WYTHOFF PAIRS

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by

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "wythoff pairs" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Manvendra Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "wythoff pairs " is a bonafide work carried out by Miss Kritali Prashant Matonkar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " wythoff pairs ." This topic is divided into five chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of what is wythoff pairs .

SECOND CHAPTER:

This chapter deals with wythoff pairs . In this topic we have discuss what are different types of wythoff pairs and how to construct different type of pairs using following definitions

THIRD CHAPTER:

In this chapter we have introduced a wythoff game basically known as the kitten and puppy game. and try to find their safe pair of sequences .

FOURTH CHAPTER:

This topic basically talks about the different heap game .

FIFTH CHAPTER.

In this chapter we study detailing about rat, ratwyt, mouse and fat mouse game and see the positioning of the sequences and see what does array mean.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my Mentor, Dr. Manvendra Tamba, who was a continual source of inspiration. He pushed me to think imaginatively and urged me to do this homework without hesitation. His vast knowledge, extensive experience, and professional competence in Number Theory enabled me to successfully accomplish this project. This endeavour would not have been possible without his help and supervision.

ABSTRACT

The Wythoff number pairs have been much discussed in the literature on Fibonacci. Bicknell-Johnson treats generalizations of Wythoff numbers which provide number triples with many interesting properties. In this paper we present three different ways to generate the Wythoff pairs, and, with some trepidation in view of the extent of the literature on them, claim that these are "new." We emphasize the notion of "generation (in contrast to "giving a formula"), and introduce Fibonacci word patterns as a tool to define n-tuple generating processes. A determinantal relation for the Wythoff pairs is described, which makes further use of the word-pattern tools. In the final section we show how similar methods can be used to generate and study sequences of integer triples. examples are given, and each is an attempt to generalize aspects of the Wythoff pairs-sequence. It is clear to us that these tools and methods hold much promise for developing a general theory of sequences of integer n-tuples which have structures.

The main aim of this article is, in general to prove some basic result concerning the above proving and stating stratergies, and, in, particular, to compute various wythoff pairs using fibonacci sequence

Keywords: fibonacci word pattern, tribonacci word pattern,p- position.

Table of contents

Notations and Abbreviations

α	alpha
β	beta
γ	gamma
σ	sigma
ϕ	phi
ω	omega
Δ_n	big delta
δ	delta

Chapter 1

INTRODUCTION

wythoff pairs is widely regarded as the most delightful branch of Mathematics. This is because of its twin nature; it contains the cleverest proofs in all the abstract reasoning and it has the most comprehensive range of applicability to any contemporary sciences. Many Mathematicians have contributed to the growth of this theory .Willem Abraham Wijthoff Wythoff was born in Amsterdam to Anna C. F. Kerkhoven and Abraham Willem Wijthoff, who worked in a sugar refinery. He studied at the University of Amsterdam, and earned his Ph.D. in 1898 under the supervision of Diederik Korteweg

Wythoff is known in combinatorial game theory and number theory for his study of Wythoff's game, whose solution involves the Fibonacci numbers. The Wythoff array, a two-dimensional array of numbers related to this game and to the Fibonacci sequence, is also named after him.

In geometry, Wythoff is known for the Wythoff construction of uniform tilings and uniform polyhedra and for the Wythoff symbol used as a notation for these geometric objects. A Fibonacci word is a specific sequence of binary digits (or symbols from any two-letter alphabet). The Fibonacci word is formed by repeated concatenation in the same way that the Fibonacci numbers are formed by repeated addition. The name

"Fibonacci word" has also been used to refer to the members of a formal language L consisting of strings of zeros and ones with no two repeated ones. Any prefix of the specific Fibonacci word belongs to L, but so do many other strings. L has a Fibonacci number of members of each possible length.

The author had been working on the safe combinations (Wythoff pairs) in Wythoffs game when there- searches of Silber came to his attention. As the two approaches differ somewhat, it is probably worthwhile to indicate briefly the author's alternative treatment, which may throw a little light on the general problem. Both Silber and the author use the fundamental idea of the canonical Fibonacci representation of an integer. While much work has been done recently on Fibonacci representation theory and on Nim-related games, we will attempt to minimize our reference list. Wythoff pairs have been analyzed in detail by Carlitz, Scoville and Hoggatt, e.g., though without specific reference to Wythoffs game. For a better understanding of the principles used in our reasoning which follows, it is desirable to present a description of the nature and strategy of Wythoffs game.

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Chapter 2

THE ALPHA AND OMEGA OF WYTHOFF PAIR

2.1 Fibonacci word pattern:

([1] is used in all this subsections of chapter 2)A sequence of words $W_1, W_2, ..$ the words are found using characters from given letter set such as {0,1} or {a,b,c} . The basic word pattern is obtained repeatedly using recurrence given by $W_{n+2} = W_n W_{n+1}$ with $W_1 = A, W_2 = B$ Then $F(A,B) = A, B, AB, BAB, ABBAB, ...$

example

If A=1 and B=0 Then $F(1,0) = 101001010010.....$ Which is denoted by α

$F(1,01) = 10110101101......$ This is denoted by ω

The tribonacci word pattern with $W_1 = A, W_2 = B, W_3 = C$

with $F(A,B,C) = A, B, C, ABC, BCABC, ...$

where $W_{n+3} = W_n W_{n+1} W_{n+2}$

2.2 Set Sequence:

Let $\{S_n\}$ be set of sequences and $\{a_n\}$ be set of integers. The set sequence is formed using the recurrence$S_{n+2} = S_n U S_{n+1} + a_n$.

Let set $\{a_1, a_2, ...\}$
Then + operation is to be carried out as $\{a_1, a_2, ...\} + a$
=$\{a_1 + a, a_2 + a, ...\}$

2.3 Addmerging and submerging:

A merging operation and its inverse
Let S and T be any monotone increasing sequences, then the addmerge of S and T written as S M T is obtained by taking the multiunion of two sequences and sorting them into monotonic increasing order.
example
S=$\{1, 6, 8, 10, 15, ...\}$
T=$\{6, 9, 11, 16, ...\}$
SMT=$\{1, 6, 6, 8, 9, 10, 11, 15, 16, ...\}$ The inverse of addmerge is submerge denoted by SWT, thus operation simply removes the sequences T from S .
example
S=$\{1, 6, 9, 10, 16, ...\}$
T=$\{6, 9, 11, 16, ...\}$
SWT=$\{1, 10, ...\}$.

2.4 Sequence notation:

$n = \{1,2,3,....\}$. which is denoted by $\mathbb{N}$

$n^{+} = \{0,1,2,3,....\}$

$f = \{1,1,2,3....\}$ Fibonacci integers denoted by $\{F_n\}$.

$f^{'} = \{1,2,3,5,....\}$ denoted by $\{F_{n+1}\}$.

$f^{''} = \{2,3,5,8,13,....\}$ denoted by $\{F_{n+2}\}$

2.5 MEMBERS OF WYTHOFF PAIRS

First member of wythoff pair is $[n\alpha]$ where $\alpha = 1+\sqrt{5}/2 = 1.618$

$n = 1$

$$[1 \times 1.618] = [1.618] = 1$$

$n = 2$

$$[2 \times 1.618] = [3.236] = 3$$

and so on so $\omega_1 = \{1,3,4,6,8,....\}$

Second member of wythoff pairs is$[n\alpha^2]$ $n = 1$

$$[1 \times 2.617] = [2.617] = 2$$

$n = 2$

$$[2 \times 2.617] = [5.234] = 5$$

and so on so $\omega_1 = \{2,5,7,10,13....\}$
Wythoff pairs of sequences

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} ....$$

2.6 Binary Word pattern:

Let B be sequence of 0's and 1's $B = b_1, b_2, ...$ and $S_n = s_1, s_2, ...$ which belongs to (0,1). Then $S * B =$ the subsequence from S whose elements are in the positions where the 1's occur in S
example
$B = \{010101....\}$then
$n * B = \{1,2,3,4,...\} * \{010101,...\}$

2.7 Generation of wythoff pairs using:

2.7.1 Different ways of generating wythoff pairs.

Use of omega sequence

The binary word pattern $\omega = F(1,01) = \{10110101101......\}$
where 1's in binary word pattern occurs in ω_1
so $\omega_1 = \{1,3,4,...\}$
and where 0's in binary word pattern occurs in ω_2
so $\omega_2 = \{2,5,7,...\}$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} n * F(1,01) \\ n * F(0,10) \end{pmatrix}$$

example

$n = \omega_1 \mathbb{M} \omega_2$

$= n * [F(1,01) + F(0,10)]$

$= n * [\{10110101101....\} + \{01001010010....\}]$

$= n * \{1111111.....\}$

$= n$

2.8 Use of Fibonacci magic matrices

A square matrices all of whose elements are Fibonacci integers, whose diagonal,row and column sums are fibonacci integers and whose powers also posses this properties called magic . The spectral radius of these matrices is α , which is the golden mean.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -\alpha & 1 \\ 1 & 1-\alpha \end{pmatrix}$$

$= -\alpha(1-\alpha) - 1$

$= -\alpha + \alpha^2 - 1$

$= (1 \pm \sqrt{1-4(1)(-1)})/2$

$= (1 \pm \sqrt{5})/2$

so we only take the positive one so here we get the golden ratio.

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

=$1-\lambda(2-\lambda)-1$

=$-3\lambda+\lambda^2+1$

=$(3\pm\sqrt{9-4(1)(1)})/2$

=$(3\pm\sqrt{5})/2$

so $\lambda = \alpha^2$ **Determinantal Relation For Wythoff Pairs** $\begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix}$

$= (-1)^{n+1}$ consider the wythoff pair sequence $\omega_1 = a = \{a_i\}$

$\omega_2 = b = \{b_i\}$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} .... = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} ....$$

Now we try to compute determinant

consider

$\begin{vmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{vmatrix}$

= $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$

=5-6

=-1

= $\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}$

=21-20

=1

computing so on we get, -1,1,-2,-2,3,-3,4,-4,-4,6,-5,-5,8,-6,9,-7,-7,11,-8

There is clearly an interesting pattern to the sequence, but how can we capture it in a formula? It is here that our word pattern notation becomes really useful. Let us submerge the negative and the positive elements, to find

-1,-2,-2,-3,-4,-4,..... is equal to

$$(-n) * F(1,2)$$

where$(-n) = \{-1,-2,-3,......\}$

and $F(1,2) = \{1212212....\}$

now $(-n) * F(1,2)$

$= \{-1,-2,-3,......\} * \{1212212....\}$

And the positive sequence is just ω_1

2.8.1 GENERATION OF WYTHOFF PAIRS.

Any Fibonacci word pattern which uses binary letter set can be used to generate pair sequence known as alpha sequence

$\alpha = F(1,0) = \{101001010010...\}$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} ....$$

$\alpha_1 = \{1,3,6,8,.....\} = \omega_1 W[\omega_1 * F(0,01)]$

$= \omega_1 W[\{1,3,4,6,8.....\} * \{00100101001.....]$

$= \omega_1 W[\{4,9,12,17,....\}]$

$= \omega_1 * [\omega_1 * F(1,10)]$

$\alpha_2 = \{2,4,5,7,9,10....\} = \omega_2 M[\omega_1 * F(0,01)]$

$= \omega_2 M[\{1,3,4,6,8.....\} * \{00100101001.....]$

$= \omega_2 M[\{4,9,12,17,....\}]$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \omega_1 W[\omega_1 * F(1,01)] \\ \omega_2 M[\omega_1 * F(0,01) \end{pmatrix}$$

$\omega_1 = \alpha_1 M[\alpha_1 * F(1,10)]$

$\omega_2 = \alpha_2 W[\alpha_2 * F(0,10]$

And so the alpha pair-sequence can be expressed in terms of the omega (Wythoff) pair-sequence; and vice-versa. It is evident that by such means an infinite number of pair-sequences can be generated, and their properties studied by establishing relationships be- tween them and the fundamental Wythoff pairs. A new kind of number theory could be developed, based upon the sequences ω_1 and ω_2 and related to the "ordinary" number theory based on n through the functions [nα] and [nα^2] . Finally, we give an indication of how these methods can be extended to study sequences of triples.

2.9 GENERATION OF TRIPPLE SEQUENCE

2.9.1 Use of Fibonacci word pattern

F(a,bc)={abcabcbcabc.....} we check positions of abc

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdots\cdots\cdots\cdots$$

$\omega_1 * \alpha$

= $\{1,3,4,6,8,.....\} * \{101001010010......\}$

then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \omega_* \alpha \\ \omega_2 \\ \omega_2 + 1 \end{pmatrix}$$

where $\alpha = F(1,0)$ Note also that $a_n and b_n$ are each strictly increasing sequences, they are non-intersecting, and their union equals $\mathbf{Z}^+$ Wythoff pairs-sequence. Their proof is immediate from the way in which $F(a_n, b_n)$ is expanded. Another interesting point is that the parity of the terms in a is alternately odd and even. And then, since the sum $(b_n + c_n)$ is always odd, we have the sum$(a_n + b_n + c_n)$ also of alternating parityof the terms in a is alternately odd and even. And then, since the sum $(b_n + c_n)$ is always odd, we have the sum$(a_n + b_n + c_n)$ also of alternating parity.

2.9.2 Use of tribonnaci word pattern

F(a,b,c)={abcabcbcabc....} we check positions of a b c

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \dots\dots\dots\dots$$

This triple-sequence again clearly has the property that each element sequence is monotone increasing, and the three sequences partition $\mathbb{Z}^+$. When we first studied this sequence, we hoped that we would find simple relationships between a, b, and c respectively, $[n\beta]$,$[n\beta^2]$,$[n\beta^3]$

where β is the positive root of the tribonacci equation$\beta = 1.839$

$$\begin{pmatrix} [n\beta] \\ [n\beta^2] \\ [n\beta^3] \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix} \dots\dots\dots\dots$$

2.9.3 Wythoff sequence

DEFINITION:[2]

We call a sequence of pairs of integers $\{A_n, B_n\}_{n \geq n_0}$ is a wythoff sequence if $n_0 > 0$ and there exist a finite set of integers T such that

$A_n = mex(\{A_i, B_i, n0 \leq i < n\} UT)$

$B_n = A_n + n$ and $B_n \cap T = \phi$

A special wythoff sequence is a wythoff sequence such that there exist integers N and α such that $n > N$ and

$A_n = [n\phi] + \alpha + \mathbb{E}_m$ where $\mathbb{E}_m$ belongs to $\{0, 1, -1\}$

Theorem 2.9.3.1. *[2] For a wythoff sequence $\{A_n, B_n\}_{n \geq n_0}$ let N be an integer such thatn $\geq N, A_n > mex(T)$ then*

- $1 \leq A_{n+1} - A_n \leq 2$
- $2 \leq B_{n+1} - B_n \leq 3$

proof: T is finite and N exist , A_n is defined using mex function A_n is the smallest available in

$\mathbb{Z} \geq 0$ so $\mathbb{Z} - \{T - \{A_i, B_i\} i < n\}$ so $\{A_n\}$is a increasing sequence .

$B_n = A_n + n$ must be increasing if $\mathbb{Z} - T - B_n$ is eventually must be in A_n i.e any non negative integers not in T much be either in A or B

1) IF $A_{n+1} - A_n \geq 2$

since A_{n+1} and A_{n+2} are not in A's they must be in B's

consider $B_{m1} = A_n + 1$ there exist distint integers $m_1$and m_2 s.t $B_{m1} = A_n + 1$ and $B_{m2} = A_n + 2$

$A_{m1} - A_{m2} = m_1 - m_2 - 1$ Since $\{A_n\}$ is increasing, the two sides of the last equation cannot have the same signs.A contradiction

2) $2 \leq B_{n+1} - B_n \leq 3$

for each $n \leq B_{n+1} - B_n = A_{n+1} + n + 1 - (A_n + n)$

Corollary 2.9.3.2. *[2]*

Given $\{(A_n, B_n)\} n \geq 0$ define $S_2 = \mathbb{Z} \geq 0 - \{B_i, A_i : i > 0\}$

proof:

$S_2 C\mathbb{Z} \geq 0$

$\{B_i, A_i : i > 0\}$ *by the definition of* B_n

secondly if there exist $d \in \mathbb{Z} \geq 0 \{B_i, A_i : i > 0\}$

assume $B_n \neq A_n + d$ *for all n*

so $B_i - A_i > d$ *for all* $i \geq N$

since for any $n \geq N (A_n, A_n + d) \notin (A_i, B_i), i > 0$

$A_n + d \in \{A_i\} for 0 < i < n$

this is impossible because $\{A_i\}$ $0 < i < n$*is increasing* $B_i - A_i = d$ *for some* $i < n$ *but this is impossible of our assumption on d gives contraction.*

Chapter 3

WYTHOFF GAMES

3.1 WYTHOFF GAMES

([3] is used in all the subsection and Theorem of chapter 3) Wythoff's Game, named after Willem Abraham Wythoff, who is well known amongst number theorists. Dr. Wythoff, who received a Ph. D. in mathematics from the University of Amsterdam in 1898, described this game in his words: "The game is played by two persons. Two piles of counters are placed on a table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter or counters, wins." This game was previously known in China as tsyan-shidzi ("choosing stones"), but was reinvented by Wythoff. Wythoff published a full analysis of this game in 1907. Nearly a half a century later, around 1960, Rufus P. Isaacs, a mathematician from Johns Hopkins University, created another description of this same game. Completely unaware of Wythoff's game, Isaacs described the same game in terms of the moves of a chess queen allowed only to travel south, west or southwest on a chessboard. This game was often called “Queen’s Move” or “Cornering the Queen”. In this game the Queen is initially placed in the far right column or in the top row of the chessboard. The player who gets the queen to the lower left corner is the winner.Under the name Last Biscuit, this same game is played by removing cookies from two jars, either from a single jar, or the same number from both jars. Also, this game has been named the Puppies and Kittens Game in The Teacher’s Circle: Finite Games by Paul Zeitz. To begin researching this game and its strategies, it only makes sense to play the game in small cases and look for patterns.

3.1.1 RULES OF THE GAME

unspecified numbers of counters occur in each of two heaps in each draw a player may freely choose counters either from

* one heap

* two heaps

provided in this case he/she must take same number from each .

example

heaps of 1 and 2 can be reduced to 0 and 2

1 and 1

1 and 0

The player who takes the last counter wins the game.

The safe combinations for Wythoffs game are known to be the pairs:

n = 1 2 3 4 5 6 7 8 ....

are the corresponding wythoff pairs

(1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), (12,20)....
A safe pair may also be called a Wythoff pair.

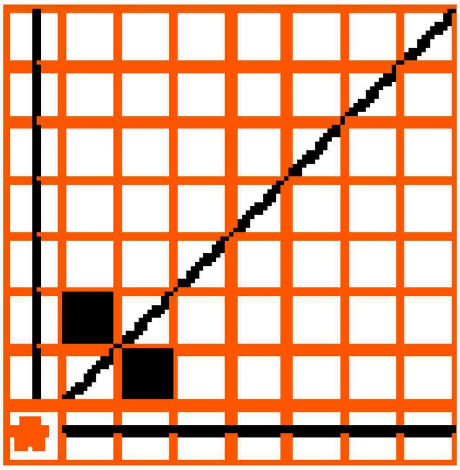
- There are several interesting things about the integers occurring in these safe combinations. They are:
- Members of the first pair of integers differ by 1, of the second pair by 2, of the third by 3,..., of the $n^t h$ pair by n
- the nth pair is $(a_n, b_n) = ([n\alpha], [n^2])$ where the symbol [x] denotes the greatest integer which is less than, or equal to x and α "golden section" number which is a root of $x^2 - x - 1 = 0$. Note that$b_n = a_n + n$
- In every pair of a safe combination, the smaller number is the smallest integer not already used and the larger number is chosen so that the difference in nth pair is n.

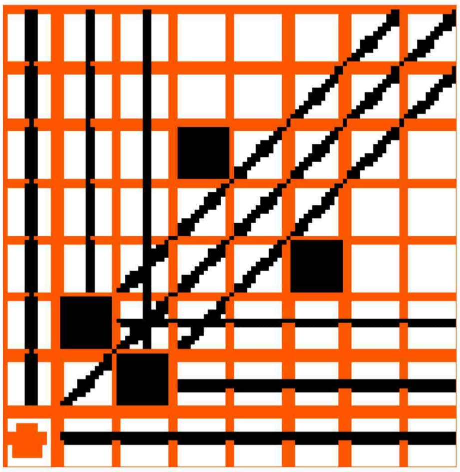
It might reasonably be asked: How does the "golden section" number a come into the solution of Wythoff's game?
Wythoff himself obtained his solution "out of a hat" without any mathematical justification). Basically, the answer to our query, as given by Hyslop and Ostrowski quoted in , depends on the occurrence of the equations $1/x + 1/y = 1$ $y = x + 1$ which, when is eliminated, yield our quadratic equation$x^2 - x - 1 = 0$.

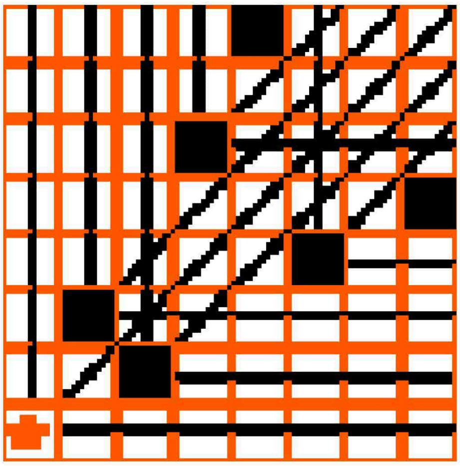
kitten and puppy game

Mathematically speaking, if one starts with p(puppies) = 0, k (kittens) = 0, or p = k, then by rules of the game player 1 wins. Thus, I started with one puppy (p=1) and two kittens (k=2). In this case, no matter what, player 2 wins. If player 1 takes the puppy, player 2 will win by taking all the kittens. Just the same, if player 1 takes all the kittens, player 2 will win by taking the puppy. If player 1 takes one puppy and one kitten, player 2 wins by taking the remaining kitten. Likewise, if player 1 takes one kitten, player 2 will win by taking the remaining puppy and kitten. This same reasoning applies if you start with two puppies and one kitten; player two will win every time. So what happens if p=3 and k=1?Strategically,player 1 is only one move away from p=2, k=1, which I just examined. Player 1 has the winning edge. Player 1 just needs to take one puppy, which forces player two to be the first player to play the p=2, k=1, which is a fatal situation to be in. With this strategy, player 1 can win for k=1 and any number of puppies greater than two. Similarly, player 1 can win for p=1 and any number of kittens greater than two. This strategy also holds true when taking an equal number of puppies and kittens in a turn. If k=3, p=2 (or vise versa), player 1 would take one from each side forcing player two to be the first player to play k=2, p=1. Similarly, if k=4, p=3, player 1 would take two from each side, again forcing player two to be the first player to play k=2, p=1. In general, player 1 can win if you start k=1, p=k+1 (or p=1, k=p+1) when k is greater than or equal to two. Therefore, in small cases of Puppies and Kittens, the position (2,1) or its reflection (1,2) is safe for the player that leaves the game in this state at the end of his or her turn.

These safe positions seem more applicable to the “Queen’s Move” on a chessboard in Isaacs’s interpretation. Isaac constructed a winning strategy for cornering the queen on boards of unbounded size by letting the cell (square) in the lower left corner, the winning position, be the origin (0, 0) and working backwards. In this way, the chessboard can correspond to the first quadrant in a coordinate plane. If the queen is in the row, column or diagonal containing (0, 0), the person who has the move can win at once. This is represented on illustration A where three straight lines mark these cells.







The shaded cells in figure 1 represent “safe” positions. They are “safe” because if you occupy either one when it is your opponents turn, your opponent is forced to move to a cell that enables you to win on the next move. Notice that the shaded cells in figure A are at (2,1) and (1,2), where the starred cell is (0,0). These are the same “safe” positions identified when playing Puppies and Kittens. second Figure represents the next step. By adding six more lines to mark all the rows, columns and diagonals containing the “safe” cells of (2,1) and (1,2), two more “safe” cells are shown at (3,5) and (5,3). If you occupy either one of these “safe” cells at the end of your turn, your opponent is forced to move, so that on your next move you can either win at once or move to another safe cell closer to (0,0). Again repeating this procedure in figure 3 a third set of “safe” cells is discovered at (7,4) or (4,7). From these illustrations, it is clear that Player 1 can always win by placing the queen on a “safe” cell. The strategy thereafter would be to move to another “safe” cell. If player 1 fails to move to a “safe” cell, player 2 can win by the same strategy.

The comparisons between Wythoff’s game and Queen’s Move are obvious. The games are mathematically equivalent even though the context is different. Each cell can be assigned a coordinate pair (x,y). These pairs correspond to the number of counters in piles x and y of Wythoff’s game. When the queen moves west, pile x diminishes.

When the queen moves south, pile y diminishes. When the queen moves diagonally or southwest, both piles diminish by the same amount. Moving the queen to cell (0,0) is the same as reducing both piles to zero. If the starting piles are "safe", player 2 should win. Player 1 has no choice but to leave an unsafe pair of piles, which the opponent can always reduce to a "safe" pair on his or her next move. If the game begins with unsafe numbers, the first player can always win by reducing the piles to a"safe" pair and continuing to play to "safe" pairs. From the analysis of the game thus far, it is obvious that winning strategies can be determined by finding "safe" cells. Inevitably however, one must ask, can we mathematically compute where the safe cells or combinations are?

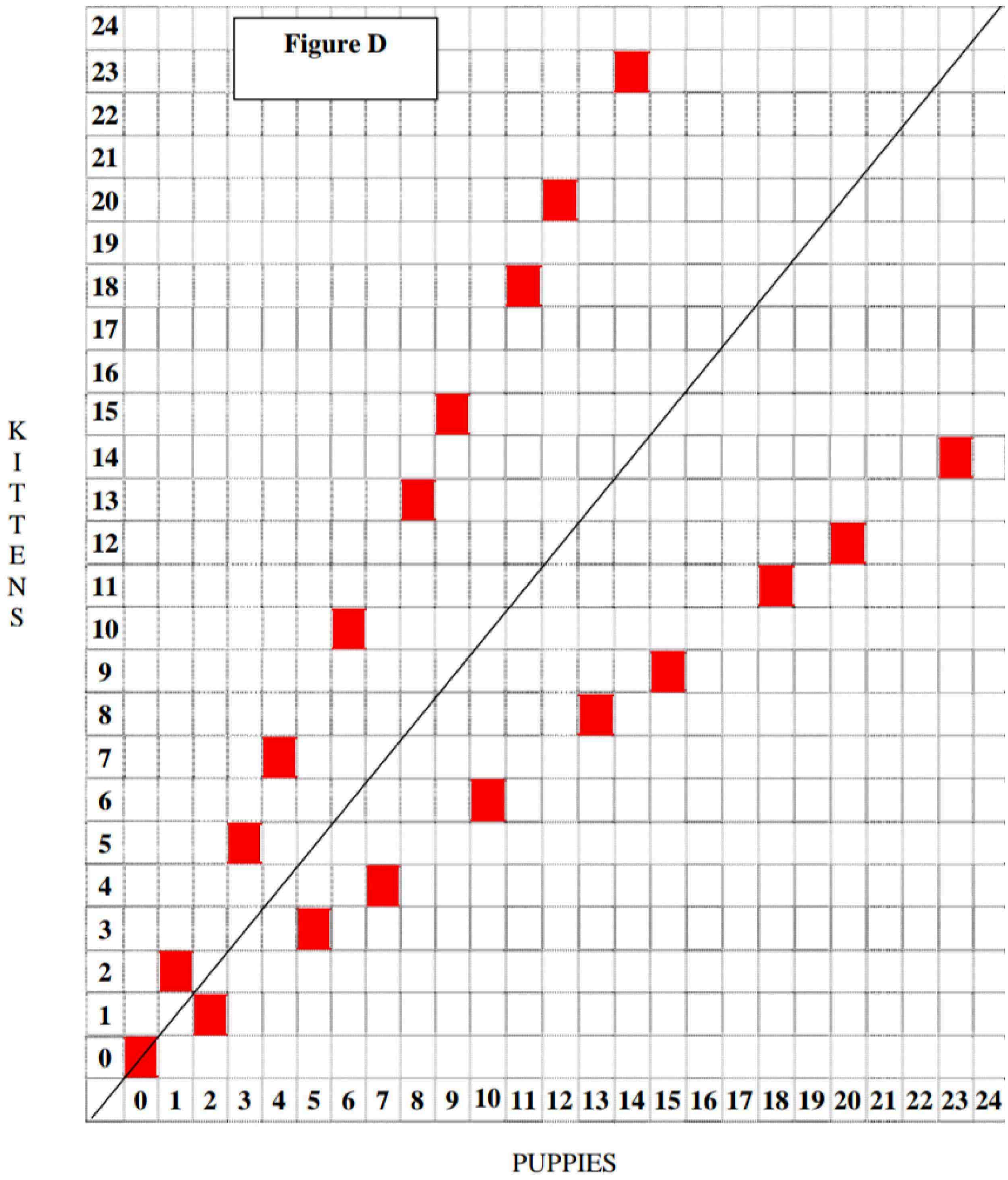


Figure D
KITTENS
PUPPIES
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

By looking at figure , it can be observed that the order of the two numbers in a "safe" pair is not important. This condition says that if a position is safe, then its mirror image when reflected across the main diagonal is also safe, i.e. the collection of safe pairs is symmetric with respect to the main diagonal. Ordered pairs (a, b) and (b, a) are symmetric about the main diagonal; they have the same coordinate numbers, one pair being the reverse order of the others. Therefore, when arranging the safe pairs in a table, one only needs to focus on the ordered pairs (cells) above the diagonal that correspond to "safe" locations, with n representing position in the sequence, A representing the sequence of the top numbers of the safe pairs, and B representing the sequence of the bottom numbers of the safe pairs.

3.1.2 Sequence of safe pairs

The Sequence of Safe Pairs from Figure D

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
A	0	1	3	4	6	8	9	11	12	14	…
B	0	2	5	7	10	13	15	18	20	23	…

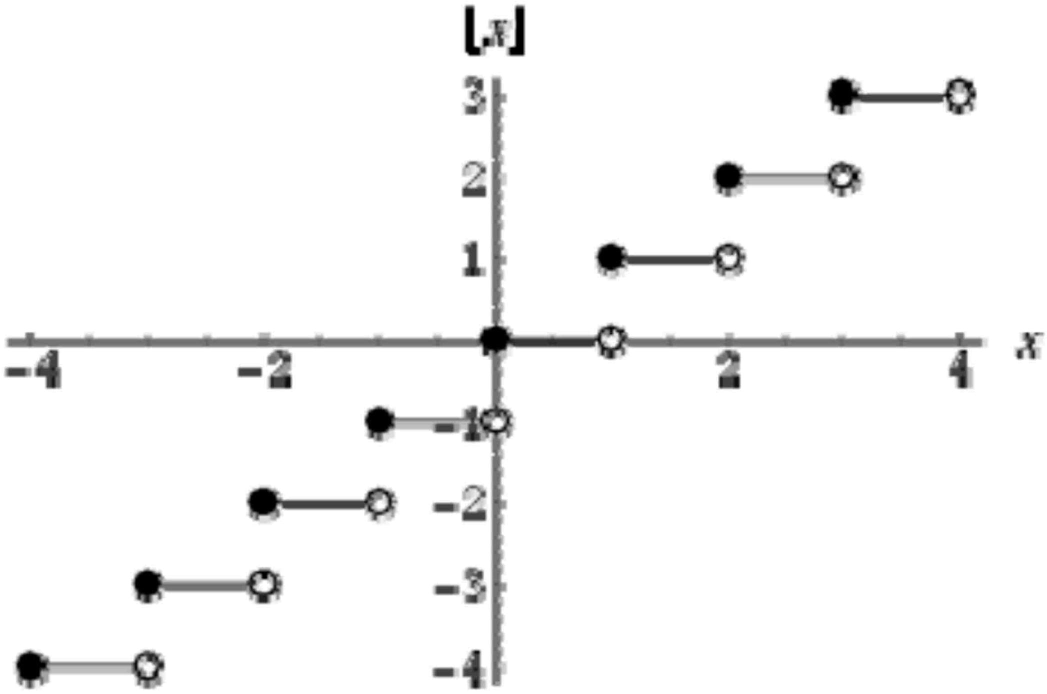
In studying this table, many numerical patterns can be observed in the sequences. The first pattern being that each B value is the sum of it's A value and its position number n. The second pattern is when an A value is added to its B value, the sum is an A value that appears in the A sequence at a position number equal to B. For example, 4 + 7 = 11, and the 7th number of the A sequence is 11. Using this idea, 6 + 10 = 16, so when n = 10, sequence A is 16. The sequences can be generated by a recursive pattern. To begin assign 1 as the top number (the A value) of the first safe pair. Add this to its position number to obtain 2 as the bottom number. The top number of the next pair is the smallest integer not previously used,which, in this case, is 3.The corresponding bottom number is then 5, which is the sum of 3 and its position number, 2. For the top of the third pair, we need to find the smallest positive integer not yet used, which, in this case, is 4. The sum of 4 and 3 is then 7, which is the bottom number. The top of the 4th pair would be 6 (the only positive integer not used yet). Below it goes 10, the sum of 6 and 4. Continuing in this way will generate all safe pairs for Wythoff's game that every positive integer must appear once and only once somewhere in the two sequences. Using this procedure, we can generate additional "safe" pairs which strategically would guarantee the next player a win in this game.

The Sequence of Safe Pairs in Wythoff's Game																									
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
A	0	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27	29	30	32	33	35	37	38
B	0	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44	47	49	52	54	57	60	62

3.1.3 Golden ratio

: The Golden Ratio, denoted by ϕ is an irrational number. It can be defined as the number which is equal to its own reciprocal plus one $\phi = 1/\phi + 1$ Multiplying both sides of this same equation by the Golden Ratio derives the interesting property that the square of the Golden Ratio is equal to the number itself plus one: $\phi^2 = \phi + 1$
Since that equation can be written as $\phi^2 - \phi - 1 = 0$ the value of the Golden Ratio can be derived from the quadratic formula, $x = (-b \pm \sqrt{b^2 - 4ac})/2a$ In this case a = 1, b = -1, and c = -1, which then implies that$\phi = (1 \pm \sqrt{5})/2$. This is approximately either 1.618033989 or -0.618033989. The first number is usually regarded as the Golden Ratio. Rounding down integers can be realized as the floor function. The floor function

$f(x) = [x]$, also called the greatest integer function, gives the largest integer less than or equal to x . Formally this means:$[x] = max\{n \in \mathbb{Z} : n\}$Therefore, $[3.7] = 3, [11.5] = 11$, and $[-1.4] = -2$
A graph of the floor function is shown below.



Theorem 3.1.3.1. *If α and β are irrational 1/α+1/β=1 then {[α],[2α],...}and {[β],[2β],...} are disjoint and their union is all integers.*

proof:

let $\alpha = \phi$

$\beta = \phi^2$

we know ϕ *is irrational and since* $\phi^2 = \phi + 1$ *,* ϕ^2 *is also irrational*

we need to show $1/\phi + 1/\phi^2 = 1$

$1/\phi + 1/\phi + 1 = 1$

= $(\phi + 1 + \phi)/\phi + \phi^2$

= $\phi + \phi^2/\phi + \phi^2$

=*1*

Theorem 3.1.3.2. *If* $E(x)$ *denote the greatest integer not greater than x then the combination* $E[1k/2(1+\sqrt{5})], E[1k/2(3+\sqrt{5})]$ *k being zero or positive integer is a safe* *combination, by successively substituting k =0,1,2,....*

proof:

$E[1k/2(3+\sqrt{5})]-E[1k/2(1+\sqrt{5})]$

$=[1k/2(3+\sqrt{5})]-[1k/2(1+\sqrt{5})]$

$=3k/2+\sqrt{5}k/2-1k/2-\sqrt{5}k/2$

$=k$

so if k=0,1,2,.... we obtain series of combination , hence it will be sufficient to prove this substitution produces once and not more than once any arbitrarily positive integer

let x denote such an integer

let α *and* β *be smallest quantities which must be added to n* $\alpha = 1/2(1+\sqrt{5}), \beta = 1/2(3+\sqrt{5})$

we have

$\alpha = 1/2p(1+\sqrt{5})-n----(1)$

$\beta = 1/2q(3+\sqrt{5})-n----(2)$

where p and q are integers

$0<\alpha<1/2(1+\sqrt{5})----(3)$

$0<\beta<1/2(3+\sqrt{5})----(4)$

multiply (1) by$1/2(-1+\sqrt{5})$ *and (2) by*$1/2(3-\sqrt{5})$ *and adding them we get*

$1/2\alpha(-1+\sqrt{5})+1/2\beta(3-\sqrt{5}) = 1/2p(1+\sqrt{5})1/2(-1+\sqrt{5})+1/2q(3+\sqrt{5})1/2(3-\sqrt{5})-n(-1+\sqrt{5})/2-n(3-\sqrt{5})/2$

$=p(-1+\sqrt{5}-\sqrt{5}+5)/4+q(-1+\sqrt{5}-\sqrt{5}+5)/4-(n-n\sqrt{5}-3n+n\sqrt{5})/2$

$=p+q-n$

therefore we get

$1/2\alpha(-1+\sqrt{5})+1/2\beta(3-\sqrt{5}) = p+q-n$

multiply (3) by$1/2(-1+\sqrt{5})$ *and (4) by*$1/2(3-\sqrt{5})$

we get$0<1/2\alpha(-1+\sqrt{5})+1/2\beta(3-\sqrt{5})<2$

hence the integer $p+q-n$ can be no other than 1

first forty wythoff pairs

n	a_n	b_n
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
11	17	28
12	19	31
13	21	34
14	22	36
15	24	39
16	25	41
17	27	44
18	29	47
19	30	49
20	32	52
21	33	54
22	35	57
23	37	60
24	38	62
25	40	65
26	42	68
27	43	70
28	45	73
29	46	75
30	48	78

31	50	81
32	51	83
33	53	86
34	55	89
35	56	91
36	58	94
37	59	96
38	61	99
39	63	102
40	64	104

interesting to observe that the first few Fibonacci numbers occur paired with

other Fibonacci numbers:

$(a_1,b_2) = (1,2),(a_2,b_2) = (3,5),(a_5,b_5) = (8,13),(a_13,b_13) = (21,34),(a_34,b_34) = (55,89)$.

It is not difficult to establish that this pattern continues throughout the sequence of Wythoff pairs, using the

fact that $lim(bn/an) = a$and also$lim(F_n+1/F_n) = a$ and $a_n+b_n = a_bn----(1)$

Theorem 3.1.3.3. *[4]*
Let $G_1G_2,G_3,....$be the Fibonacci sequence generated by a Wythoff pair$\{a_n,b_n\}$. Then every pair $(G_1,G_2)(G_2,G_3),.....$is again a Wythoff pair.

Proof:
By construction, every Wythoff pair satisfies

$a_k + k = b_k - - - - - - (2)$

Consider the first four terms of the generated Fibonacci sequence:

$a_n, b_n, a_n + b_n, a_n + 2b_n.$

According to Equation. (1)

$a_n + b_n = a_b n$

adding both sides b_n

$a_n + b_n + b_n = a_b n + b_n$

$a_n + 2b_n = a_b n + b_n$

if $k = b_n$*in (2)*

$a_b n + b_n = b_b n$

so $a_n, b_n, a_b n, b_b n$

therefore(G_3, G_4) *is a wythoff pair.*

Chapter 4

N PILES /HEAP GAME

([2] is used in all subsections and theorems of chapter 4)

Given N $\geq$ 2 piles of finitely many tokens $p_1,....,p_N$. The moves are to take any positive number of token from a single rule or $(a_1,a_2,a_3,.....,a_N) \in \mathbf{Z}^N \geq 0$ from all piles a_i from ith pile such that

1)$a_i > 0$ for some i

2) $a_i \leq p_i$

3)$a_1 \bigoplus a_2 \bigoplus .... \bigoplus a_N = 0$

The player making the last move wins and the opponent looses

for $N \geq 3$

denote the p positions for N piles by $(A^1,......,A^{N-2},A_n^{N-1},A_n^N)$ where $A^{N-2} \leq A_n^{N-1} \leq A_n^N$ and $A_n^{N-1} < A_{n+1}^{N-1} n \geq 0$

so given N piles of tokens whose sizes are$A^1 \leq ....^N$

A player can remove any number of tokens from a single pile or remove $(a_1,a_2,a_3,.....,a_N)$ tokens from all providing $0 \leq a_i \leq A^i$

$\sum_{i=1}^{N} a_i > 0$

$a_1 \bigoplus a_2 \bigoplus .... \bigoplus a_N = 0$

where $\bigoplus$ is nim addition denote the p postions by above equations

so given two conjectures on game when $A^1, ..., A^{N-2}$ are fixed

1) there exists an integer N s.t when $n > N$,

$A_n^N = A_n^{N-1} + n$

2) there exists integerN_2 and α_2 s.t $n > N_2$, $A_n^{N-1} = [n\phi] + \in_n + \alpha_2$ and $A_n^N = A_n^{N-1} + n$

where$-1 \leq \in_n \leq 1$

further more $A_n^{N-1} = mex\{A_i^{N-1}, A_i^N : 0 \leq i < n\} UT$

where T is small set of integers

we know that p positions of wythoff game can be written as $[n\phi], [n\phi^2]$

Theorem 4.0.0.1. *[5]*

In the two conjectures of N heaps $A_n^{N-1} = mex\{A_i^{N-1}, A_i^N : 0 \leq i < n\} UT$

where T is a finite set only on depending on $A^1 \leq ....^{N-2}$

in fact T = {a: there exist b , k such that $A^{k-1} \leq b \leq A^k$ *and*$(A^1, ..., A^{k-1}, b, , A^k, ...A_n^{N-2}, a)$

is a p position

proof:

By definition of $T = \mathbf{Z} \geq 0 - \{A_i^{N-1}, A_i^N : 0 \leq i < n\}$

write $T^{'}$ *as the last set*

Claim: $T = T^{'}$

we first prove that $T^{'} \subset T$

we want to show that any $a \in T^{'}, (A^1, ..., A^{N-2}, a, b)$ *is an N position for all* $N \geq A^{N-2}$.

This is by defintion of $T^{'}$ we can always remove tokens from the last pile to create a P-position.

Secondly given that $a \in T, (A^1, ..., A^{N-2}, a, b)$ is an N position for all $b \geq a$. by the definition of T. There are several kind of moves from this position to find a P-position:

1) remove $(a_1, a_2, a_3, ..., a_N)$ tokens from all corresponding piles where $a_1 \bigoplus a_2 \bigoplus ...... \bigoplus a_N = 0$ so that

$$(A^1 - a_1, ..., A^{N-2} - a_{N-2}, a - a_{N-1}, b - a_N)$$

is a P-position.

2) remove $a_k \leq A^k$ tokens from k piles so that

$$(A^1, ..., A^{k-1} A^k - a_k, A^{k+1} ...., A_{N-2}, a, b)$$

is a P-position.

3) remove $a_{N-1} \leq a$ tokens from (N-1) th pile so that

$$(A^1, ..., A^{N-2}, a - a_{N-1}, b)$$

is a p position .

4)remove $a_N \leq b$ tokens from N th pile so that

$$(A^1, ..., A^{N-2}, a, b - a_N)$$

is a p position .

There are only finitely many possible moves using the first three kinds of moves, but there are infinitely many choices of b. So there are cases for the fourth kind of move, i.e.,there

exists an integer b_1, b_2 *such that* $(A^1, ..., A^{N-2}, a, b_1 - b_2)$*is a P-position. Again by the definition of T, we must have* $b_1 - b_2 \leq A^{N-2}$ *otherwise* $a \in \{A^{N-1}\}$
therefore $a \in T^{'}$ *thus* $T \subset T^{'}$
Since there are only finitely many choices of

$$b = b_1 - b_2$$

and each choices of b can yield at most one corresponding a as in the definition of $T^{'}, T = T^{'}$ *must be finite.*
To prove the equation take $a = mex\{A_i^{N-1}, A_i^N : 0 \leq i < n\}UT$

$A_n^{N-1} \geq a$ *as a is the smallest integer*
Supposedly$A_n^{N-1} \neq a$
since we assume $A_i - 1^{N-1} < A_i^{N-1}$ *for all i so a did not appear in*$\{A_i - 1^{N-1}\}_{i<n}$ *and can no longer appear in* $\{A_i - 1^{N-1}\}_{i \geq n}$ *thus a cannot be in*$\{A_i - 1^N\}$ *either*
therefore $T = \mathbf{Z} \geq 0 - \{A_i^{N-1}, A_i^N : 0 \leq i < n\}$ *but this is contrary to the definition of a.*

Theorem 4.0.0.2. *Every Wythoff's sequence is special.[2]*

Proof. Let $x_n = A_n - \lfloor n\phi \rfloor$. By definition, we only need to prove that if m and n are large enough, $| x_m - x_n |$ is at most 2.

By Lemma that α_n below we know x_n is bounded, say, by M.

Define a function $\Delta(n) = B_{A_n+C+1} + C + 1$.

For integers $m, n \geq \Delta^{2M-1}(n_0)$, we can construct two sequences $a_1, \ldots\ldots, a_k$ and $b_1, \ldots\ldots b_k$ such that $k \geq 2M-1, a_k = m, b_k = n, A_{n_0} \leq \min(a_1, b_1) < B_{A_{n_0}+C+1} + C + 1$, and $a_i = B_{A_{a_{i-1}}+C+\varepsilon_{a_2}^{(i)}}, +\varepsilon_{a_1}^{(i)}$, $b_i = B_{A_{b_{i-1}}+C+\varepsilon_{b_2}^{(i)}} + \varepsilon_{b_1}^{(i)}$, where $1 < i \leq k, \varepsilon_{a_1}^{(i)}, \varepsilon_{b_1}^{(i)} \in \{0, \pm 1\}$ and $\varepsilon_{a_2}^{(i)}, \varepsilon_{b_2}^{(i)} \in \{0, 1\}$. we know

$$| x_{a_i} - x_{b_i} | \leq \max\ (| x_{a_{i-1}} - x_{b_{i-1}} | - 1, 2).$$

Hence

$$\begin{aligned} | x_m - x_n | &= | x_{a_k} - x_{b_k} | \\ &\leq \max(| x_{a_1} - x_{b_1} | - (k-1), 2) \\ &\leq \max(2M - (2M-2), 2) \\ &= 2 \end{aligned}$$

□

4.0.1 The Game NIM(a, b)

[5] For any positive integer a and b, a game NIM(a, b) is as follows.

Two piles contain x and y matches. Two players alternate turns.By one move, it is allowed to take x' and y' matches from these two piles such that

$$0 \leq x' \leq x, 0 \leq y' \leq y$$

$$0 < x' + y'$$

either $|x' - y'| < a$ or $min(x', y') < b$

In other words, a player can take "approximately equal" (differing by at most a-1)numbers of matches from both piles or any number of matches from one pile but at most b-1 from the other. This game, NIM(a, b), extends further the game $NIM(a) \equiv NIM(a, 1)$ position of NIM(a, b) is a non-negative integer pair (x, y). Due to obvious symmetry, positions (x, y) and (y, x) are equivalent. By default, we will assume that $x \leq y$. Obviously, (0, 0) is a unique terminal position. By definition, the player entering this position is the winner in the normal version of the game and (s)he is the loser in its mis´ere version.

The normal version of NIM(a, b) was solved in It was shown that the P-positions (x_n, y_n) are characterized by the recursion:

$$x_n = mex_b(\{x_i, y_i | 0 \leq i < n\}), y_n = x_n + a_n; n \geq 0$$

where $x_n \leq y_n$ and the function mex_b is defined as follows: Given a finite non-empty subset $S \subset \mathbf{Z}^+$ of m non-negative integers, let us order S and extend it by $s_{m+1} = \infty$ and by $s_0 = -b$, to get the sequence $s_0 < s_1 < \ < s_m < s_{m+1}$. Obviously there is a unique minimum i such that $s_{i+1} - s_i > b$. By definition, let us set $mex_b(S) = s_i + b$; in particular, $mex_b(\phi) = 0$.

It is easily seen that mexb is well-defined and for b = 1 it is exactly the classic minimum excludant mex, which assigns to S the (unique) minimum non-negative integer missing in S. Thus,$mex_1 = mex$.

Furthermore, Fraenkel solved the recursion for NIM(a, 1) and got the following explicit formula for (x_n, y_n)

let

$$\alpha_a = 1/2(2 - a + \sqrt{a^2 + 4})$$

be the (unique) positive root of the quadratic equation

$z^2 + (a-2)z - a = 0$, or equivalently,

$$1/z + 1/z + a = 1$$

in particular

$$\alpha_1 = 1/2(1 + \sqrt{5})$$

is the golden section and

$$\alpha_2 = \sqrt{2}$$

then it follows that

$$n \in \mathbf{Z}^+$$

$$x_n = [\alpha_a n]$$

and

$$y_n = x_n + a_n \equiv [n(\alpha_a + a)]$$

This recursion implies the asymptotic

$$\liminf x_n(a)/n = \alpha_a$$

$$\liminf y_n(a)/n = \alpha_a + a$$

4.1 Tribonnaci game

[5]: Wythoff Nim as the “Fibonacci game”. Duch^ene and Rigo in-troduced a new characterization of the P-positions of Wythoff Nim, which deals with the Fibonacci word. Given a two-letter alphabet $\{a,b\}$ take the morphism

$$\phi : \{a,b\}ß\{a,b\}^*$$

defined as follows:

$$\phi(a) = ab$$

$$\phi(b) = a$$

By iterating this morphism from a, we get the famous Fibonacci word$w_F = (w_n)_n \geq 1 = \lim_{n\to\infty} \phi^n(a)$

$$w_F = abaababaabaababaababaabaababa....$$

In this word, we will use the convention that the first letter has index 1. For X = A, B (resp. x = a, b), we define the sets $X = \{X_1 < X_2 < ....\} = \{n \in N | w_n = x\}$Roughly speaking, the indices of the letters a (resp. b) in$w_F$correspond to the sequence(A_n) (resp. (B_n)). In addition, we set$A_0 = B_0 = 0$. According to this definition, the P-positions of Wythoff Nim exactly correspond to the sequence (A_n, B_n).
For the P-positions of k-Wythoff using the morphism

$$\phi^{'}(a) = a^k b$$

$$\phi^{'}(b) = a$$

Since the P-positions of Wythoff Nim are correlated to the Fibonacci word, a natural question arose: does there exist a 3-heap game whose P-positions can becoded by the so-called Tribonacci word w_T , defined as the unique fixed-point of the morphism

$$\sigma : \{a, b, c\}ß\{a, b, c\}^*$$

defined as follows:

$$\sigma(a) = ab$$

$$\sigma(b) = ac$$

$$\sigma(c) = a$$

Hence w_T starts with:

$$w_T = abacabaabacababacabaabacabacabaabacababaca......$$

The first values of the sequence (A_n, B_n, C_n) derived from the Tribonacci word are a 3-heap game is built, whose P-positions exactly correspond to the sequence (A_n, B_n, C_n) (with all their permutations adjoined). This game has been called the Tribonacci game

Chapter 5

GAMES AND P POSITION

5.1 Rat game:

[6] The Rat game is played on 3 piles of tokens by 2 players who play alternately. Positions in the game are denoted throughout in the form (x, y, z), with $0 \leq x \leq y \leq z$, and moves in the form $(x,y,z) \mapsto (u,v,w)$, where of course also $0 \leq u \leq v \leq w$. The player first unable to move — because the position is (0, 0, 0) — loses; the opponent wins.
There are 3 types of moves:
1) Take any positive number of tokens from up to 2 piles.
2)Take $l > 0$ from the x pile,$k > 0$ from the y pile, and an arbitrary positive number from the z pile, subject to the constraint $|k-l| < a$,where

$$a = \begin{cases} 1, & \text{if} y-x \not\equiv 0(mod7) \\ 2, & \text{if} y-x \equiv 0(mod7) \end{cases}$$

3) Take $l > 0$ from the x pile,$k > 0$from the z pile, and an arbitrary positive number from the y pile, subject to the constraint $|k-l| < b$, where b = 3 if w = u; otherwise,

$$b = \begin{cases} 5, & \text{if} u-w \not\equiv 4(mod7) \\ 6, & \text{if} u-w \equiv 04(mod7) \end{cases}$$

In a move of type (2) we permit the permutation $x \mapsto v, y \mapsto w, z \mapsto u(sol = x-v, k = y-w)$, in addition to $x, y \mapsto v, z \mapsto w(l = x-u, k = y-v)$. No other permutations are allowed for (2), and none (except$x \mapsto u, y \mapsto v, z \mapsto w$) for (3). For (1), any rearrangement is possible. When we write $(x,y,z) \mapsto (u,v,w), wealwaysmeanx \mapsto u, y \mapsto v, z \mapsto w$. Note that in (2), the congruence conditions depend only on 2 of the piles moved from: the smallest and the intermediate; whereas in (3) they depend only on 2 of the piles moved to: the smallest and the largest. The case w = u in (3) is an initial condition, to accommodate the end position (0, 0, 0).

Example:

Given the position

$$p_1 = (1,2,4).$$

If player I takes one of the piles in its entirety, player II wins with a type (1) move to ϕ := (0, 0, 0) . If player I moves

$$p_1 \mapsto (1,2,t), t \in 1,2,3,$$

player II wins with a type (3) move to ϕ If player I moves

$$p_1 \mapsto (1,1,4)$$

, player II wins with a type (2) move to ϕ It's straightforward to see that if player I makes a move of type (1), or (2) or (3), then player II can win by moving to ϕ. It follows that p_1 is a P-position.

Consider now the position

$$p_2 = (3,6,10)$$

. Then player I can make a type (3) move top_1, so $p_2$is an N-position. Indeed,

$$(10-4)-(3-1) = 4 < 5 = b.$$

5.2 Ratwyt game

[5] This is another game played with rational numbers (Rat–rational,wyt–Wythoff). Given a rational number p/q in lowest terms, a step is defined by

$$p/q \mapsto (p-q)/q$$

otherwise $p/q \geq 1$

$$p/q \mapsto p/(q-p)$$

is played on a pair of reduced rational numbers $(p_1/q_1, p_2/q_2)$. A move consists of either

(i) doing any positive number of steps to precisely one of the rationals, or

(ii) doing the same number of steps to both. The first player unable to play (because both numerators are 0) loses.

5.3 Two characterizations of the P-positions

([6] is used in following subsections ,Theorems and lemmas) Let $T \subset\neq Z \geq 0$ Define the mex operator by $mex(T) = min(Z \geq 0\ T)$= smallest nonnegative integer not in T. Recall that the set of P-positions of a game is the set of positions for which the Previous (second) player can win, and the set of N-positions is the set of positions for which the Next (first) player can win. We begin with a recursive characterization of the P-positions of the Rat game.

Theorem 5.3.0.1. *The P-positions of the Rat game are given by (0, 0, 0), and for* $n \in \mathbf{Z}_{\geq 1}$
$a_n = mex\{a_i, b_i, c_i : 0 \leq i < n\} b_n = a_n + [(7n-2)/4], c_n = b_n[(7n-3)/2]$

$$R = \bigcup_{n=1}^{\infty} \{(a_n, b_n, c_n)\}$$

Theorem 5.3.0.2. *Let* $p, q \in \mathbf{Z}_{\geq 1}$ *with* $p > q, s \in \mathbf{Z}$*. Then*
(i) For every $t \in \mathbf{Z}$*, the q values* $[(pn+s)/q], n \in \{t+1, ..., t+q\}$ *are distinct (mod p).*
(ii) For every $k \in \mathbf{Z}, [(p(n+kq)+s)/q] = [(pn+s)/q] + kp$*.*

proof:
(i) Let $n_1, n_2 \in \{t+1, ..., t+q\}, n_1 \neq n_2, say n_2 > n_1$*. Then* $0 < (p/q) - 1 < [(pn_2 + s)/q] - [(pn_1 + s)/q]$
$< (pn_2 + s - pn_1 - s)/q$

$< p(n_2 - n_1)/q$

let $n_2 - n_1 = q - 1$

$< p(q-1)/q$

$< p(q-1)/q+1$

$< q$

(ii) It follows that for $n \in \{1,...,q\}, [(pn+s)/q]$ *contains distinct residues* $r_1 < ... < r_q(mod p)$*, for* $n \in \{q+1,...,2q\}$ *it contains* $p+r_1,...,p+r_q,...,$
for $n \in q+1,...,2q$ *it contains* $p+r_1,...,p+r_q,...,$
for $n \in kq+1,...,(k+1)q$ *it contains* $kp+r_1,...,kp+r_q$.

Theorem 5.3.0.3. *For all* $n \in \mathbf{Z}_{\geq 1}$

$$d_n = [(7n-2)/4], \delta_n = [(7n-3)/2], \Delta n = d_n + \delta_n$$

.

proof:

The assertion for d_n *is seen to hold for n = 1, 2, 3, 4. Therefore it holds for all* $\mathbf{Z}_{\geq 0}$ *by above theorem Similarly, the assertion about* δ_n *is seen to hold for n = 1, 2, therefore it holds for all* $\mathbf{Z}_{\geq 0}$*. Finally,*

$$\Delta n = C_n - A_n = (C_n - B_n) + (B_n - A_n) = \delta n + d_n.$$

Lemma 5.3.0.4. *(i) Each of the sequences* d_n, δ_n, Δ_n*is strictly increasing.*
(ii) For $n \in \mathbf{Z}_{\geq 1}$*,* $d_n < \delta_n < \Delta_n$
(iii) For $n \in \mathbf{Z}_{\geq 1}$*,* d_n*and* δ_n *are disjoint . In fact*

$$d_n \equiv \{1,3,4,6\}(mod 7)$$

$\delta_n \equiv \{2,5\}(mod7)$, *and each of the residues (mod 7) of* d_n*and* δ_n *are assumed for infinitely many n. Also* $d_n > \delta_0 = -2$ $\delta_n > d_0 = -1$ *and* $\Delta_n \equiv \{3,1,6,4\}(mod7)$.

(iv) $\bigcup_{n=1}^{\infty}(d_n \bigcup \delta_n) = \mathbf{Z}_{\geq 1}/\bigcup_{n=1}^{\infty}\{7i\}$

proof:

(i) and (ii) follows from the theorem

For all $n \in \mathbf{Z}_{\geq 1}$

$$d_n = [(7n-2)/4], \delta_n = [(7n-3)/2], \Delta n = d_n + \delta_n$$

. (iii) By inspection, this holds for n = 1, 2, 3, 4. It follows in general from theorem Let $p,q \in \mathbf{Z}_{\geq 1}$ *with* $p > q, s \in \mathbf{Z}$*. Then*

(i) For every $t \in \mathbf{Z}$*, the q values* $[(pn+s)/q], n \in \{t+1,...,t+q\}$ *are distinct (mod p).*

(iv) From (iii) we see that $\bigcup_{n=1}^{4}(d_n \bigcup \bigcup_{n=1}^{2} \delta_n) = \{1,2,3,4,5,6\}$ *now it follows from theoreom For every* $k \in \mathbf{Z}$, $[(p(n+kq)+s)/q] = [(pn+s)/q] + kp$.

Theorem 5.3.0.5. *For all* $n \in \mathbf{Z}_{\geq 0}$ *,*$a_n = A_n, b_n = B_n, c_n = C_n$ *.In other words, the set of triples* $R = \bigcup_{n=0}^{\infty}\{(a_n, b_n, c_n)\}$

proof:

We note that

$$(a_0, b_0, c_0) = (A_0, B_0, C_0) = (0,0,0)$$

. we know already that

$$(a_n, b_n, c_n) = (A_n, B_n, C_n)$$

for all $n < N$.

the sets A, B, C partition $\mathbf{Z}_{\geq 0}$*, and clearly*

$$A_n < B_n < C_n$$

for all $n \in \mathbf{Z}_{\geq 1}$ *Therefore*

$$A_N = mex_1\{A_i, B_i, C_i : 1 < N\}$$

. Otherwise A_N *would never be attained in the complementary sets A, B, C. Thus*$A_N = a_N$

.

Now

$$B_n - A_n = [(7n-2)/4]$$

, and

$$C_n - B_n = [(7n-3)/2]$$

for all$n \in \mathbf{Z}_{\geq 0}$ *is same as in the recursive definition of the triples*

$$(a_n, b_n, c_n)$$

. Hence also

$$B_N = b_N$$

, and

$$C_N = c_N$$

.

n	a_n	b_n	c_n
0	0	0	0
1	1	2	4
2	3	6	11
3	5	9	18
4	7	13	25
5	8	16	32
6	10	20	39
7	12	23	46
8	14	27	53
9	15	30	60
10	17	34	67
11	19	37	74
12	21	41	81
13	22	44	88
14	24	48	95
15	26	51	102

5.4 The mouse game

([6]is used in following) The Mouse game is played on 2 piles of tokens by 2 players who play alternately. Analogously to the Rat game, positions are denoted in the form (x, y), with $0 \leq x \leq y$, and moves in the form $(x,y) \mapsto (u,v)$, where of course also $0 \leq u \leq v$. The player first unable to move — because the position is (0, 0) — loses; the opponent wins. There are 2 types of moves:
1) Take any positive number of tokens from a single pile.
2)Take $l > 0$ from the x pile,$k > 0$ subject to the constraint $|k-l| < a$,where

$$a = \begin{cases} 1, & \text{if} y-x \not\equiv 0(mod 3) \\ 2, & \text{if} y-x \equiv 0(mod 3) \end{cases}$$

then we have

Theorem 5.4.0.1. *The P-positions of the Rat game are given by (0, 0) and for* $n \in \mathbf{Z}_{\geq 1}$ $A_n = mex\{A_i, B_i : 0 \leq i < n\} B_n = A_n + [(3n-1)/2]$

Theorem 5.4.0.2. *The P-positions of the Mouse game are given by (0, 0), and for* $n \geq 1, A_n = [3n/2], B_n = 3n-1.$

5.4.1 P-positions of the Mouse game with their differences d_n.

n	A_n	d_n	B_n
0	0	0	0
1	1	1	2
2	3	2	5
3	4	4	8
4	6	5	11
5	7	7	14
6	9	8	17
7	10	10	20
8	12	11	23
9	13	13	26
10	15	14	29
11	16	16	32
12	18	17	35
13	19	19	38
14	21	20	41
15	22	22	44

5.5 The Fat Rat Game

[6] The Fat Rat game is the case of the Rat Game played on an arbitrary number of $m \in \mathbf{Z} \geq 2$ piles. The games for $m \in \{2,3\}$ were analyzed in the previous sections. The P-positions for m = 4 are given down This follows the general rule of $A_n^k = [(2^m - 1)n/2^{m-k}] - 2^{k-1} + 1, k = 1,2,.....mn \geq$. It is not hard to see that for m = 4, the differences $d_n^i = A_n^{i+1} - A_n^i$ are

$d_n^1 = [15n - 4/8]$
$d_n^2 = [15n - 6/4]$
$d_n^3 = [15n - 7/2]$
and
$\bigcup_{n=1}^{\infty} \{d_n^1, d_n^2, d_n^3, \{15n\}\} = \mathbf{Z}_{\geq} 1$

for the Fat Rat game,

$2^m - 1/2^{m-1} = [1, 1, 2^{m-1} - 1]$. So we have the P positions. But what are the game rules? Even for the case m = 4, there are various possibilities to be checked out. It appears that 4 types of moves are required. Perhaps the case m = 4 will point to the game rules for general m. It appears that the transition from 3 to 4 piles is a stumbling block in a number of games. This seems to be the case, for example, for the 3-pile Tribonacci game, based on the Tribonacci word .The P-positions of the 3-pile Raleigh game are, for all $n \in \mathbf{Z}_{\geq} 0$

$$A_n = [[n\Phi]\Phi]$$

$$B_n = [n\Phi^2]$$

$$C_n = [[n\Phi^2]\Phi]$$

where $\phi = (1+\sqrt{5})/2$ (golden section). A natural generalization to $m > 3$piles may also be nontrivial.

n	A_n^1	d_n^1	A_n^2	d_n^2	A_n^3	d_n^3	A_n^4	Δ_n
1	1	1	2	2	4	4	8	7
2	3	3	6	6	12	11	23	20
3	5	5	10	9	19	19	38	33
4	7	7	14	13	27	26	53	46
5	9	8	17	17	34	34	68	59
6	11	10	21	21	42	41	83	72
7	13	12	25	24	49	49	98	85
8	15	14	29	28	57	56	113	98
9	16	16	32	32	64	64	128	112
10	18	18	36	36	72	71	143	125
11	20	20	40	39	79	79	158	138
12	22	22	44	43	87	86	173	151
13	24	23	47	47	94	94	188	164
14	26	25	51	51	102	101	203	177
15	28	27	55	54	109	109	218	190
16	30	29	59	58	117	116	233	203

5.6 Wythoff arrays , shuffles and interspersions

([7] Is used in following subsections and Theorems and lemmas in 5.6) wythoff arrays: A sequence a of positive integers is arrayable if the following properties hold:

(i) a(1)= 1,

(ii) a is strictly increasing

(iii) the complement with respect to the set of positive integers of the set $\{a(n)\}$ is infinite.

(iv) Suppose a is an arrayable sequence. Let b be the sequence formed by placing in increasing order the complement of the set $\{a(n)\}$. The Wythoff array of a is the array

W(i,j) with elements w(i,j) defined by

$$w(i,1) = a(a(i)) \equiv a^2(i)$$

,

$$w(i,2) = b(a(i)) = ba(i),$$

$$w(i,j) = \begin{cases} a(w(i,j-1)), & \text{if } j \text{ is odd and } j \geq 3 \\ b(w(i,j-2)), & \text{if } j \text{ is even and } j \geq 4 \end{cases}$$

for i=l,2, . . .

Lemma 5.6.0.1. *Let* $W = W(i,j)$ *be the Wythoff array of an arrayable sequence a. The rows of W are given by*

$$w(i,2t-1) = ab^{t-1}a(i)$$

and

$$w(i,2t) = b^t a(i)$$

for all $i,t \geq 1$

Proof:

For any positive integer i

$$w(i,1) = a(a(i)) \equiv a^2(i)$$

,

$$w(i,2) = b(a(i)) = ba(i),$$

as claimed for t = 1.

for $t \geq 1$ $w(i,2(t+1)-1)$

$= w(i, 2t+1)$

$= aw(i, 2t)$

$= ab^t a(i)$

$w(i, 2(t+1))$

$= w(i, 2t+2)$

$= bw(i, 2t)$

$= b^{t+1}a(i)$

Theorem 5.6.0.2. *Suppose $W = W(i, j)$is the Wythoff array of an arrayable sequence a. The sequences $\{w(i, j) : j \geq 1\}$,for i=l,2,3 ,..., partition the set of positive integers*

Proof:

We shall show inductively that every n does occur in W. by Definition (iii) there is a smallest i_0 for which $a(i_0) > i_0$. Thus $w(i, 1) = a(i) = i$ for $1 \leq i \leq i_0$.

Now suppose, for arbitrary $k \geq i_0$, that every$1 < k$ is in W. Since a and b are complementary, there exists m such that $k = a(m)$ or $k = b(m)$. Since$m < k$, the induction hypothesis implies, by Lemma that m has the form $ab^t a(i)$ or $= b^t a(i)$ for $t \geq O$ and i<m, so that there are four cases:

Case 1:

$k = a(m)$ *and* $m = ab^t a(i)$

$k = a(ab^t a(i))$

$= a^2(b^t a(i))$

$= w(b^t a(i), 1)$

Case 2:

$k = a(m)$ *and* $m = b^t a(i)$

$k = a(b^t a(i))$

$= ab^t a(i)$

$= w(i, 2t+1)$

Case 3:

$k = b(m)$ *and* $m = ab^t a(i)$

$k = b(ab^t a(i))$

$= ba(b^t a(i))$

$= w(b^t a(i), 2)$

Case 4:

$k = b(m)$ *and* $m = b^t a(i)$

$k = b(b^t a(i))$

$= bb^t a(i)$

$= b^{t+1} a(i)$

$= w(i, 2t+2)$

Thus k is in W and, by induction, every positive integer is in W. Next we show that the numbers w(i, j) are distinct. by Lemma it suffices to show that none of the following equations can hold:

2) $ab^{p-1}a(i) = b^p a(j)$

3)$ab^{s-1}a(i) = ab^{t-1}a(j)$

4)$b^s a(i) = b^t a(j)$

for positive integers i, j, p, q, s and t with $s < t$. *Now (2) contradicts the disjointness of a and b. Since a and b are increasing, (3) implies*

$$a(i) = b^{t-s} a(j)$$

, and the same is implied by (4), which again contradicts the disjointness of a and b.

- Shuffles and interspersions Suppose A =A(i,j) is an array of distinct real numbers. For any positive integer $k > 2$, we define an array $\phi_k(A)$ called the k-shuffle of A. Beginning with k = 2, separate every row (r_i) of A into two sequences: the sequence $r_1, r_3, r_5, ...$ of terms with ascending odd indices, and the sequence $r_2, r_4, r_6, ...$ of terms with ascending even indices. Arrange the set of all such sequences in rows so that the sequence of their first terms forms an increasing sequence. The array thus determined is the 2-shuffle of A.
  Let s be the ordered complement of the first column of an interspersion A. In it is shown that a(i,j) = s(a(i, j- 1)) for all $i \geq 1$ and all $j > \geq 2$. Because of this property, A is said to be the dispersion of s.

Lemma 5.6.0.3. *Suppose $A = A(i, j)$is an interspersion. if $a(i, j) < a(i', j')$then*

$a(i, j+1) < a(i', j'+1)$ for every positive integer 1.

Proof. if p=a(i,j) and $q = a(i', j')$ Let s denote the sequence of which A is the dispersion. Since $p < q$and s is strictly increasing, we have $s(p+m) - s(p+m-1) \geq 1$ for m= 1,2, ... q-p. Adding these q-p inequalities gives
=$s(p+q-p) - s(p+q-p-1)$
=$s(q) - s(q-1)$
we know p=q-1
=$s(q) - s(p) \geq q-p$
implies $s(a(i, j)) - a(i, j) < s(a(i', j')) - a(i', j')$

$a(i,j+1)-a(i,j) < a(i',j'+1)-a(i',j')$

now by induction

$a(i,j+l)-a(i,j) < a(i',j'+l)-a(i',j')$

$a(i,j+l) < a(i',j'+l)-a(i',j')+a(i,j)$

for every positive integer 1. □

Chapter 6

ANALYSIS AND CONCLUSIONS

In **Chapter 2** we have seen examples and seen how we can generate wythoff pairs using various kinds of sequences like the fibonacci word pattern, the tribonacci word pattern . we have seen that the word pattern we get is increasing sequence which is termed as the alpha and omega generation of pairs.

In **Chapter 3** we have introduced the puppy kitten games which is famous as queen cornering game which basically talks about stratergies of wining the game using a certain pair of sequence called the safe pairs.basically finding safe pairs of sequence using the golden ratio.

In **Chapter 4** talks about the n piles of heap what are the stratergies concerning us to take a particular move using the definition of n piles and proving theorem based on it saying every wythoff sequence is special .further we add on to the tribonacci game which is played using three piles.

Chapter 5 here we focus on different types of games which has similar type of name the rat, ratwyt,mouse and fat rat game so the difference is rat game uses three piles and mouse game uses two piles where as ratwyt ant fat rat games are basically of the type p/q.

here basically all chapters focus on winning stratergies on how a player can make sure the number he or she is taking is correct.

Structural properties and behaviors of Wythoff's sequence are investigated. The main result is that for such a sequence, there always exists an integer such that when n is large enough, A_n where $\alpha = (1+\sqrt{5})/2$, the golden section. The value of can also be easily determined by a relatively small number of pairs in the sequence. As a corollary, the two conjectures on the N-heap Wythoff's.

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