $f^s(x)$ exists. Since

$$\begin{aligned}
f^s(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{2h} + \lim_{h\to 0}\frac{f(x)-f(x-h)}{2h} \\
&= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) \\
&= f'(x)
\end{aligned}$$

□

However ,the converse is not true as seen in following example

Example: 4.1.0.3. *The function $f(x) = |x|$ is symmetrically differentiable at zero but it is not differentiable at zero. To see this evaluate the limits*

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

and

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

at the point $x = 0$. Evaluating the first limit, we get

$$
\begin{aligned}
f^{'(0)} &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0} \frac{|h|-|0|}{h} \\
&= \lim_{h\to 0} \begin{cases} \frac{h}{h} & \text{if } h \geq 0 \\ \frac{-h}{h} & \text{if } h < 0 \end{cases}
\end{aligned}
$$

$$
= \begin{cases} 1 & \text{if } h \geq 0 \\ -1 & \text{if } h < 0 \end{cases}
$$

Hence the limit does not exist at 0. Therefore f is not differentiable at 0. Now we examine the symmetric differentiablity of f. for this, we compute

$$
\begin{aligned}
f^{s}(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0-h)}{2h} \\
&= \lim_{h\to 0} \frac{|h|-|h|}{2h} \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}
$$

Hence the limit exists and is equal to 0. Therefore f is symmetrically differentiable at 0. The symmetric derivative of $f(x) = |x|$ is

$$
f^{s}(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
$$

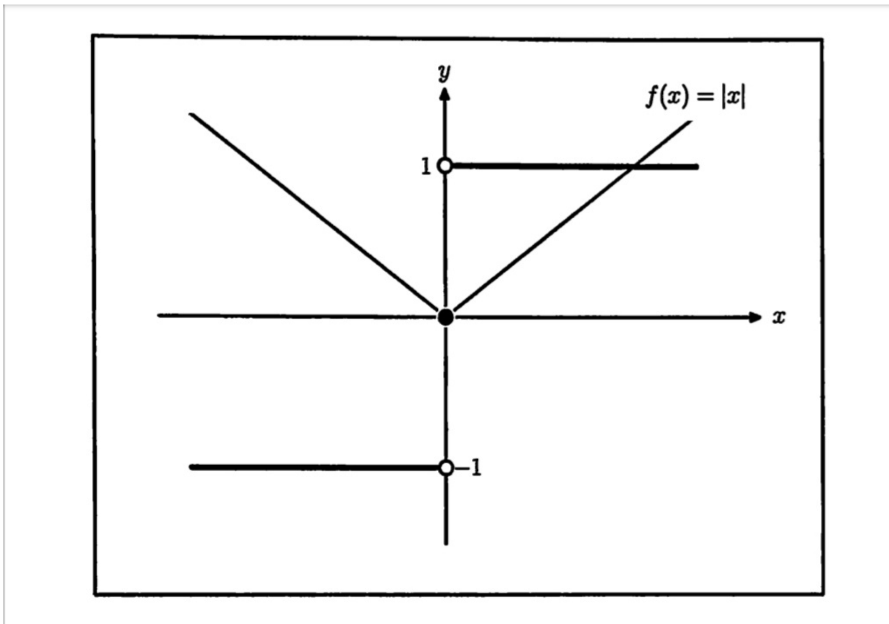

Figure 4.2: Graph of |x| and its symmetric derivative

4.2 A Quasi- Mean Value Theorem

In this section we establish a Quasi-mean value theorem for functions with symmetric derivatives. Further we will show that every continuous function whose symmetric derivative has the Darboux property obeys the ordinary mean value theorem of Lagrange.

Lemma 4.2.0.1. *Let f be continuous on the interval $[a,b]$ and let f be symmetrically diffrentiable on (a,b).If $f(b) > f(a)$, then there exists a point $\eta \in (a,b)$ such that*

$$f^s(\eta) \geq 0$$

Further, if $f(b) < f(a)$, then there exists a point η in (a,b) such that

$$f^s(\varepsilon) \leq 0$$

Proof. Suppose $f(b) > f(a)$. Let k be a real number such that $f(a) < k < f(b)$, The set

$$\{x \in [a,b]/f(x) > k\}$$

is bounded from below by a. Since it is a subset of $\mathbb{R}$ it has a greatest lower bound , say, η. Since f is continuous and k satisfies $f(a) < k < f(b)$, therefore η is different from a and b. Let $(\eta - h, \eta + h)$ be an arbitrary neighborhood of η in $[a,b]$. Since η is the greatest lower bound of the set $x \in [a,b]/f(x) > k$ there are points in $(\eta - h, \eta + h)$ such that

$$f(x+h) > k$$

and

$$f(x-h) \leq k$$

Therefore,

$$f^s(\eta) = \lim_{h \to 0} \frac{f(\eta + h) - f(\eta - h)}{2h} \geq 0$$

Similarly, it can be shown that if $f(a) > f(b)$, then there exists $\varepsilon \in (a,b)$ such that

$$f^s(\varepsilon) \leq 0.$$

The proof is now complete □

Theorem 4.2.0.2. *Let f be continuous on [a,b] and symmetrically differentiable on (a,b). Suppose $f(a) = f(b) = 0$. Then there exists η and ε in (a,b) such that*

$$f^s(\eta) \geq 0$$

and

$$f^s(\varepsilon) \leq 0$$

Proof. If $f \equiv 0$, then the theorem is obviously true. Since f is continuous and $f(a) = f(b) = 0$, there are points x_1 and x_2 such that

$$f(x_1) > \text{ and } f(x_2) < 0 \qquad (*)$$

or

$$f(x_1) < 0 \text{ and } f(x_2) > 0 \qquad (*')$$

or

$$f(x_1) > 0 \text{ and } f(x_2) > 0 \qquad (*'')$$

or

$$f(x_1) < 0 \text{ and } f(x_2) < 0 \qquad (*''')$$

if the inequalities in (*) are true, then we apply (Lemma 4.2.0.1) to f on the interval $[a, x_1]$ to obtain

$$f^s(\eta) \geq 0$$

for some $\eta \in (a,x_1) \subset (a,b)$.Again by applying (Lemma 4.2.0.1) to f on the interval$[a,x_2]$, we obtain

$$f^s(\varepsilon) \leq 0$$

for some $\varepsilon \in (a,x_2) \subset (a,b)$. The other cases can be handled in a similar manner and the proof of the theorem is now complete. □

Theorem 4.2.0.3. *Let f be continuous on [a,b] and symmetrically differentiable on (a,b). Then there exist η and ε in (a,b) such that*

$$f^s(\eta) \leq \frac{f(b)-f(a)}{b-a} \leq f^s(\varepsilon)$$

Proof. Define g by

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then $g(a) = g(b) = 0$ Applying the above theorem to g on the interval [a,b],we get

$$g^s(\eta) \leq 0 \text{ and } g^s(\varepsilon) \geq 0 \qquad (*'''')$$

From (*"") and the definition of g, we obtain

$$f^s(\eta) \leq \frac{f(b)-f(a)}{b-a} \leq f^s(\varepsilon)$$

The proof is now complete. □

Definition 4.2.0.4. A real valued function f defined on the interval $[a,b]$ is said to have a **Darboux property** if whenever η and ε in $[a,b]$, and y is any number between $f(\eta)$ and $f(\varepsilon)$, then there exists a number γ between η and ε such that $y = f(\gamma)$

Theorem 4.2.0.5. *Let f be continuous on $[a,b]$ and symmetrically differentiable on (a,b). If the symmetric derivative of f has the Darboux property, then there exits γ in (a,b) such that*

$$f^s(\gamma) = \frac{f(b)-f(a)}{b-a}$$

Proof. By the above theorem ,we obtain η and ε in (a,b) such that

$$f^s(\eta) \leq \frac{f(b)-f(a)}{b-a} \leq f^s(\varepsilon)$$

since $f^s(x)$ has the Darboux property, there exists γ in (a,b) such that

$$f^s(\gamma) = \frac{f(b)-f(a)}{b-a}$$

This completes the proof. □

4.3 An Application

Since symmetrically differentiable functions are not necessarily differentiable, some additional conditions should be imposed on the function to make it differentiable. We have seen that continuity of the function along with symmetric differentiability does not imply differentiability. In this section, we show that if $f(x)$ and $f^s(x)$ are both continuous, then f is differentiable.

Theorem 4.3.0.1. *Let $f(x)$ be continuous and symmetrically differentiable on (a,b), If the symmetric derivative of f is continuous on (a,b) then $f'(x)$ exists and*

$$f'(x) = f^s(x) \tag{4.1}$$

Proof. Choose h to be sufficiently small so that $a < x+h < b$. Since f^s is continuous, it has the Darboux property. Applying the mean value theorem to f on $[x, x+h]$, we have

$$f^s(\eta) = \frac{f(x+h) - f(x)}{h} \tag{4.2}$$

for some $\eta \in (x, x+h)$. Taking limit on both sides as $h \to 0$ and knowing that the limit of the right side exists, one obtains

$$f^s(x) = f'(x)$$

This completes the proof. □

Chapter 5

INTEGRAL MEAN VALUE THEOREMS AND RELATED TOPICS

As we have seen that the Mean value theorem of differential calculus is very important theorem another important theorem in calculus is the integral mean value theorem. The main goal of this chapter is to present the integral mean value theorem and some generalizations of this theorem. Also by using the integral mean value theorem we present integral representations of several means such as arithmetic, geometric,logarithmic and identric.

5.1 The Integral MVT and Generalizations

The mean value theorem for integrals is established using the second fundamental theorem of calculus, which states that if f(x) is continuous on an interval [a,b] and $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$, then $F'(x) = f(x)$ for all $x \in (a,b)$.

Theorem 5.1.0.1. *If f(x) is continuous on* $[x,y]$*, then there exists a point* ε *in* (x,y) *depending on x and y such that*

$$f(\varepsilon(x,y)) = \frac{\int_x^y f(t)dt}{y-x} \tag{5.1}$$

Proof. Let us define

$$F(z) = \int_a^z f(t)dt$$

where 'a' is a constant in (x,y) and $z \in (x,y)$. Since f is continuous on the interval $[x,y]$, the function F is also continuous on $[x,y]$ and F is also differentiable on the open (x,y). By the second fundamental theorem of calculus, we have $F'(x) = f(x)$.

Now we apply the mean value theorem of differentiable calculus to the function F. Then there exists a point ε in the interval (x,y) depending on x and y such that

$$\frac{F(y)-F(x)}{y-x} = F'(\varepsilon(x,y)),$$

that is

$$\frac{\int_a^y f(t)dt - \int_a^x f(t)dt}{y-x} = f(\varepsilon(x,y)).$$

Hence, we have the asserted statement

$$f(\varepsilon(x,y)) = \frac{\int_x^y f(t)dt}{y-x}$$

The proof is now complete.

□

Theorem 5.1.0.2. *If f and g are continuous on [a,b] and g is never zero on (a,b), then there exists a number ε in (a,b) depending on a and b such that*

$$f(\varepsilon(a,b)) = \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)dt}$$

Proof. We define two functions H(x) and G(x) in [a,b] by

$$H(x) = \int_a^x f(t)g(t)dt$$

$$G(x) = \int_a^x g(t)dt$$

consider the function

$$D(x) = \begin{vmatrix} H(x) & G(x) & 1 \\ H(a) & G(a) & 1 \\ H(b) & G(b) & 1 \end{vmatrix}$$

Since $G(a) = 0 = H(a)$, We get

$$D(x) = \begin{vmatrix} H(x) & G(x) & 1 \\ 0 & 0 & 1 \\ H(b) & G(b) & 1 \end{vmatrix}$$

Note that $D(a) = 0 = H(a)$. Applying Rolle's theorem to D(x) on the interval [a,b], we get

$$D'(\varepsilon) = 0$$

for some $\varepsilon \in (a,b)$ that is

$$\begin{aligned} 0 &= D'(\varepsilon) \\ &= \begin{vmatrix} H'(x) & G'(x) & 1 \\ 0 & 0 & 1 \\ H(b) & G(b) & 1 \end{vmatrix} \\ &= \begin{vmatrix} f(\varepsilon)g(\varepsilon) & g(\varepsilon) & 0 \\ 0 & 0 & 1 \\ H(b) & G(b) & 1 \end{vmatrix} \\ &= g(\varepsilon)[-f(\varepsilon)G(b) + H(b)] \end{aligned}$$

Hence

$$f(\varepsilon) = \frac{H(b)}{G(b)}$$

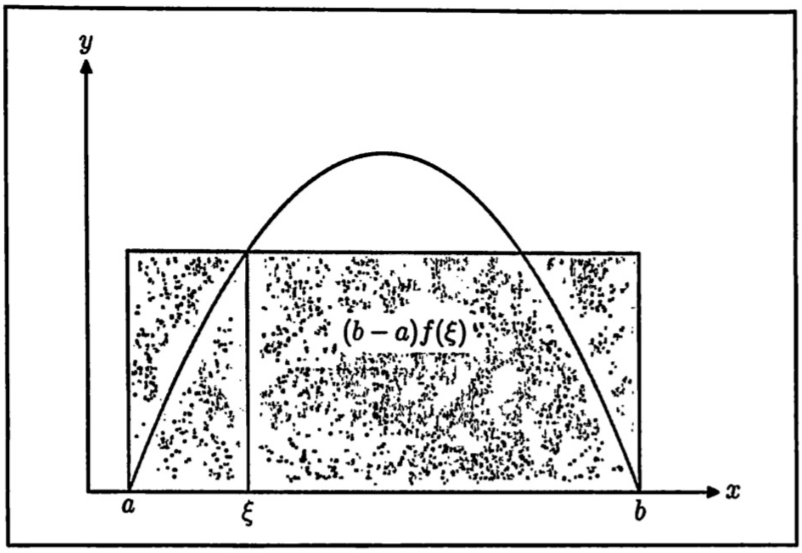

Figure 5.1: A Geometrical Illustration of the Integral Mean Value Theorem

that is

$$f(\varepsilon(a,b)) = \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)dt}$$

This completes the proof.

□

Definition 5.1.0.3. Let $f : [a,b] \in \mathbb{R}$ be a continuous function. If $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, then the set

$$P_n = \{x_0, x_1, \ldots, x_n\}$$

is called a partition of the interval $[a,b]$.

Definition 5.1.0.4. Given a partition $P_n = \{x_0, x_1, .., x_n\}$ of [a,b], the norm of P_n, denoted by $||P_n||$, is defined as the length of the subinterval of maximum length.

Definition 5.1.0.5. Given a partition $P_n = \{x_0, x_1, .., x_n\}$ of [a,b], let $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, .., n$. The Riemann sum for f over [a,b] is defined as

$$R_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ In the above Reimann sum, there are many ways of choosing the sample points $c_1, c_2, .., c_n$. We will choose these sample points in such a way that they divide the corresponding subintervals in a fixed proportion. In other words, for a partition $P_n = x_0, x_1, .., x_n$ there is a fixed number t in $[0, 1]$ such that

$$c_i = x_{i-1} + t\Delta x_i$$

for $i = 1, 2.., n$. If we choose the sample points $c_1, c_2, .., c_n$ in this manner then the above Reimann sum becomes a function of t for a fixed partition and we denote this by

$$\begin{aligned} R_n(t) &= \sum_{i=1}^{n} f(c_i)\Delta x_i \\ &= \sum_{i=1}^{n} f(x_{i-1} + t\Delta_i)\Delta x_i \end{aligned}$$

Theorem 5.1.0.6. *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Let $p_n = \{x_0, x_1, .., x_n\}$ be a fixed partition of [a,b]. For $t \in [0, 1]$, let the points be given by $c_i = x_{i-1} + t\Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$. Then there exists a point $\eta \in (0, 1)$ such that*

$$R_n(\eta) = \int_a^b f(x)dx$$

Proof. If $n = 1$, then this is just the integral mean value theorem. Suppose $n \geq 1$.

The function $R_n(t)$ is continuous on the interval $[0, 1]$ since the function $f(x_{i-1} + t\Delta x_i)$

are composites of the continuous function f(x) with the continuous function $x_{i-1}+t\Delta x_i$. Hence, the function $R_n(t)$ is integrable, and

$$\begin{aligned}
\int_0^1 R_n(t)dt &= \int_0^1 [\sum_{i=1}^{n} f(x_{i-1}+t\Delta x_i)\Delta x_i]dt \\
&= \sum_{i=1}^{n}\int_0^1 f(x_{i-1}+t\Delta x_i)\Delta x_i dt \\
&= \sum_{i=1}^{n}\int_{x_i}^{x_{i-1}} f(c_i)dc_i \\
&= \int_a^b f(x)dx
\end{aligned}$$

where $c_i = x_{i-1}+t\Delta x_i$. Now applying the integral mean value theorem to continuous function $R_n(t)$ on the interval $[0,1]$, we get

$$R_n(\eta) = \int_0^1 R_n(t)dt$$

for some $\eta \in (0,1)$. Since the right hand integral is equal to $\int_a^b f(x)dx$, we obtain

$$R_n(\eta) = \int_a^b f(x)dx$$

and the proof of the theorem is now complete.

□

Theorem 5.1.0.7. *(Bonnet's mean value theorem) Let f be integrable and g be nonnegative, increasing function on the closed interval $[a,b]$. Then there exists at least one point η in (a,b) such that*

$$\int_a^b f(t)g(t)dt = g(b)\int_\eta^b f(t)dt$$

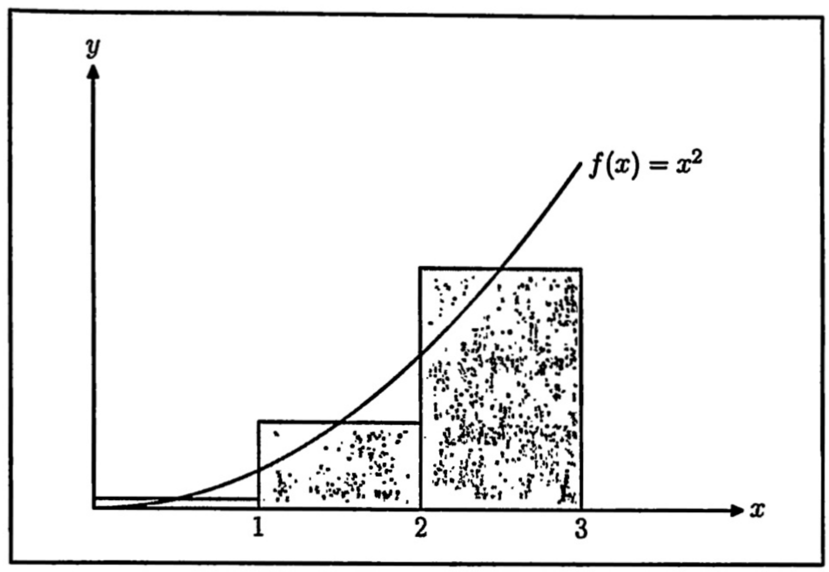

Figure 5.2: An Illustration of Theorem 5.1.0.6.

Proof. Let $P = a = t_0, t_1, .., t_n = b$ be an arbitrary partition of the interval [a,b]. The integral $\int_a^b f(t)g(t)dt$ can be approximated by the right sum

$$s(P) = \sum_{k=1}^{n} f(t_k)g(t_k)(t_k - t_{k-1}) \tag{5.2}$$

Let

$$S_m(P) = \sum_{k=m+1}^{n} f(t_k)(t_k - t_{k-1}) \tag{5.3}$$

The partial sum S_m approximates the integral $\int_{x_m}^b f(t)dt$. Let A and B be the upper and lower bounds of the integral $\int_x^b f(t)dt$ for each x in $[a,b]$,

that is

$$A \leq \int_x^b f(t)dt \leq B \tag{5.4}$$

for each x in [a,b].Let $A(P)$ and $B(P)$ be the lower and upper bounds of $S_m(P)$ for all $m = 1,2,..,n$. Hence

$$A(P) \le S_m(P) \le B(P) \tag{5.5}$$

for each m. From (5.3), We see that

$$f(t_k)(t_k - t_{k-1}) = S_{k-1}(P) - S_k(P) \tag{5.6}$$

Using (5.6) in (5.2) and simplifying, we get

$$S(P) = g(t_1)S_0(P) + [g(t_2) - g(t_1)]S_1(P) + .. + [g(t_{n-1}) - g(t_{n-2})]S_{n-2}(P) + g(b)S_{n-1}(P) \tag{5.7}$$

Since, g is nonnegative and increasing on $[a,b]$, We obtain from (5.7)

$$S(P) \le g(t_1)B(P) + [g(t_2) - g(t_1)]B(P) + .. + [g(t_{n-1}) - g(t_{n-2}]B(P) + g(b)B(P) \tag{5.8}$$

which is

$$S(P) \le g(b)B(P)$$

Similarly, from (5.7),we also have

$$S(P) \ge g(t_1)A(P) + [g(t_2) - g(t_1)]A(P) + .. + [g(t_{n-1}) - g(t_{n-2})]A(P) + g(b)A(P) \tag{5.9}$$

that is

$$S(P) \ge g(b)A(P)$$

Therefore, we obtain

$$g(b)A(P) \leq S(P) \leq g(b)B(P)$$

As the partition becomes finer, the sum S(P) tends to the integral $\int_a^b f(t)g(t)dt$, that is

$$\lim_{\|P_n\| \to 0} S(P) = \int_a^b f(t)g(t)dt$$

and

$$\lim_{\|P_n\| \to 0} A(P) = A$$

and

$$\lim_{\|P_n\| \to 0} B(P) = B$$

Hence, we have

$$g(b) \lim_{\|P\| \to 0} A(P) \leq \lim_{\|P\| \to 0} S(P) \leq \lim_{\|P\| \to 0} g(b)B(P)$$

that is

$$Ag(b) \leq \int_a^b f(t)g(t)dt \leq Bg(b)$$

Now for $x \in [a,b]$, we define $\Phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) = g(b) \int_x^b f(t)dt$$

Then, clearly ϕ is continuous on [a,b] and $Ag(b) \leq \phi(x) \leq Bg(b)$. Applying the intermediate value theorem to ϕ, we have

$$\phi(\eta) = \int_a^b f(t)g(t)dt$$

for some $\eta \in (a,b)$. This implies that there exists a point η in (a,b) such that

$$g(b)\int_\eta^b f(t)dt = \int_a^b f(t)g(t)dt$$

and the proof of the theorem is now complete.

□

5.2 Integral Representation of Means

As we have seen that Lagrange's mean value theorem of differential calculus can be used for generating various means between two positive real numbers a and b. In this section we illustrate how the mean value theorem of integral calculus can be used for finding the integral representation of various means.

Definition 5.2.0.1. A continuous function $M : \mathbb{R}_+^2 \to \mathbb{R}$ is said to be a mean of two numbers a and b if and only if

1. $min\{a,b\} \leq M(a,b) \leq max\{a,b\}$
2. $M(a,b) = M(b,a)$
3. $M(\lambda a, \lambda b) = \lambda M(a,b)$

for all $a, b \in \mathbb{R}_+$

The property (1) is called internality while (2) and (3) are called symmetry and homogeneity, respectively. The condition (1) is the absolutely essential part of definition. The condition (2) and (3) are often unnecessary. However, we will treat them to be equally important in defining a mean. Recall that the mean value theorem of the integral calculus says that if $f : [a,b] \to \mathbb{R}$ is a continous function, then there exists an intermediate value η in (a,b) depending on a and b such that

$$f(\eta(a,b)) = \frac{1}{b-a}\int_a^b f(x)dx \tag{5.10}$$

It is easy to see that $\eta(a,b)$ satisfies the property (1) and (2). Now by selecting an appropriate function f, which ensures the homogeneity condition on η, it is possible to generate various existing means. To construct the arithmetic mean between two positive numbers a and b consider the function $f(x) = x$. Then using the integral mean value theorem, we obtain

$$\eta(a,b) = f(\eta(a,b)) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{b-a}\int_a^b xdx = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2}$$

Thus, the arithmetic mean $A(a,b)$ of two positive numbers a and b has following integral representation

$$A(a,b) = \frac{1}{b-a}\int_a^b xdx \tag{5.11}$$

Likewise an integral representation of the geometric mean can be obtained as follows. Let $f(x) = \frac{1}{x^2}$ and compute

$$\begin{aligned}
\frac{1}{\eta^2(a,b)} &= f(\eta(a,b)) \\
&= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{b-a}\int_a^b \frac{1}{x^2}dx \\
&= \frac{1}{b-a}[\frac{-1}{x}]_a^b \\
&= \frac{1}{b-a}[\frac{1}{a}-\frac{1}{b}] \\
&= \frac{1}{ab}
\end{aligned}$$

Thus, we have

$$\eta^2(a,b) = ab$$

or

$$\eta(a,b) = \sqrt{ab}$$

This shows that the geometric mean, $G(a,b)$, has the following integral representation

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{1}{x^2}dx$$

Selecting $f(x) = \ln x$, we can find an integral representation for the identric mean, $I(a,b)$, between two positive numbers a and b. To see this compute

$$\begin{aligned}
\ln\eta(a,b) &= f(\eta(a,b)) \\
&= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{b-a}\int_a^b \ln x dx \\
&= \frac{1}{b-a}[x\ln x - x]_a^b \\
&= \frac{1}{b-a}[b\ln b - a\ln a - (b-a)] \\
&= \frac{blnb - a\ln a}{b-a} - 1 \\
&= \ln(\frac{b^b}{a^a})^{\frac{1}{b-a}} - \ln e \\
&= \ln(\frac{1}{e}[\frac{b^b}{a^a}]^{\frac{1}{b-a}})
\end{aligned}$$

Therefore

$$\eta(a,b) = \frac{1}{e}[\frac{b^b}{a^a}]^{\frac{1}{b-a}}$$

which is identric mean. Thus, the identric mean can be represented as follows

$$\ln(I(a,b)) = \frac{1}{b-a}\int_a^b \ln x dx$$

Note that the identric mean like other means is symmetric and homogenous of degree one. The identric mean, arithmetic mean and the geometric mean satisfy the following ordering

$$min\{a,b\} \leq G(a,b) \leq I(a,b) \leq A(a,b) \leq max\{a,b\}$$

Chapter 6

GENERALIZATIONS OF FLETT'S THEOREM

In this chapter we prove generalization of Flett's theorem and note its geometric interpretations. Several other mean value theorems extending further the result, which involves both real and complex functions, are also proved.

6.1 Flett's Theorem in $\mathbb{R}$

proposition 6.1.0.1. *If f(x) is differentiable in $a \leq x \leq b$ and $f'(a) = f'(b)$ then there exists a point η $a < \eta < b$ such that*

$$f'(\eta) = \frac{f(\eta) - f(a)}{\eta - a}. \tag{6.1}$$

Proof. Without loss of generality, we shall assume that $f'(a) = f'(b) = 0$. If this is not the case we work with $f(x) - xf'(a)$. Consider the function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \in (a,b] \\ f'(a) & \text{if } x = a \end{cases} \tag{6.2}$$

Evidently g is continuous on $[a,b]$ and differentiable on $(a,b]$. Further, from (6.2) we have

$$g'(x) = -\frac{f(x)-f(a)}{(x-a)^2} + \frac{f'(x)}{x-a}$$

which is

$$g'(x) = -\frac{g(x)}{x-a} + \frac{f'(x)}{x-a} \tag{6.3}$$

for all $x \in (a,b]$.In view of (6.2), to establish the theorem we have to show that there exists a point $\eta \in (a,b)$ such that $g'(\eta) = 0$. From (6.2), we see that $g(a) = 0$. If $g(b) = 0$, then by Rolle's theorem there exists $\eta \in (a,b)$ such that $g'(\eta) = 0$ and the theorem is established. If $g'(b) \neq 0$, then either $g(b) > 0$ or $g(b) < 0$. Suppose $g(b) > 0$. Then from (6.3), we see that

$$g'(b) = -\frac{g(b)}{b-a} < 0$$

Since g is continuous and $g'(b) < 0$,there exists a point $x_1 \in (a,b)$ such that

$$g(x_1) > g(b)$$

Hence, we have $g(a) < g(b) < g(x_1)$ and by intermediate value theorem there exits a $x_0 \in (a,x_1)$ such that $g(x_0) = g(b)$. Now applying the Rolle's theorem to the function g

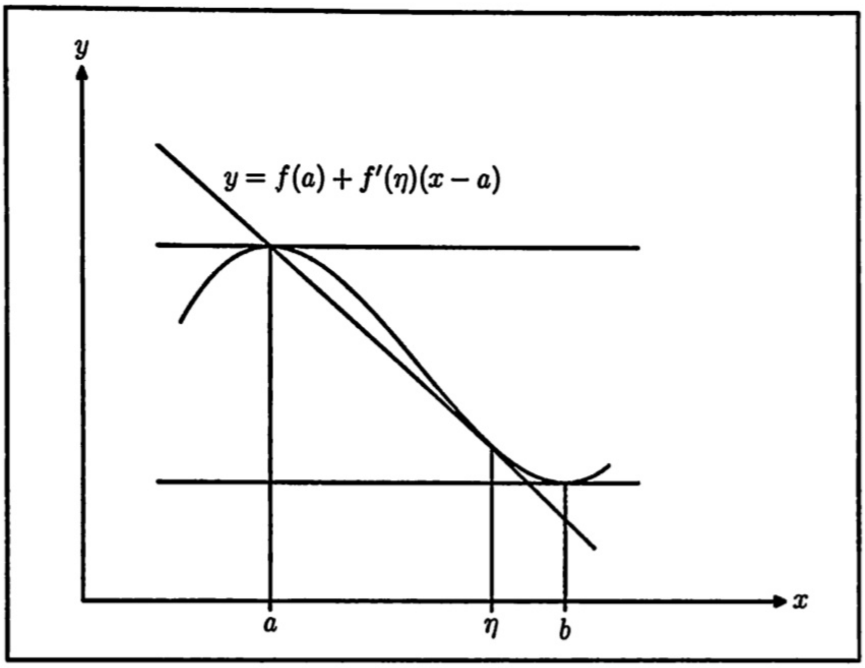

Figure 6.1: A Geometrical Illustration of Flett's Theorem

on the interval $[x_0, b]$ we have $g'(\eta) = 0$ for some $\eta \in (a, b)$.

A similar argument applies if $g(b) < 0$, and now the proof of the theorem is complete.

$\square$

Flett's theorem is in reallity a statement about the existence of two points: under the aforementioned assumptions, there exist ε, $\eta \in (a, b)$ *for which*

$$f'(\varepsilon) = \frac{f(\varepsilon) - f(a)}{\varepsilon - a}, \quad f'(\eta) = \frac{f(b) - f(\eta)}{b - \eta}.$$

Saho and Riedel developed further Flett's result by removing the boundary condition on the derivatives, proving the following:

proposition 6.1.0.2. *(Generalized Flett's mean value theorem)*
Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable. Then $\exists\, \varepsilon \in (a,b)$, such that

$$f'(\varepsilon) = \frac{f(\varepsilon) - f(a)}{\varepsilon - a} + \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(\varepsilon - a). \tag{6.4}$$

Proof. Defining an auxillary function $\psi : [a,b] \to \mathbb{R}$ as

$$\Psi(x) = f(x) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(x - a)^2$$

We see that ψ is differentiable on [a,b] and

$$\psi'(x) = f'(x) - \frac{f'(b) - f'(a)}{b - a}(x - a)$$

here note that,

$$\psi'(a) = \psi'(b) = f'(a).$$

Applying Flett's mean value theorem to ψ, we get.

$$\psi(\varepsilon) - \psi(a) = (\varepsilon - a)\psi'(\varepsilon)$$

for some $\varepsilon \in$ (a,b). Using the definition of auxillary function we get the asserted result.

$$f(\varepsilon)-f(a)=(\varepsilon-a)f'(\varepsilon)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\varepsilon-a)^2$$

$\square$

proposition 6.1.0.3. *Let $f,g:[a,b]\to\mathbb{R}$ be two differentiable functions. Suppose that $g'(x)\neq 0$ for $x\in[a,b]$, and*

$$\frac{f'(a)}{g'(a)}=\frac{f'(b)}{g'(b)}.$$

Then $\exists\varepsilon\in(a,b)$, *such that*

$$\frac{f(\varepsilon)-f(a)}{g(\varepsilon)-g(a)}=\frac{f'(\varepsilon)}{g'(\varepsilon)} \tag{6.5}$$

Theorem 6.1.0.4. *(A Cauchy-type of Flett's theorem):*

Let $f,g:I\supset[a,b]\to\mathbb{R}$ be differentiable, and suppose that

$$f'(a)g'(b)=f'(b)g'(a) \tag{6.6}$$

then $\exists$ $\varepsilon\in$ (a,b),such that

$$g'(a)[f'(\varepsilon)-\frac{f(\varepsilon)-f(a)}{\varepsilon-a}]=f'(a)[g'(\varepsilon)-\frac{g(\varepsilon)-g(a)}{\varepsilon-a}] \tag{6.7}$$

Proof. consider the function $\phi(x)=g'(a)f(x)-f'(a)g(x)$. Suppose first that f'(a) and g'(a) are not both zero.Then (6.4) becomes

$$f'(a)g'(b) = f'(b)g'(a),$$

and

$$\phi(x) = g'(a)f(x) - f'(a)g(x).$$

A direct calculation shows that

$$\phi'(a) = 0 = \phi'(b),$$

and so by Flett's theorem, there exists $\varepsilon \in$ (a,b) such that

$$\phi'(\varepsilon) = \frac{\phi(\varepsilon) - \phi(a)}{\varepsilon - a},$$

or

$$g'(a)f'(\varepsilon) - f'(a)g'(\varepsilon) = \frac{g'(a)f(\varepsilon) - f'(a)g(\varepsilon) - g'(a)f(a) + f'(a)g(a)}{\varepsilon - a},$$

which is easily rearranged to give

$$g'(a)[f'(\varepsilon) - \frac{f(\varepsilon) - f(a)}{\varepsilon - a}] = f'(a)[g'(\varepsilon) - \frac{g(\varepsilon) - g(a)}{\varepsilon - a}]$$

as desired.

In the case when $f'(a) = 1 = g'(a)$,(6.6) becomes g'(b)=f'(b),and $\phi(x) = f(x) - g(x)$.

Again we have $\phi'(a) = 0$ $\Phi'(b) = 0$,and Flett's theorem applies:there exists $\varepsilon \in (a,b)$, such that

$$\phi'(\varepsilon) = \frac{\phi(\varepsilon) - g(\varepsilon)}{\varepsilon - a},$$

which now is

$$f'(\varepsilon) - g'(\varepsilon) = \frac{f(\varepsilon) - g(\varepsilon) - f(a) + g(a)}{\varepsilon - a}$$

or

$$f'(\varepsilon) - \frac{f(\varepsilon) - f(a)}{\varepsilon - a} = g'(\varepsilon) - \frac{g(\varepsilon) - g(a)}{\varepsilon - a}$$

$\square$

Example: 6.1.0.5. *Take $f(x) = sinx$ and $g(x) = cosx$ on $[\frac{\pi}{4}, \frac{5\pi}{4}]$. Standard calculations show that (6.4) is met, and (6.5) becomes*

$$sin\varepsilon + cos\varepsilon - \sqrt{2} = (cos\varepsilon - sin\varepsilon)(\varepsilon - \frac{\pi}{4})$$

or

$$\big(1 + \varepsilon - \frac{\pi}{4}\big)sin\varepsilon + \big(1 - \varepsilon + \frac{\pi}{4}\big) = \sqrt{2}$$

Consider the function $Q(x) = (1 + x - \frac{\pi}{4})sinx + (1 - x + \frac{\pi}{4})cosx - \sqrt{2}$
Since $Q(\frac{\pi}{2}) = 1 + \frac{\pi}{4} - \sqrt{2} > 0$ *and* $Q(\pi) = \frac{3\pi}{4} - 1 - \sqrt{2} < 0$, *there is a zero* ε *between* $\frac{\pi}{2}$ *and* π, *which is the point guaranteed by the Theorem 1.*

Theorem 6.1.0.6. *Let* $f, g : I \supset [a,b] \to \mathbb{R}$ *be differentiable. Then* $\exists \varepsilon \in (a,b)$, *such that*

$$g'(a)[f'(\varepsilon) - \frac{f(\varepsilon) - f(a)}{\varepsilon - a}] - f'(a)[g'(\varepsilon) - \frac{g(\varepsilon) - g(a)}{\varepsilon - a}] = \frac{1}{2}\frac{f'(b)g'(a) - f'(a)g'(b)}{b - a}(\varepsilon - a) \tag{6.8}$$

Proof. As in the proof of theorem 1,consider the function
$\phi(x) = g'(a)f(x) - f'(a)g(x)$, and apply the generalized Flett's mean value theorem (proposition 6.1.0.2) $\exists \varepsilon \in (a,b)$,such that

$$\frac{\phi(\varepsilon) - \phi(a)}{\varepsilon - a} - \phi'(\varepsilon) = \frac{1}{2}\frac{\phi'(b) - \phi'(a)}{b - a}(\varepsilon - a).$$

this gives

$$\frac{g'(a)f(\varepsilon) - f'(a)g(\varepsilon) + g'(a)f(a) + f'(a)g(a)}{\varepsilon - a} - g'(a)f'(\varepsilon) + f'(a)g'(\varepsilon)$$

$$= -\frac{1}{2}\frac{g'(a)f'(b) - f'(a)g'(b) - g'(a)f'(a) + f'(a)g'(a)}{b - a}(\varepsilon - a)$$

$$= -\frac{1}{2}\frac{g'(a)f'(b) - f'(a)g'(b)}{b-a}(\varepsilon - a)$$

$\square$

theorem (6.1.0.4) does not hold true for complex functions.

Example: 6.1.0.7. *Let* $f(s) = e^s$, $g(s) = s^3 - 3\pi i s^2 - 3s$,*on* $[0, 2\pi i]$. *condition (6.4) is met, and we need to show that the equation*

$$(-3)\left[\frac{e^s - 1}{s} - e^s\right] = \frac{s^3 - 3\pi i s^2 - 3s}{s} - (3s^2 - 6\pi i s - 3) \tag{6.9}$$

has no solution in $(0, 2\pi i)$. *The last equation simplifies to*

$$se^s - e^s + 1 = -\frac{2}{3}s^3 + \pi i s^2, \tag{6.10}$$

and since $s = iy$*, it is reduced to*

$$1 - cosy - ysiny + i(ycosy - siny) = i\left[\frac{2}{3}y^3 - \pi y^2\right] \tag{6.11}$$

thus, we must consider the system

$$1 - cosy - ysiny = 0, \quad ycosy - siny = \frac{2}{3}y^3 - \pi y^2. \tag{6.12}$$

The first of these gives

$$y = \frac{1-cosy}{siny} = tan(\frac{y}{2}).$$

A standard analysis of the function $y \to y - tan(\frac{y}{2})$*, defined on* $(0, 2\pi)$*, shows that it has a single zero* $\tau \in (\frac{\pi}{2}, \frac{3\pi}{2})$*.(its approximate numerical value is* $\tau = 2.33112$*) on the other hand, substituting* $y = \frac{1-cosy}{siny}$ *on the left-hand side of the second equation in (4) gives*

$$\frac{1-cosy}{siny}cosy - siny = \frac{(1-cosy)cosy - sin^2 y}{siny} = \frac{cosy-1}{siny} = -y \qquad (6.13)$$

so the equation becomes $\frac{2}{3}y^3 - \pi y^2 + y = 0$*. Its solutions are as follows:*

$$y = \frac{3\pi \pm \sqrt{9\pi^2 - 24}}{4}$$

none of which is in $(\frac{\pi}{2}, \frac{3\pi}{4})$*. Thus, the system (4) has no solution in* $(0, 2\pi)$*.*

Theorem 6.1.0.8. *(Complex Rolle's Theorem) Let f be a holomorphic function defined on an open convex subset* $\mathscr{D}_f$ *of* $\mathbb{C}$*.Let a,b* $\in \mathscr{D}_f$ *be such that* $f(a) = f(b) = 0$ *and* $a \neq b$*. Then there exists* $z_1, z_2 \in (a, b)$ *such that* $\Re(f'(z_1)) = 0$ *and* $\Im(f'(z_2)) = 0$*.*

Proof. Let $a_1 = \mathscr{R}(a)$, $a_2 = \mathscr{I}(a)$, $b_1 = \mathscr{R}(b)$, $b_2 = \mathscr{I}(b)$ and let $u(z) = \mathscr{R}(f(z))$ and $v(z) = \mathscr{I}(f(z))$ for $z \in \mathscr{D}_f$. We define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = (b_1 - a_1)u(a + t(b-a)) + (b_2 - a_2)v(a + t(b-a))$$

for every $t \in [0,1]$. Since $f(a) = f(b)$, we get

$$u(a) = u(b) = v(a) = v(b) = 0$$

Consequently, $\phi(0) = 0 = \phi(1)$. Applying Rolle's theorem to ϕ on [0,1], we obtain

$$\phi'(t_1) = 0$$

for some $t_1 \in (0,1)$. Letting $z_1 = a + t_1(b-a)$, we get from the above equation

$$(b_1 - a_1)[(b_1 - a_1)\frac{\partial u(z_1)}{\partial x} + (b_2 - a_2)\frac{\partial u(z_1)}{\partial y}]$$
$$+(b_2 - a_2)[(b_1 - a_1)\frac{\partial v(z_1)}{\partial x} + (b_2 - a_2)\frac{\partial v(z_1)}{\partial y}] = 0$$

By the Cauchy-Riemann equations, that is

$$\frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$$

and

$$\frac{\partial u(z)}{\partial y} = -\frac{\partial v(z)}{\partial x}$$

it follows that

$$\frac{\partial u(z_1)}{\partial x}[(b_1 - a_1)^2 + (b_2 - a_2)^2] = 0$$

Thus

$$\frac{\partial u(z_1)}{\partial x} = 0$$

which is

$$\mathscr{R}(f'(z_1)) = 0$$

By applying this first part of the theorem to the function $g = -if$ we see that there exists a $z_2 \in (a,b)$ such that

$$0 = \mathscr{R}(g'(z_2)) = \frac{\partial v(z_2)}{\partial x} = -\frac{\partial u(z_2)}{\partial y} = \mathscr{I}(f'(z_2))$$

This completes the proof of the theorem.

$\square$

Theorem 6.1.0.9. *(Complex Mean Value Theorem) Let f be a holomorphic function defined on an open convex subset $\mathscr{D}_f$ of $\mathbb{C}$. Let a and b be two distinct points in $\mathscr{D}_f$ of $\mathbb{C}$.Then there exists $z_1, z_2 \in$ (a,b) such that*

$$\Re(f'(z_1)) = \Re(\frac{f(b) - f(a)}{b-a})$$

and

$$\Im(f'(z_2)) = \Im(\frac{f(b) - f(a)}{b-a})$$

Proof. Let

$$g(z) = f(z) - f(a) - \frac{f(b) - f(a)}{b-a}(z-a)$$

for every $z \in \mathscr{D}_f$. Obviously $g(a) = g(b) = 0$. By the above theorem, there exists $z_1, z_2 \in (a,b)$ such that

$$\mathscr{R}(g'(z_1)) = 0$$

and

$$\mathscr{I}(g'(z_2)) = 0$$

Using the definition of g, we get

$$g'(z) = f'(z) - \frac{f(b) - f(a)}{b - a}$$

for every $z \in \mathscr{D}_f$. Therefore

$$0 = \mathscr{R}(g'(z_1)) = \mathscr{R}(f'(z_1)) - \mathscr{R}(\frac{f(b) - f(a)}{b - a})$$

and

$$0 = \mathscr{I}(g'(z_2) = \mathscr{I}(f'(z_2)) - \mathscr{I}(\frac{f(b) - f(a)}{b - a})$$

The proof of the theorem is now complete. □

Corollary 6.1.0.10. *Let f be a holomorphic function defined on an open connected subset $\mathscr{D}_f$ of $\mathbb{C}$ such that $f'(z) = 0$ for every $z \in \mathscr{D}_f$. Then f is constant.*

Proof. Let $f(z) = u(x,y) + iv(x,y)$ Then

$$\begin{aligned} f'(z) &= u_x(x,y) + iv_x(x,y), \quad z \in \mathscr{D}_f \\ &= 0 + i(0), \quad z \in \mathscr{D}_f \end{aligned}$$

$\therefore u_x(x,y) = 0$ and $v_x(x,y) = 0, \quad \forall z \in \mathscr{D}_f \therefore$ u and v are independent of x in $\mathscr{D}_f$ Also, since f is differentiable in $\mathscr{D}_f$, hence C-R equations are satisfied.

$\therefore u_x = v_y = 0$ and $u_y = -v_x = 0 \quad \forall\, z \in \mathscr{D}_f$

$\therefore$ u and v are independent of y in $\mathscr{D}_f$

$\therefore$ u and v are constant in $\mathscr{D}_f$

$\therefore f(z)$ is constant in $\mathscr{D}_f$ $\square$

Example: 6.1.0.11. *Let $f(z) = e^z - 1$ and note that $f(z) = 0$ for $z = 2k\pi i$ for every integer k. Since $f'(z) = e^z = e^x cosy + ie^x siny$, $\Re(f'(z)) = 0$ if $y = (2k+1)\frac{\pi}{2}$ and $\Im(f'(z)) = 0$ if $y = k\pi$.Therefore the zeros of the real and imaginary parts of f' are straight lines both separating the zeros of f.*

6.2 A Cauchy type of extension for Flett's theorem in $\mathbb{C}$

Theorem 6.2.0.1. *(A complex Cauchy-type extension of Flett's theorem) Let $s = x + iy$ be a complex variable and let $f(s)$ and $g(s)$ be two holomorphic functions defined on the open convex set $\mathscr{D} \in \mathbb{C}$. Suppose $A = a_1 + ia_2 \in \mathscr{D}$, $B = b_1 + ib_2 \in \mathscr{D}$ and denote by (A,B) the open segment connecting A and B.Suppose that*

$$f'(A)g'(B) = f'(B)g'(A) \tag{6.14}$$

then $\exists z_1, z_2 \in (A,B)$ such that

$$\Re\{g'(A)\left[f'(z_1) - \frac{f(z_1) - f(A)}{z_1 - A}\right]\} = \Re\{f'(A)\left[g'(z_1) - \frac{g(z_1) - g(A)}{z_1 - A}\right]\} \tag{6.15}$$

$$\Im\{g'(A)[f'(z_2) - \frac{f(z_2) - f(A)}{z_2 - A}]\} = \Im\{f'(A)[g'(z_2) - \frac{g(z_2) - g(A)}{z_2 - A}]\} \tag{6.16}$$

Proof. Let $\phi(s) = g'(A)f(s) - f'(A)g(s)$. As mentioned earlier, the condition (6.14) means that $\phi(s) = g'(A)f(s) - f'(A)g(s)$ if f'(A) and g'(A) are not both zero, and

$\phi(s) = f(s) - g(s)$ if $f'(A) = 1 = g'(A)$ as in Real case. we have $\phi'(A) = \phi'(B)$. there exist $z_1, z_2 \in (A, B)$
such that,

$$\Re[\phi'(z_1)] = \Re[\frac{\phi(z_1) - \phi(A)}{z_1 - A}], \quad \Im[\phi'(z_2)] = \Im[\frac{\phi(z_2) - \phi(A)}{z_2 - A}]$$

or

$$\Re\{g'(A)f'(z_1) - f'(A)g'(z_1)\}$$
$$= \Re\{\frac{g'(A)f(z_1) - f'(A)g(z_1) - g'(A)f(A) + f'(A)g(A)}{z_1 - A}\}$$
$$\Im\{g'(A)f'(z_2) - f'(A)g'(z_2)\}$$
$$= \Im\{\frac{g'(A)f(z_2) - f'(A)g(z_2) - g'(A)f(A) + f'(A)g(A)}{z_2 - A}\}$$

since $\Re\{\alpha \pm \beta\} = \Re\{\alpha\} \pm \Re\{\beta\}$ and $\Im\{\alpha \pm \beta\} = \Im\{\alpha\} \pm \Im\{\beta\}$ for any α, $\beta \in \mathbb{C}$,the last two relations can be rearranged, obtaining (6.15) and (6.16)

□

Example: 6.2.0.2. *(revisited complex case) Returning to our example and setting $s = iy$ in (6.10), we get*

$$(-3)(\frac{e^{iy} - 1}{iy} - e^{iy}) = -2(iy)^2 + 3\pi i(iy)$$

$$3[cosy - \frac{siny}{y} + i(\frac{cosy - 1}{siny} + siny)] = 2y^2 - 3\pi y.$$

equating the real and imaginary parts, respectively,gives

$$cosy - \frac{siny}{y} = \frac{2}{3}y^2 - \pi y \tag{6.17}$$

and

$$\frac{cosy-1}{siny}+siny=0 \tag{6.18}$$

For (9)*,we consider the function* $R(y)=cosy-\frac{siny}{y}-\frac{2}{3}y^2+\pi y$ *and note that* $R(\frac{\pi}{2})>0$*, while* $R(2\pi)<0$*.Thus, R has a zero in* $(\frac{3\pi}{2},2\pi)$*.Its numerical value is approximately 4.806302. The solution to (10) as mentioned earlier is approximately 2.33112.Thus, the two complex solutions assured by theorem 3 are* $z_1=4.806302i$ *and* $z_2=2.33112i$*.*

Theorem 6.2.0.3. *(A cauchy-type extension of the generalised Flett's theorem) Let* $s=x+iy$ *be a complex variable and let f(s) and g(s) be two holomorphic functions defined on the open convex set* $\mathscr{D}\in\mathbb{C}$*. Suppose* $A=a_1+ia_2\in\mathscr{D}$*,* $B=b_1+ib_2\in\mathscr{D}$*. and denote by* (A,B) *the open segment connecting A and B.then* $\exists z_1,z_2\in(A,B)$ *such that*

$$\Re\{g'(A)[f'(z_1)-\frac{f(z_1)-f(A)}{z_1-A}]\}-\Re\{f'(A)[g'(z_1)-\frac{g(z_1)-g(A)}{z_1-A}]\}$$
$$=\frac{1}{2}\frac{\Re\{f'(B)g'(A)-f'(A)g'(B)\}}{B-A}(z_1-A)$$

(*)

$$\Im\{g'(A)[f'(z_2)-\frac{f(z_2-f(A))}{z_2-A}]\}-\Im\{f'(A)[g'(z_2)-\frac{g(z_2)-g(A)}{z_2-A}]\}$$
$$=\frac{1}{2}\frac{\Im\{f'(B)g'(A)-f'(A)g'(B)\}}{B-A}(z_2-A)$$

(**)

Proof. If a holomorphic function $\phi(s)$ satisfies the assumptions then there exists $z_1, z_2 \in (A,B)$ such that

$$\Re\{\phi'(z_1)\} = \Re\{\frac{\Phi(z_1)-\phi(A)}{z_1-A} + \frac{1}{2}\frac{\phi'(B)-\phi'(A)}{B-A}(z_1-A)\} \tag{6.19}$$

$$\Im\{\phi'(z_2)\} = \Im\{\frac{\phi(z_2)-f(A)}{z_2-A} + \frac{1}{2}\frac{\phi'(B)-\phi'(A)}{B-A}(z_2-A)\} \tag{6.20}$$

The method of proof goes back to the paper by Edvard and Jaffari who discovered the true form of complex Rolle's theorem. The idea is to consider the real functions

$$\phi(t) = (b_1-a_1)\Re[\phi(A+t(B-A))] + (b_2-a_2)\Im[\phi(A+t(B-A))]$$

$$\psi(t) = (b_1-a_1)\Im[\phi(A+t(B-A))] - (b_2-a_2)\Re[\phi(A+t(B-A))]$$

$t \in [0,1]$ and to differentiate them, making use of the Cauchy-Riemann equations. Applying the generaliized Flett's theorem to ϕ and ψ guarantees the existence of two real numbers $t_1, t_2 \in (0,1)$ such that

$$\phi'(t_1) = \frac{\phi(t_1)-\phi(0)}{t_1} + \frac{1}{2}[\phi'(1)-\phi'(0)]t_1$$

$$\psi'(t_2) = \frac{\psi(t_2-\psi(0))}{t_2} + \frac{1}{2}[\psi'(1)-\psi'(0)]t_2$$

Then the required equations (*) and (**) follow up setting $z_1 = A + t_1(B-A)$ and $z_2 = A + t(B-A)$. Note that (6.19) and (6.20) can be rewritten as follows:

$$\Re\{\phi'(z_1) - \frac{\phi(z_1)-\phi(A)}{z_1-A}\} = \frac{1}{2}\frac{\Re\{\phi'(B)-\phi'(A)\}}{B-A}(z_1-A),$$

$$\Im\{\phi'(z_2) - \frac{\phi(z_2)-\phi(A)}{z_2-A}\} = \frac{1}{2}\frac{\Im\{\phi'(B)-\phi'(A)\}}{B-A}(z_2-A)$$

now we set $\phi(s) = g'(A)f(s) - f'(A)g(s)$ and apply the aforementioned, from required equation (*) and (**) are straight-forward manupulations.

□

With Reference to the equation $\phi(x) = g'(a)f(x) - f'(a)g(x)$ mentioned in theorem discussed earlier, we see the following result

Theorem 6.2.0.4. *Let $f,g : I \subset [a,b] \to \mathbb{R}$ be differentiable. Then $\exists\, \varepsilon \in$ (a,b), such that*

$$g'(a)[f'(\varepsilon) - \frac{f(\varepsilon)-f(a)}{\varepsilon-a}] - f'(a)[g'(\varepsilon) - \frac{g(\varepsilon)-g(a)}{\varepsilon-a}]$$

$$= \frac{1}{2}\frac{f'(b)g'(a)-f'(a)g'(b)}{b-a}(\varepsilon-a)$$

and by applying a result of Pawlikowska to above theorem

Theorem 6.2.0.5. *(Pawlikowska Result) Let $f : [a,b] \to \mathbb{R}$ be an n times differentiable function. Then there exists a point $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = \sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k!}f^{(k)}(\eta)(\eta-a)^k$$

$$+(-1)^n\frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(\eta-a)^{n+1}.$$

Theorem 6.2.0.6. *Theorem 5 Let $f,g : I \in [a,b] \to \mathbb{R}$ be n-times differentiable. then $\exists$ $\varepsilon \in (a,b)$ such that*

$$g'(a)[f(\varepsilon)-f(a)]-f'(a)[g(\varepsilon)-g(a)]$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}[g'(a)f^{(k)}(\varepsilon)-f'(a)g^{(k)}(\varepsilon)](\varepsilon-a)^k$$

$$+(-1)^n\frac{1}{(n+1)!}\frac{g'(a)[f^{(n)}(b)-f^{(n)}(a)]-f'(a)[g^{(n)}(b)-g^{(n)}(a)]}{b-a}(\varepsilon-a)^{n+1}$$

Chapter 7

ANALYSIS AND CONCLUSIONS

The Mean value theorem was originated from Rolle's theorem. The only difference is the condition on the end points. i.e. in Mean value theorem for a function defined and continuous on $[a,b]$ and differentiable on (a,b), the condition $f(a) = f(b)$ is not required which is necessary for Rolle's theorem.

Geometrically the Mean value theorem means that, the tangent line to the graph of function f at $\eta(x_1,x_2)$ is parallel to the secant line joining the points $(x_1, f(x_1))$. This important result is further applied to further obtain small results like:

1. If $f'(x) = 0$ for all x in (a,b), then f is a constant on $[a,b]$
2. If $f'(x) = g'(x)$ for all x in (a,b), then f and g differ by a constant on [a,b].
3. If $f'(x) > 0 (< 0)$ for all x in (a,b), then f is a strictly increasing (decreasing) function on [a,b]

Further it can also be used to prove Bernoulli's inequality etc.

Mean value theorem and related functional equations

In this section we illustrated functional equations arising from Mean value theorem. and then we presented systematic study of functional equations and its various generalizations.

1.) In view of divided difference the mean value theorem takes the form

$$f[x_1,x_2] = f'(\eta(x_1,x_2))$$

By using the equation of divided difference we learnt how various results of functional equations were obtained.

2.) The integral Representation of divided difference is given by

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} f^{(n)}\big(x_0 + \sum_{k=1}^{n} t_k(x_k - x_{k-1})\big) dt_n$$

$$= f[x_0,x_1,...,x_n]$$

where $n \geq 1$

and

$$min\{x_0,x_1,..,x_n\} \leq x \leq max\{x_0,x_1,..,x_n\}$$

If all the points $x_0,x_1,..,x_n$ are all distinct.

3.) By the mean value theorem for divided differences, there exists one point $\eta \in [a,b]$ such that,

$$f[b^{[n]},a^{[n]}] = \frac{f^{(2n-1)}(\eta)}{(2n-1)!}$$

Strict monotonicity of $f^{(2n-1)}(x)$ forces η to be a mean value, that is, $a < \eta < b$. Further, since $f^{(2n-1)}(x)$ is strictly monotone, such a η is also unique, and this defines a functional mean $M_f^n(a,b)$ in a and b. Hence

$$M_f^n(a,b) = (f^{(2n-1)})^{-1}\{(2n-1)!f[b^{[n]},a^{[n]}]\}.$$

Note that in the above formula $(f^{(2n-1)})^{-1}$ denotes the inverse function of $f^{(2n-1)}$. If $n = 1$, then $M_f^n(a,b)$ reduces

$$M_f^n(a,b) = (f')^{-1}(\frac{f(b)-f(a)}{b-a})$$

If $f(x) = x^m$, where m is a positive integer greater than or equal to n, then $M_f^n(a,b) = \frac{a+b}{2}$.

and

If $f(x) = \frac{1}{x}$, then $M_f^n(a,b) = \sqrt{ab}$

Hence we can conclude that if $f(x) = x^p$, where $p \in \mathbb{R}$. The first example shows that, p is a positive integer greater than or equal to n, then the functional mean $M_f^n(a,b)$ is the arithmetic mean of a and b. The second example illustrates that if $p = -1$, then the functional mean $M_f^n(a,b)$ is the geometric mean of a and b.

4.) Under Limiting Behavior of mean values we have learnt that, If x is a number in the interval (a,b) then by applying Langrange's mean value theorem to the interval $[a,x]$, it is possible to choose a number η_x in (a,x) as a function of x such that

$$f[a,x] = f'(\eta_x)$$

In this section, we examine the behavior of the mean value η_x as x approaches the left end point a of the interval [a,x].
Further with the help of two examples we have seen that as x approaches the left end point of the interval from the right, the mean value η_x approaches the midpoint between x and the left end point of the interval.
5.) Pompeiu's mean value theorem is variant of Lagrange's mean value theorem. Which geometrically means that the tangent at the point $(\varepsilon, f(\varepsilon))$ intersects on the y-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Pompeiu's mean value theorem attains a functional equation of the form

$$\frac{xf(y) - yf(x)}{x - y} = h(\varepsilon(x,y))$$

$$\forall x, y \in \mathbb{R}$$

$$x \neq y$$

Two Dimensional Mean Value Theorem

1.) The difference between the values of the function at thpoints (u,v) and (x,y) is equal to the differential at an intermediate point (η, ε) on the line-segment joining the two points.

Mean Value Theorem For Some Generalized Derivative

1.) If f is continuous on [a,b] and symmetrically differentiable on (a,b) and if the symmetric derivative of f has Darboux property, then there exists γ in (a,b) such that

$$f^s(\gamma) = \frac{f(b) - f(a)}{b - a}$$

Integral Mean Value Theorem

1.) If $f(x)$ is continuous on $[x, y]$,then there exists a point ε in (x,y) depending on x and y such that

$$f(\varepsilon(x, y)) = \frac{\int_x^y f(t)dt}{y - x}$$

2.) Using Integral Mean value theorem we further obtained arithmetic mean, geometric mean and the identric mean.

The ordering for the same is given by

$$mina, b \leq G(a, b) \leq I(a, b) \leq A(a, b) \leq maxa, b$$

Cauchy Type Of Generalization Of Flett's Theorem

1.) Geometrically Flett's Theorem means that if $f(x)$ is defined and differentiable on an open interval I, such that I contains $[a, b]$ and if f has the property that the slopes of the tangents at two points $(a, f(a))$ and $(b, f(b))$ are equal, then there is some point ε between a and b such that the tangent to the curve $y = f(x)$ at $(\varepsilon, f(\varepsilon))$ passes through $(a, f(a))$. There is nothing special about the left end point $(a, f(a))$ and that the symmetric result holds at the right endpoint; thus, Flett's theorem in reality a statement about existence of two points: under the aforementioned assumptions, there exists $\varepsilon, \eta \in (a, b)$ for which

$$f'(\varepsilon) = \frac{f(\varepsilon) - f(a)}{\varepsilon - a}$$

,

$$f'(\eta) = \frac{f(b) - f(\eta)}{b - \eta}$$

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