SOME THEOREMS IN PARTITION THEORY

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by

Mr. MOHANKUMAR SHARNAPPA LAMANI

22P0410013

ABC ID:393-199-053-097

201407987

Under the Supervisor of

Dr. Manvendra Tamba

School of Physical & Applied Sciences

Mathematics Discipline

GOA UNIVERSITY

APRIL 2024

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Some Theorems in Partition Theory" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Manvendra Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will be not responsible for the correctness of observation/experimental or other findings given in the dissertation.

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Signature:

Student Name: Mohankumar Sharnappa Lamani

Seat no: 22P0410013

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COMPLETION CERTIFICATE

This to certify that the dissertation report "Some Theorems in Partition Theory" is a bonafide work carried out by Mr. Mohankumar Sharnappa Lamani under my supervision in partial fulfilment of the requirement for the award of the degree of Masters of Sciences in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences, Goa University.

Signature:

Supervisor: Dr. Manvendra Tamba

Date: 10/05/2024

School of Physical and Applied Sciences
Goa University

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Preface

This Project Report has been prepared in the partial fulfilment of the requirement for the Subject: MAT-651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: "Some Theorems in Partition Theory". This survey is divided into six chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

First Chapter:

The Introductory stage of this Project report is based on overview of the Partition theory and a small history about Rogers-Ramanujan identity.

Second Chapter:

This chapter deals with the detail proof of first known Rogers-Ramanujan identity i.e. Modulo 5. In this chapter we have explained the modulo 5 identities in detail by Analytically and Combinatorically.

Third Chapter:

In this chapter we have introduced a Ramanujan's Theta function and some different notation which can be used to prove the different identities. Also here we have introduced about Jacobi's triple product identity and some general transformations.

Also we have introduced some identities from Lucy Slater's famous list of Rogers-Ramanujan type Identities.

Fourth Chapter:

In this chapter we have given three new partition theorems of the classical Rogers-Ramanujan type which are very much in the style of MacMahon. These are a continuation of four theorems of the same kind given recently by the second author. One of these new theorems, very similar to one of the original Rogers-Ramanujan MacMahon type theorems is as follows: Let C(n) denote the number of partition of n into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7 (mod 20)$. Let D(n) denote the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_t, where\ b_t \geq 2, b_i \geq b_{i+1}$, and, if

$$1 \leq i \leq \left[\frac{t-2}{2}\right], b_i - b_{i+1} \geq 2$$

Then C(n)=D(n) for all n.

Fifth Chapter:

In this Chapter the following theorem is proved and generalized.

The partitions of any integer, n, into parts of the forms 6m, 6m+2, 6m+3, 6m+4 are equinumerous with those partitions of n into parts ≥ 2 which neither involve sequences nor allow any part to appear more than twice.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my Mentor, Dr. Manvendra Tamba, who was a continual source of inspiration. He pushed me to think imaginatively and urged me to do this homework without hesitation. His vast knowledge, extensive experience, and professional competence in “Number Theory” enabled me to successfully accomplish this project. This endeavor would not have been possible without his help and supervision.

ABSTRACT

The most famous of the "Series = Product" identities are

For $|q| < 1$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 (mod 5)}}^{\infty} \frac{1}{1-q^n} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty}$$

And

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1 (mod 5)}}^{\infty} \frac{1}{1-q^n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q;q)_\infty}$$

Where

$$(q;q)_n = \prod_{j=1}^{n}(1-q^j) \; and \; (q;q)_\infty = \prod_{j=1}^{\infty}(1-q^j)$$

and

$$(a_1, a_2, a_3, \ldots a_s; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots \ldots (a_s;q)_\infty$$

Which are known as the celebrated original Rogers-Ramanujan Identities. These two identities have motivated extensive research over the past hundred years. They were first proved by L. J. Rogers in 1894 that was completely ignored. They were rediscovered without proof by Ramanujan sometimes before 1913. Also in 1917, these identities were rediscovered and proved independently by Issai Schur. There are now many different proofs of these identities. In the ensuing decades, numerous identities that are similar to the Rogers-Ramanujan Identities has been discovered by several eminent mathematicians like Jackson, W. N. Bailey, G. E. Andrews, L. J. Slater, A. K. Agarwal, etc.

The Rogers-Ramanujan Identities have two aspects: one analytical and the other is combinatorial. In this paper, some identities of Rogers-Ramanujan Type related to modulo 5, 7,8 and 10 has been derived by using some general transformation between Basic Hypergeometric Series and with the incorporation of some identities from Lucy Slaters famous list of 130 identities of Rogers-Ramanujan type.

Key words: Rogers-Ramanujan Identity, Slaters Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey Pair etc.

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Chapter 1: The Fundamentals of Partition Theory

(All the material in this chapter is taken from [4])

1.1 Partition of Numbers

Definition 1.1. A partition of a positive integer n is a finite non-increasing sequence of positive integers whose sum is n.

Definition 1.2. A partition of a positive integers n is finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition.

Definition 1.3. The partition function $p(n)$ denotes the number of partition of n.

Remark: Obviously $p(n) = 0$ where n is negative. We shall set $p(0) = 1$ with the observation that the empty sequence form the only partition of zero.

The following table list the partitions of n and the value of $p(n)$ for n up to 6

N	Partition of n	$p(n)$
1	1	1
2	2,1+1	2
3	3,2+1,1+1+1	3
4	4,3+1,2+2,2+1+1,1+1+1+1	5
5	5,4+1,3+2,3+1+1,2+2+1,2+1+1+1,1+1+1+1+1	7
6	6,5+1,4+2,4+1+1,3+3,3+1+1,2+1+1+1+1,2+3+1,2+2+2,2+2+1+1,1+1+1+1+1+1	11

Remark: We can also write partition of number in this way as well.

Consider the partition of 10:

$10 = 1 + 1 + 1 + 2 + 2 + 3$

This can be written as

$$3(1) + 2(2) + 1(3)$$

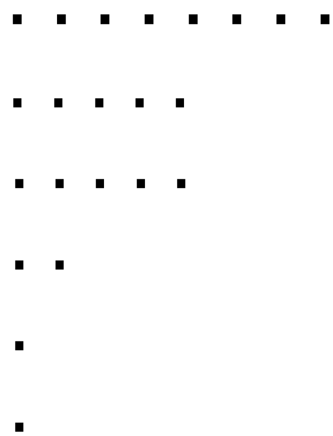
P(n) is called "usual" partition function. It counts the partitions in which "parts" are any natural numbers and "number of times parts are repeated is any whole no."

1.2 Graphical Representation of Partitions:

Another effective elementary device for studying partition is the graphical representation. To each partition λ is its graphical representation $\mathcal{G}_\lambda$ (or Ferrer's graph), which formally is the set of point with integral co-ordinate (i, j) in the plane such that if $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ $then$ $(i, j) \in \mathcal{G}_\lambda$ if and only if $0 \geq i \geq -n + 1, 0 \leq j \leq \lambda_{|i|+1} - 1$.

Rather than dwell on this formal definition, we shall by means of a few example, fully explain the graphical representation.

The graphical representation of the partition $8+5+5+2+1+1$ is



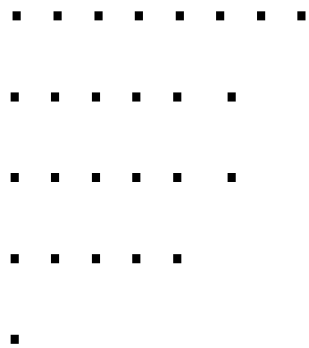
The graphical representation of the partition $7+3+3+2+1+1$ is

Note that the i^{th} row of the graphical representation of $(\lambda_1, \lambda_2, \dots, \lambda_n)$ contains λ_i points (or dots or nodes).

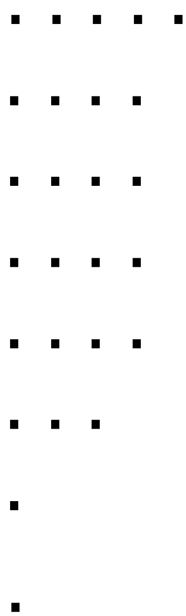
Definition 1.4. If $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ is a partition, we may define a new partition $\lambda' = \lambda_1{}', \dots, \lambda_m{}'$ by choosing $\lambda_i{}'$ as the number of parts of λ that are $\geq i$. The partition λ' is called the conjugates of λ.

While the formal definition of conjugate is not too revealing. We may better understand the conjugate by using graphical representation. From the definition, we see that the conjugate of the partition $8+6+6+5+1$ is $5+4+4+4+3+1+1$

The graphical representation of $8+6+6+5+1$ is



And the conjugate of this partition is obtained by counting the dots in successive columns; i.e., the graphical representation of the conjugate is obtained by reflecting the graph in the main diagonal. Thus the graph of the conjugate partition is



Notice that not only does the graphical representation provided a simple method by which to obtain the conjugate of λ, but it also shows directly that the conjugate partition λ' is a partition of the same integers as λ is; that is, $\sum \lambda_i = \sum \lambda_i{}'$.

Furthermore, it is clear that conjugation is an involutions of the partitions of any integer in that the conjugate of the conjugate of λ is again λ.

Let us now prove some theorems on partition, using graphical representation.

Theorem 1:

The number of partitions of n with at most m parts equals the number of partitions of n in which no parts exceeds m.

We proof this using an example

Let us consider the partition of 6, first into at most three parts and then into parts none of which exceeds 3.

We shall list conjugate opposite each other

6	1+1+1+1+1+1
5+1	2+1+1+1+1
4+2	2+2+1+1
4+1+1	3+1+1+1
3+3	3+2+1
3+2+1	3+3
2+2+2	

Theorem 1 is quite useful and shows how a graphical representation can be used directly to obtain important information.

1.3: Generating Function:

Note:

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
2. $\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots$

Consider

$$\prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdots$$

$$= \left(1 + x^{1(1)} + x^{2(1)} + x^{3(1)} + \cdots\right)\left(1 + x^{1(2)} + x^{2(2)} + x^{3(2)} + \cdots\right)\left(1 + x^{1(3)} + x^{2(3)} + x^{3(3)} + \cdots\right)\left(1 + x^{1(4)} + x^{2(4)} + x^{3(4)} + \cdots\right) \ldots$$

$$= 1 + \left(x^{1(1)}\right) + \left(x^{1(2)} + x^{2(1)}\right) + \left(x^{1(3)} + x^{3(1)} + x^{1(1)+1(2)}\right) + \ldots$$

$$= 1 + p(1)x + p(2)x^2 + p(3)x^3 + \cdots$$

$= 1 + \sum_{n=1}^{\infty} p(n)x^n$

A function $f: \mathbb{N} \longrightarrow \mathbb{C}$ which requires counting of partitions to calculate $f(n)$ is called partition function, $p(n)$ is called "usual " partition function .

The series

$$F(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n$$

Is called the "generating function" of $p(n)$.

Remarks:

1. Let $p_1(x)$ = number of partition of n into distinct parts .
   Then the generating function is
   $$p_1(x) = \prod_{n=1}^{\infty}(1 + x^n)$$
2. Let $p_2(x)$ =number of partition of n into parts repeated at most twice.
   $$p_2(x) = \prod_{n=1}^{\infty}(1 + x^n + x^{2n})$$
3. $p_3(x)$ = number of partition of n into odd parts
   $$p_3(x) = \prod_{n=1}^{\infty}\frac{1}{(1 - x^{2n-1})}$$
4. $p_4(x)$ =number of partition of n into odd and distinct parts
   $$p_4(x) = \prod_{n=1}^{\infty}(1 + x^{2n-1})$$

Remarks: To show that two partition function are equal it suffices to show that their generating function are equal.

Theorem 2:

The number of partition of n into unequal parts is equal to the number of partitions into odd parts.

Proof:

$$(1 + x)(1 + x^2)(1 + x^3) \ldots = \frac{(1-x^2)}{(1-x)} \cdot \frac{(1-x^4)}{(1-x^2)} \cdot \frac{(1-x^6)}{(1-x^3)} \ldots$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots}$$

1.4 The Rogers-Ramanujan Identities:

We end this chapter with two identities which are known as the celebrated original Rogers-Ramanujan identities:

For $|q| < 1$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 (mod 5)}}^{\infty} \frac{1}{1-q^n} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q;q)_\infty}$$

And

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1 (mod 5)}}^{\infty} \frac{1}{1-q^n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q;q)_\infty}$$

Where

$$(q;q)_n = \prod_{j=1}^{n} (1-q^j) \; and \; (q;q)_\infty = \prod_{j=1}^{\infty} (1-q^j)$$

and

$$(a_1, a_2, a_3, \ldots a_s; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots \ldots (a_s;q)_\infty$$

These two identities have motivated extensive research over the past hundred years. They were first proved by L. J. Rogers in 1894 that was completely ignored. They were rediscovered without proof by Ramanujan sometime before 1913.

Also in 1917, these identities were rediscovered and proved independently by Issai Schur. There are many different proofs of these identities. In the ensuring decades, numerous identities that are similar to the Rogers-Ramanujan identities has been discovered by several eminent mathematicians like Jackson. N. Bailey, G.E. Andrews, L. J. Slaters, A. K. Agrawal, etc.

The Rogers-Ramanujan identities have two aspects: one analytical and the other is combinatorial. Some identities of Rogers-Ramanujan type related to modulo 6,7 and 10 has been derived by using some general transformation between basic hypergeometric series and with the incorporation of some identities from Lucy Slater's famous list of 130 identities of Rogers-Ramanujan type.

Chapter 2. The Rogers-Ramanujan Identity

2.1 Introduction:

The so-called Rogers-Ramanujan identities were sent by Ramanujan to Hardy nearly 100 years ago. In the next few years, the identities were circulated amongst mathematician, but nobody, including Ramanujan, was able to prove them. Then one day, while rifling through old back copies of a journal, Ramanujan himself discovered them in an obscure paper written in 1894 by the English mathematician Rogers. This spurred both Rogers and Ramanujan to provide simpler proofs of the identities, that were published in 1919.

In these chapter will prove Rogers –Ramanujan identity modulo 5 by two ways,

1. Analytical proof and
2. Combinatorial interpretation

2.2. Analytical Proof:

Theorem 2.1:

$$1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\cdots = \frac{1}{(1-x)(1-x^6)\ldots(1-x^4)(1-x^9)\ldots}$$

i.e.

$$1+\sum_{m=1}^{\infty}\frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}=\prod_{m=0}^{\infty}\frac{1}{(1-x^{5m+1})(1-x^{5m+4})} \qquad (1)$$

Theorem 2.2:

$$1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)}+\cdots = \frac{1}{(1-x^2)(1-x^7)\ldots(1-x^3)(1-x^8)\ldots}$$

i.e.

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}=\prod_{m=0}^{\infty}\frac{1}{(1-x^{5m+2})(1-x^{5m+3})} \qquad (2)$$

Proof: Proof of Theorem 2.1 and Theorem 2.2.

We write

$$P_0 = 1, P_r = \prod_{s=1}^{r} \frac{1}{1 - x^s}, \quad Q_r = Q_r(a) = \prod_{s=r}^{\infty} \frac{1}{1 - ax^s},$$

$$\lambda(r) = \frac{1}{2} r(5r + 1),$$

And define the operator η by

$$\eta f(a) = f(ax)$$

We introduce the auxiliary function

$$H_m = H_m(a) = \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r) - mr} (1 - a^m x^{2mr}) P_r Q_r \qquad (i)$$

Where $m = 0, 1, 2$. Our object is to expand H_1 *and* H_2 in powers of a.

We first prove that

$$H_m - H_{m-1} = a^{m-1} \eta H_{3-m} \quad (m = 1,2) \qquad (ii)$$

Where

$$H_m = \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r) - mr} (1 - a^m x^{2mr}) P_r Q_r$$

$$H_{m-1} = \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r) - (m-1)r} \left(1 - a^{m-1} x^{2(m-1)r}\right) P_r Q_r$$

$$H_m - H_{m-1} = \left(\sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r) - mr} (1 - a^m x^{2mr}) P_r Q_r \right) - \left(\sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r) - (m-1)r} \left(1 - a^{m-1} x^{2(m-1)r}\right) P_r Q_r \right)$$

$$= \sum_{r=0}^{\infty} (-1)^r a^{2r} \left\{ x^{\lambda(r)-mr} (1 - a^m x^{2mr}) - x^{\lambda(r)-(m-1)r} \left(1 - a^{m-1} x^{2(m-1)r}\right) \right\} P_r Q_r$$

$$= \sum_{r=0}^{\infty} (-1)^r a^{2r} \left\{ x^{\lambda(r)} . x^{-mr} (1 - a^m x^{2mr}) - x^{\lambda(r)} x^{-(m-1)r} (1 - a^{m-1} x^{2(m-1)r}) \right\} . P_r Q_r$$

$$= \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)} \left\{ x^{-mr} - a^m x^{-mr+2mr} - x^{-(m-1)r} + a^{m-1} . x^{-(m-1)r} . x^{2(m-1)r} \right\} . P_r Q_r$$

$$= \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)} \left\{ x^{-mr} - a^m x^{mr} - x^{-(m-1)r} + a^{m-1} . x^{(m-1)r} \right\} . P_r Q_r$$

$$\Longrightarrow H_m - H_{m-1} = \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda(r)} C_{mr} P_r Q_r \qquad (*)$$

Where

$$C_{mr} = x^{-mr} - a^m x^{mr} - x^{-(m-1)r} + a^{m-1} . x^{(m-1)r}$$

$$C_{mr} = a^{m-1} x^{r(m-1)} (1 - a x^r) + x^{-mr} (1 - x^r)$$

Now,

$$Q_r = \prod_{s=r}^{\infty} \frac{1}{1 - a x^s}$$

$$= \frac{1}{(1 - a x^r)(1 - a x^{r+1}) \dots}$$

$$Q_{r+1} = \prod_{s=r+1}^{\infty} \frac{1}{(1 - a x^{r+1})(1 - a x^{r+2}) \dots}$$

$$(1 - a x^r) Q_r = Q_{r+1} \qquad (a)$$

$$P_r = \prod_{s=1}^{r} \frac{1}{1 - x^s} = \frac{1}{(1 - x)} . \frac{1}{(1 - x^2)} . \frac{1}{(1 - x^3)} \dots \frac{1}{(1 - x^r)}$$

$$P_{r-1} = \prod_{s=1}^{r-1} \frac{1}{1 - x^s} = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \dots (1 - x^{r-1})}$$

$$P_r = \frac{1}{(1 - x^r)} . P_{r-1}$$

$$(1-x^{r})P_{r}=P_{r-1},(1-x^{0})P_{0}=0 \qquad (b)$$

and so now (*) using (a) and (b)

$$H_{m}-H_{m-1}=\sum_{r=0}^{\infty}(-1)^{r}a^{2r}x^{\lambda(r)}C_{mr}P_{r}Q_{r}$$

$$=\sum_{r=0}^{\infty}(-1)^{r}a^{2r+m-1}x^{\lambda(r)+r(m-1)}.P_{r}Q_{r+1}+\sum_{r=0}^{\infty}(-1)^{r}a^{2r}x^{\lambda(r)-mr}P_{r-1}Q_{r}$$

In the second sum on the right-hand side of this identity we change r into r+1

Thus

$$H_{m}-H_{m-1}=\sum_{r=0}^{\infty}(-1)^{r}a^{2r+m-1}x^{\lambda(r)+r(m-1)}.P_{r}Q_{r+1}$$
$$+\sum_{r=0}^{\infty}(-1)^{r+1}a^{2(r+1)}x^{\lambda(r+1)-m(r+1)}P_{(r+1)-1}Q_{r+1}$$

$$=\sum_{r=0}^{\infty}(-1)^{r}\left[a^{2r+m-1}.x^{\lambda(r)+r(m-1)}-a^{2(r+1)}.x^{\lambda(r+1)-m(r+1)}\right]P_{r}Q_{r+1}$$

$$H_{m}-H_{m-1}=\sum_{r=0}^{\infty}(-1)^{r}D_{mr}P_{r}Q_{r+1}$$

Where

$$D_{mr}=a^{2r+m-1}.x^{\lambda(r)+r(m-1)}-a^{2(r+1)}.x^{\lambda(r+1)-m(r+1)}$$
$$=a^{2r+m-1}.x^{\lambda(r)+r(m-1)}-a^{2r+2}.x^{\lambda(r+1)-m(r+1)}$$
$$=a^{m-1+2r}.x^{\lambda(r)+r(m-1)}(1-a^{3-m}x^{(2r+1)(3-m)})$$
$$=a^{m-1}\eta\left\{a^{2r}x^{\lambda(r)-r(3-m)}(1-a^{3-m}x^{2r(3-m)})\right\}$$

$$\because \lambda(r+1)-\lambda(r)=\frac{1}{2}(r+1)[5r+5+1]-\frac{1}{2}r(5r+1) \qquad (by\ definition)$$
$$=\frac{1}{2}\{(r+1)(5r+6)-r(5r+1)\}$$
$$=\tfrac{1}{2}\{5r^{2}+6r+5r+6-5r^{2}-r\}$$
$$=\tfrac{1}{2}\{11r+6-r\}$$
$$=\tfrac{1}{2}\{10r+6\}$$

$$= 5r + 3$$

Also $Q_{r+1} = \eta Q_r$ and so

$$H_m - H_{m-1} = a^{m-1}\eta \sum_{r=0}^{\infty} (-1)^r a^{2r} x^{\lambda r - r(3-m)}.(1 - a^{3-m} x^{2r(3-m)}) P_r Q_r$$

$$H_m - H_{m-1} = a^{m-1}\eta H_{3-m} \quad \text{(Hence Proved (ii))}$$

If we put $m = 1\ and\ m = 2\ in\ (ii)$

$$H_1 - H_0 = \eta H_2$$

$$H_1 = \eta H_2 \quad \because H_0 = 0 \qquad (iii)$$

$$H_2 - H_1 = a\eta H_1$$

So that

$$H_2 = \eta H_2 + a\eta^2 H_2 \qquad (iv)$$

We use this to expand H_2 in powers of a

If

$$H_2 = c_0 + c_1 a + c_2 a^2 + \cdots$$
$$= \sum c_s a^s$$

where the c_s are independent of a, then $c_0 = 1$ and (iv) gives

$$H_2 = \eta\left(\sum c_s a^s\right) + a\eta^2\left(\sum c_s a^s\right)$$
$$= \sum c_s a^s x^s + a \sum c_s (ax^2)^s$$
$$= \sum c_s a^s x^s + \sum c_s a^{s+1}.x^{2s}$$
$$\Rightarrow \sum c_s a^s = \sum c_s a^s x^s + \sum c_s a^{s+1}.x^{2s}$$

$$\Rightarrow c_0 a^0 + c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + \cdots$$
$$= c_0 a^0 x^0 + c_1 a x + c_2 a^2 x^2 + c_3 a^3 x^3 + \cdots + c_0 a x^0 + c_1 a^2 x^2 + c_2 a^3 x^6 + c_3 a^4 x^8 + \cdots$$

$$\Rightarrow 1 + c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + \cdots$$
$$= 1 + c_1 a x + c_2 a^2 x^2 + c_3 a^3 x^3 + \cdots + a + c_1 a^2 x^2 + c_2 a^3 x^6 + c_3 a^4 x^8 + \cdots$$

Hence, equating the co-efficient of a^s, we have

$$c_1 a = c_1 a x + a$$

$$c_1 a = a(c_1 x + 1)$$

$$c_1 = c_1 x + 1 \quad \text{(cancelling } a \text{ on both side)}$$

$$c_1 - c_1 x = 1$$

$$c_1 = \frac{1}{1-x}$$

Similarly

$$c_2 a^2 = c_2 a^2 x^2 + c_1 a^2 x^2$$

$$c_2 = c_2 x^2 + c_1 x^2$$

$$c_2 = c_2 x^2 + \frac{x^2}{1-x}$$

$$c_2(1-x^2) = \frac{x^2}{1-x}$$

$$c_2 = \frac{x^2}{(1-x^2)(1-x)}$$

Similarly

$$c_3 a^3 = c_3 a^3 x^3 + c_2 a^3 x^6$$

$$c_3(1-x^3) = \frac{x^2 . x^6}{(1-x)(1-x^2)}$$

$$c_3 = \frac{x^2 . x^6}{(1-x)(1-x^2)(1-x^3)}$$

Continuing in this way we get

$$c_s = \frac{x^{2s-2}}{1-x^s} . c_{s-1}$$

$$= \frac{x^{2+4+6+\cdots+2(s-1)}}{(1-x)(1-x^2)\ldots(1-x^s)} = x^{s(s-1)} P_s$$

Hence

$$H_2(a) = \sum_{s=0}^{\infty} a^s x^{s(s-1)} P_s$$

If we put $a = x$, the right-hand side of this is the series in (2.1).

Also $P_r Q_r(x) = P_\infty$ and so, by (i)

$$H_2(x) = P_\infty \sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)} \left(1 - x^{2(2r+1)}\right)$$

$$= P_\infty \left\{ \sum_{r=0}^{\infty} (-1)^r x^{\lambda(r)} + \sum_{r=1}^{\infty} (-1)^r x^{\lambda(r-1)+2(2r-1)} \right\}$$

$$\Rightarrow H_2(x) = P_\infty \left\{ 1 + \sum_{r=1}^{\infty} (-1)^r \left(x^{\frac{1}{2}r(5r+1)} + x^{\frac{1}{2}r(5r-1)} \right) \right\}$$

Hence by the Theorem 356 [by Hardy and Wright pg. 376 See [4]] i.e. (special case of Jacobi identity)

$$\prod_{n=0}^{\infty} \{(1 - x^{5n+2})(1 - x^{5n+3})(1 - x^{5n+5})\} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+1)}$$

$$H_2(x) = P_\infty \prod_{n=0}^{\infty} \{(1 - x^{5n+2})(1 - x^{5n+3})(1 - x^{5n+5})\}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1 - x^{5n+1})(1 - x^{5n+4})}$$

This completes the proof of theorem 2.1.

Again, by (iii)

$$H_1(a) = \eta H_2(a)$$

$$H_2(ax) = \sum_{s=0}^{\infty} a^s x^{s^2} P_s$$

And, for $a = x$, the right-hand side becomes the series in (2.2). Using (i) and Theorem 355, we complete the proof of Theorem 2.2 in the same way as we did that of Theorem 1

{Theorem 355: [by Hardy and Wright pg. 376 see [4]]

$$\prod_{n=0}^{\infty} \{1 - x^{5n+1})(1 - x^{5n+4})(1 - x^{5n+5})\} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+3)} \ \}.$$

2.3. Combinatorial interpretation of Theorem 2.1 and 2.2:

Consider Theorem 2.1

We can exhibits any square m^2 as

$$m^2 = 1 + 3 + 5 + \cdots + (2m - 1)$$

Or as shown by the black dots in the graph, in which m=6.

If we now take any partition of $n - m^2$ into m parts at most , with the parts in descending order, and add it to the graph as shown by the stars, where $m = 6$ and $n = 6^2 + 13 = 36 + 13 = 49$

We obtain a partition of n $(here\ n = 49 = 14 + 12 + 10 + 7 + 5 + 1)$ into parts without repetitions or sequence or parts whose minimal difference is 2. The left-hand side of (2.1) enumerates this type of partition of n.

$$m^2 = 1 + 3 + 5 + \cdots + (2m - 1)$$

$$m^2 = 6^2 = 36 = 1 + 3 + 5 + 7 + 9 + 11$$

$$11 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \underbrace{* * * *}_{4}$$

$$9 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \underbrace{* * *}_{3}$$

$$7 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot\cdot \underbrace{* * *}_{3}$$

$$5 \rightarrow \cdot\cdot\cdot\cdot\cdot \underbrace{* *}_{2}$$

$$3 \rightarrow \cdot\cdot\cdot \underbrace{* *}_{2}$$

$$1 \rightarrow \cdot$$

On other hand, the right-hand side enumerates partitions into numbers of the form $5m + 1\ and\ 5m + 4$.

Hence Theorem 2.1 may be restated as a purely combinatorial theorem viz

Theorem 2.3:

The number of partition of n with minimal difference 2 is equal to the number of partitions of n into parts of the forms $\boldsymbol{5m + 1\ and\ 5m + 4}$.

Similarly we know $m(m + 1) = 2 + 4 + 6 + \cdots + 2m$

For $m = 5$

$m(m + 1) = 5(5 + 1) = 5 * 6 = 30 = 10 + 8 + 6 + 4 + 2$

$$10 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \ \underbrace{*\ *\ *}_{3}$$

$$8 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \ \underbrace{*\ *\ *}_{3}$$

$$6 \rightarrow \cdot\cdot\cdot\cdot\cdot\cdot \ \underbrace{*\ *}_{2}$$

$$4 \rightarrow \cdot\cdot\cdot\cdot \ \underbrace{*\ *\ *}_{3}$$

$$2 \rightarrow \cdot\cdot$$

so the equivalent of Theorem 2.2 is

Theorem 2.4:

The number of partitions of n into parts not less than 2, and with minimal difference 2, is equal to the number of partitions of n into parts of the forms

$\boldsymbol{5m + 2 \ and \ 5m + 3}$.

Chapter 3: On Identities of Rogers-Ramanujan Type

3.1. Introduction:

For $|q| < 1$ the q-factorial is defined by (see [10])

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^k) for\ n \geq 1$$

$$and \qquad (a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k)$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1, a_2, a_3, \dots a_m; q)_n = (a_1;q)_n (a_2;q)_{nn} \dots \dots (a_m;q)_n$$

$$(a_1, a_2, a_3, \dots a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \dots \dots (a_m;q)_\infty$$

The Basic Hyper Geometric Series is

$$_{p+1}\varphi_{p+r}\begin{pmatrix} a_1,a_2,\dots,a_{p+1};q;x \\ b_1,b_2,\dots,b_{p+r} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q,q)_n (b_1;q)_n (b_2;q)_n \dots (b_{p+r};q)_n}$$

The series $_{p+1}\varphi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when $|x| < 1$.

Some definitions:

Ramanujan's Theta Function:

Ramanujan's Theta function ([2], P.11, Eq.(1.1.5)) is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \text{for } |ab| < 1$$

It is called a theta function, despite the lack of a theta in the notation, because it is equivalent, via change of variable, to the theta function of Jacobi.

$$\vartheta(z,w) := \sum_{n=-\infty}^{\infty} (-1)^n w^{n^2} e^{2niz} \text{ where } |w| < 1.$$

The following special case of $f(a,b)$ arise so often that they were given their own notation by Ramanujan([2],P.11):

$$\varphi(q) = f(q,q)$$

$$\psi(q) = f(q,q^3)$$

$$f(-q) = f(-q,-q^2)$$

Ramanujan further defines

$$\chi(q) \coloneqq \frac{f(-q^2;-q^2)}{f(-q)}$$

One of the most important results in the theory of theta functions is that they can be expressed as infinite products.

Jacobi's triple product identity: (see [7], P.2, Eq. (1.1.7))

For $|ab| < 1$, $f(a,b) = (-a,-b,ab;ab)_\infty$

Where $(a;w)_n = \prod_{n=1}^{\infty}(1-aw^n)$, $and\ (a_1,a_2,a_3,\dots a_r;w)_\infty = (a_1;w)_\infty(a_2;w)_\infty \dots\dots (a_r;w)_\infty$

An immediate corollary ([7], P-2, Eq. (1.1.8) (1.1.9), (1.1.10)) of this identity is thus:

$$f(-q) = (q;q)_\infty$$

$$\varphi(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

Sometimes a linear combination of two theta series can be expressed as a single infinite product as follows:(see [7], p-2, Eq. (1.1.12))

Quintuple Product identity:

$$(Q;x) = f(-wx^3,-w^2x^{-3}) + xf(-wx^{-3},-w^2x^3)$$

$$= \frac{f(wx^{-1},x)f(-wx^{-2},-wx^2)}{f(-w^2;-w^4)}$$

$$= (-wx^{-1},-x,w;w)_\infty(wx^{-2},wx^2;w^2)_\infty$$

3.2 Elementary Series product identities:

Remark : If $|q| < 1, |t| < 1$, then

$$1 + \sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\dots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty}\frac{(1-atq^n)}{(1-tq^n)} \qquad (3.2.1)$$

Euler Identity:

For $|t| < 1, |q| < 1$

$$\mathbf{1 + \sum_{n=1}^{\infty} \frac{t^n}{(1-q)(1-q^2)(1-q^3)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n}} \qquad \mathbf{(3.2.2)}$$

$$\mathbf{1 + \sum_{n=1}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(1-q)(1-q^2)(1-q^3)\dots(1-q^n)} = \prod_{n=0}^{\infty} (1+tq^n)} \quad \mathbf{(3.2.3)}$$

Proof:

To obtain equation (3.2.2) we put a=0 in (3.2.1)

$$1 + \sum_{n=1}^{\infty} \frac{(1-0)(1-0.q)\dots(1-0.q^{n-1})t^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-0.tq^n)}{(1-tq^n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{t^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-tq^n)}$$

To obtain (3.2.3) we replace a by a/b and t by bz in (3.2.1)

$$1 + \sum_{n=1}^{\infty} \frac{\left(1-{}^{a}/_{b}\right)\left(1-{}^{a}/_{b}\,q\right)\dots(1-{}^{a}/_{b}\,q^{n-1})(bz)^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\left(\frac{b-a}{b}\right)\left(\frac{b-aq}{b}\right)\dots\left(\frac{b-aq^{n-1}}{b}\right)(bz)^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(b-a)(b-aq)\dots(b-aq^{n-1})z^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{(1-\frac{a}{b}.bz.q^n)}{(1-bz.q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{(1-a.z.q^n)}{(1-bz.q^n)} \qquad (3.2.4)$$

Now set b=0 a=-1 in (3.2.4)

$$1 + \sum_{n=1}^{\infty} \frac{(0-(-1))(0-(-1)q)\dots(0-(-1)q^{n-1})z^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-(-1).z.q^n)}{(1-0.z.q^n)}$$

$$1+\sum_{n=1}^{\infty}\frac{(1.q.q^2\dots q^{n-1})z^n}{(1-q)(1-q^2)\dots(1-q^n)}=\prod_{n=0}^{\infty}(1+z.q^n)$$

$$1+\sum_{n=1}^{\infty}\frac{t^n q^{\frac{1}{2}n(n-1)}}{(1-q)(1-q^2)\dots(1-q^n}=\prod_{n=0}^{\infty}(1+z.q^n)$$

Hence proved.

3.3 Mod 2 identities

1.

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n}=\frac{f(-q,-q)}{f(-q)}$$

$$=\frac{(q;q;q^2;q^2)_\infty}{(q;q)_\infty}$$

$$=\frac{(q;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty}{(q;q)_\infty}$$

$$=\frac{\prod_{j=0}^{\infty}\left(1-qq^{2j}\right).(1-q.q^{2j})(1-q^2q^{2j})}{\prod_{j=0}^{\infty}(1-q.q^j)}$$

$$=\frac{\prod_{j=0}^{\infty}\left(1-q^{2j+1}\right).(1-q^{2j+1})(1-q^{2j+2})}{\prod_{j=0}^{\infty}(1-q^{j+1})}$$

$$=\prod_{j=1}^{\infty}(1-q^{1+2j})$$

2.

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(-q,q^2)_n}{(q^4;q^4)_n}=\frac{f(-q,-q)}{\psi(-q)}$$

$$=\frac{f(-q;-q)}{\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}}$$

$$=f(-q;-q)\times\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(q;q;q^2;q^2)_\infty\times(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q;q^2)_\infty (q;q^2)_\infty (q^2;q^2)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \prod_{j=1}^{\infty} \left(1 - qq^{2j}\right).(1 - q.q^{2j})(1 + qq^{2j})$$

$$= \prod_{j=1}^{\infty} \left(1 - q^{2j+1}\right).(1 - q^{2j+1})(1 + q^{2j+1})$$

$$= (1-q^3)(1-q^5)(1-q^7)\dots(1-q^3)(1-q^5)(1-q^7)\dots(1+q^3)(1+q^5)(1+q^7)\dots$$

3.

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-q,q^2)_n}{(q)_{2n}} = 2\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q,q^2)_n}{(q)_{2n+1}}$$

$$= \frac{f(1,q^2)}{\psi(-q)}$$

$$= \frac{f(1;q^2)}{\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}}$$

$$= f(1;q^2) \times \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(-1;-q^2;q^2;q^2)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(-1;q^2)_\infty (-q^2;q^2)_\infty (q^2;q^2)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= (-1;q^2)_\infty \prod_{j=1}^{\infty} \left(1 + q^2 q^{2j}\right)(1 + qq^{2j})$$

$$= (-1;q^2)_\infty \prod_{j=1}^{\infty} \left(1 + q^{2j+2}\right).(1 + q^{2j+1})$$

$$= (-1;q^2)_\infty (1+q^4)(1+q^6)(1+q^8)\dots(1+q^3)(1+q^5)(1+q^7)\dots$$

3.4 Mod 3 identities:

1.

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n}=\frac{f(-q)}{\varphi(-q)}$$

$$=\frac{(q;q)_\infty}{\frac{(q;q)_\infty}{(-q;q)_\infty}}$$

$$=(q;q)_\infty\times\frac{(-q;q)_\infty}{(q;q)_\infty}$$

$$=(-q;q)_\infty$$

$$=\prod_{j=1}^{\infty}(1+q^{1+j})$$

$$=(1+q^2)(1+q^3)(1+q^4)\dots$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-1)^n}{(q)_n(q;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q)_n}{(q)_n(q;q^2)_{n+1}}=\frac{f(q;q^2)}{f(-q)}$$

$$=\frac{(-q;-q^2;q^3;q^3)_\infty}{(q;q)_\infty}$$

$$=\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}{(q;q)_\infty}$$

$$=\frac{\prod_{j=1}^{\infty}(1+q^{1+3j})(1+q^{2+3j})(1-q^{3+3j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$=\frac{(1+q^4)(1+q^7)(1+q^{10})\dots(1+q^5)(1+q^8)(1+q^{11})\dots(1-q^6)(1-q^9)\dots.}{(1-q^2)(1-q^3)(1-q^4)\dots..}$$

2

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(q;q^2)^2{}_n}{(q^2;q^2)_{2n}}=\frac{f(q;q^2)}{\psi(q)}$$

$$=\frac{f(q;q^2)}{\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}}$$

$$= f(q;q^2) \times \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(-q;-q^2;q^3;q^3)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{\prod_{j=1}^{\infty}(1+q^{1+3j})(1+q^{2+3j})(1-q^{3+3j})(1+q^{1+2j})}{\prod_{j=1}^{\infty}(1-q^{2+2j})}$$

$$= \frac{(1+q^4)(1+q^7)(1+q^{10}) \dots (1+q^5)(1+q^8)(1+q^{11}) \dots (1-q^6)(1-q^9)(1-q^{12}) \dots (1+q^3)(1+q^5)..}{(1-q^4)(1-q^6)(1-q^8)..}$$

Similarly, we can do

$$\frac{f(-q;q^2)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(q)_{2n}}$$

3.5 Mod 4 Identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q;q^2)_n}{(q^4;q^4)_n} = \frac{f(-q^2;-q^2)}{\psi(-q)}$$

$$= \frac{f(-q^2;-q^2)}{\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}}$$

$$= f(-q^2;-q^2) \times \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q^2;q^2;q^4;q^4)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(q^2;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty \times (-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{\prod_{j=1}^{\infty}(1-q^{2+4j})(1-q^{2+4j})(1-q^{4+4j})(1+q^{1+2j})}{\prod_{j=1}^{\infty}(1-q^{2+2j})}$$

$$= \frac{(1-q^6)(1-q^{10})(1-q^{14}) \dots (1-q^6)(1-q^{10})(1-q^{14}) \dots (1-q^8)(1-q^{12})(1-q^{16}) \dots (1+q^3)(1+q^5)..}{(1-q^4)(1-q^6)(1-q^8)..}$$

Similarly, we can do the following identities

$$\frac{f(-q;-q^3)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}}$$

$$\frac{f(-q^2;-q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1)_n}{(q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_{n+1}}{(q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^2)_n}{(q)_{2n}}$$

$$\frac{f(q,q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}}$$

$$\frac{f(q;q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1)_{2n}}{(q^2;q^2)_n(q^2;q^4)_n}$$

$$\frac{f(q;-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}}$$

$$\frac{f(-q;q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}}$$

$$\frac{f(q;q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_{2n}}{(q;q^2)_{n+1}(q^4;q^4)_n}$$

$$\frac{f(q;-q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^4)_n}{(q)_{2n+1}(-q;q^4)_n}$$

$$\frac{f(-q^2;-q^2)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^2;q^2)_{n+1}}{(-q^3;q^3)_{n+1}(q)_n}$$

We now list some general transformations. These can be derived as limiting case of transformations between basic hyper geometric series.

Let a, b, c, d, γ *and* $q \in \mathbb{C}, |q| < 1$ then

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n q^{\frac{n(n-1)}{2}} ({-c\gamma}/{ab})^n}{(c,\gamma,q;q)_n} = \frac{(c\gamma/ab;q)_\infty}{(\gamma;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/a\,,c/b\,;q)_n q^{\frac{n(n-1)}{2}} (-\gamma)^n}{(c,c\gamma/ab\,,q;q)_n} \qquad (1)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (-\gamma)^n}{(b;q)_n (q;q)_n} = (\gamma;q)_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{(3n^2-3n)}{2}} (-b\gamma)^n}{(q;q)_n (b;q)_n (\gamma;q)_n} \qquad (2)$$

$$(-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{q^{\frac{(m^2+m)}{2}} a^m}{(-bq^n;q^n)_m (q;q)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{q^{\frac{n(m^2+m)}{2}} (b)^m}{(-aq;q)_{nm} (q^n:q^n)_m} \qquad (3)$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n^2-n} (-b)^n}{(q;q)_n (ab;q^2)_n} = \frac{(b;q^2)_n}{(ab;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n q^{n^2-n} (-bq)^n}{(q^2;q^2)_n (b;q^2)_n} \qquad (4)$$

The transformation (1) is a limiting case of q-analogue of the Kummer Thomae-Whipple formula (see [5] p-72, equation 3.2.7) or (see [7] p-40, equation (6.1.2)). The proof of transformation (2) is found in [1]. This transformation (2) is also appears in [7], (equation (6.1.11) p-41). A limiting case of a transformation due to Andrews [2] leads to the identity (3). This transformation (3) is also appears in [7], (equation (6.1.14) p-41). The identity (4) follows from a result of Andrews in [1]. It is also appearing as equation (6.1.19) in [7].

We now again list some general transformations. These can also be derived as limiting case of transformations between basic hyper geometric series.

Let a, b, c, d, γ *and* $q \in \mathbb{C}, |q| < 1$ then

$$\sum_{n=0}^{\infty} \frac{q^{n^2} \gamma^n}{({q}/{b}\,;q)_n (q;q)_n} = (-\gamma q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} \gamma^n (-\frac{q}{b};q)_{2n}}{(q^2;q^2)_n (\frac{q^2}{b^2};q^2)_n (-\gamma q^2;q^2)_n} \qquad (5)$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_{2n} q^{n^2-n} \left(-\frac{c^2}{d^2}\right)^n}{(q^2;q^2)_n (c;q)_{2n}} = \frac{(\frac{c^2}{d^2};q^2)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n} \gamma^n (-c)^n}{(q;q)_n (-\frac{c}{d};q)_n} \qquad (6)$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2-2n}(-a^2)^n}{(q^2;q^2)_n(a;q)_{2n}} = \frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-a)^n}{(q;q)_n} \qquad (7)$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n^2-n}(-b)^n}{(q;q)_n(ab;q^2)_n} = \frac{(b;q^2)_n}{(ab;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n q^{n^2-n}(-bq)^n}{(q^2;q^2)_n(b;q^2)_n} \qquad (8)$$

$$\sum_{n=0}^{\infty} \frac{(a^2;q)_n q^{3n^2+n/2}(a)^{2n}}{(q;q)_n} = \frac{(a^2q;q)_\infty}{(-aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-a;q)_n q^{n^2-n/2}(aq)^n}{(aq,q;q)_n} \qquad (9)$$

The transformation (5) is appeared as (6.1.12) on page 41 in [7]. The transformation (6) follows from(3.5.4) on pages 77-78 in [5], after replacing c with ${}^{aq}/_{c}$, then letting $a \to 0$ and finally letting $b \to \infty$. It also appeared as (6.1.17) on page 41 in [7]. The transformation (7) follows from (6) upon letting $d \to \infty$, and then replacing c with a. this transformation is also appeared as (6.1.18) on page 41 in [7]. The transformation (8) follows from a result of Andrews in [1] (see corollary 1.2.3. of [2], where it follows after replacing t by ${}^{t}/_{b}$, then letting $b \to \infty$ and finally replacing t by b). Finally, the transformation (9) is appeared as (6.1.21) on page 42 in [3].

Now we introduce some identities from Lucy Slater's famous list [9] of Rogers-Ramanujan Type identities. Each of them below that appears [8] is designated with the a "Slaters number" S.n.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_n}{(q;q^2)_{n+1}(q)_n} = \frac{f(-q;-q^5)}{\varphi(-q)}, \ \ (see\ [7], equation\ (2.6.5), p-13); (S_{22}) \qquad (i)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{3n(n+1)}{2}}}{(q;q^2)_{n+1}(q)_n} = \frac{f(-q^2;-q^8)}{f(-q)}, \ (see\ [7], equation\ (2.10.4), p-17); (S_{44}) \qquad (ii)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(3n-1)}{2}}}{(q;q^2)_{n+1}(q)_n} = \frac{f(-q^4;-q^6)}{f(-q)}, (see\ [7], equation\ (2.10.5), p-17); (S_{46}) \qquad (iii)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(-q)_n}{(q)_{2n+1}} = \frac{Q(q^7;-q^2)}{\varphi(-q)}, (see\ [7], equation\ (2.14.5), p-19); (S_{80}) \qquad (iv)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(-q)_n}{(q)_{2n}} = \frac{Q(q^7;-q)}{\varphi(-q)}, (see\ [7], equation\ (2.14.4), p-19); (S_{81}) \qquad (v)$$

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+3)}{2}}(-q)_n}{(q)_{2n+1}} = \frac{Q(q^7;-q^3)}{\varphi(-q)}, (see\ [7], equation\ (2.14.6), p-19); (S_{82}) \qquad (vi)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n+1}} = \frac{Q(q^{10};-q^3)}{f(-q)}, (see\ [7], equation\ (2.20.5), p-23); (S_{94}) \qquad (vii)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q)_{2n+1}} = \frac{Q(q^{10};-q^4)}{f(-q)}, (see\ [7], equation\ (2.20.6), p-23); (S_{96}) \qquad (viii)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n}} = \frac{Q(q^{10};-q)}{f(-q)}, (see\ [7], equation\ (2.20.3), p-22); (S_{99}) \qquad (ix)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{f(-q;-q^4)}{f(-q)}, see\ [7], equation\ (2.5.1) p-11); (S_{14}) \qquad (x)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(-q;q^2)_n (q^4;q^4)_n} = \frac{f(-q^2;-q^3)}{f(-q^2)}, see\ [7], equation\ (2.5.7) p-12); (S_{19}) \qquad (xi)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q;q)_{2n+1}} = \frac{f(-q^2;-q^{10})}{f(-q)}, see\ [7], equation\ (2.12.2) p-17); (S_{50}) \qquad (xii)$$

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}}=\frac{f(q;q^7)}{f(-q^2)}, see\ [7], equation\ (2.8.9)p-15);(S_{38}) \qquad (xiii)$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{f(q^3;q^5)}{f(-q^2)}, see\ [7], equation\ (2.8.10)p-15);(S_{39}) \qquad (xiv)$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q^4;q^4)_n}=\frac{f(-q;-q^5)}{\psi(-q)}, see\ [7], equation\ (2.6.2)p-13) \qquad (xv)$$

3.6. Some Identities of Rogers-Ramanujan Type Related to Modulo 7:

Replacing q by q^2 in (3) we get

$$(-bq^{2n};q^{2n})_\infty\sum_{m=0}^{\infty}\frac{q^{m^2+m}a^m}{(-bq^{2n};q^{2n})_m(q^2;q^2)_m}$$

$$=(-aq^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{n(m^2+m)}(b)^m}{(-aq^2;q^2)_{nm}(q^{2n}:q^{2n})_m} \qquad (3.6.1)$$

The equation (3.6.1) for $n=2, a=q^2, and\ b=-q^2$ gives

$$(q^6;q^4)_\infty\sum_{m=0}^{\infty}\frac{q^{m^2+3m}}{(q^6;q^4)_m(q^2;q^2)_m}=(-q^4;q^2)_\infty\sum_{m=0}^{\infty}\frac{(-1)^m q^{2(m^2+2m)}}{(-q^4;q^2)_{2m}(q^4:q^4)_m}$$

This, after some simplification gives

$$\frac{(-q^4;q^2)_\infty}{(1-q^2)(q^6;q^4)_\infty}\sum_{m=0}^{\infty}\frac{(-1)^m q^{2(m^2+2m)}}{(-q^4;q^2)_{2m}(q^4:q^4)_m}=\sum_{m=0}^{\infty}\frac{q^{m^2+3m}(-q^2;q^2)_m}{(q^2;q^2)_{2n+1}}$$

Now taking $q\to q^{\frac{1}{2}}$ $and\ writing\ n$ in place of m, we obtain the following identity:

$$\frac{(-q^2;q)_\infty}{(1-q)(q^3;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+2n)}}{(-q^2;q)_{2n}(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{\frac{(n^2+3n)}{2}}(-q;q)_n}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^7;-q^2)}{\varphi(-q)}, \text{ on using (vi)}$$

$$=\frac{(q^4,q^3,q^7;q^7)_\infty(q,q^{13};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}$$

$$= (q, q^{13}; q^{14})_\infty (-q; q)_\infty \frac{(q^4; q^7)_\infty (q^3; q^7)_\infty (q^7; q^7)_\infty}{(q; q)_\infty}$$

$$= (q, q^{13}; q^{14})_\infty (-q; q)_\infty \frac{\prod_{j=1}^{\infty}(1 - q^{4+7j})(1 - q^{3+7j})(1 - q^{7+7j})}{\prod_{j=1}^{\infty}(1 - q^{1+j})}$$

$$= (q, q^{13}; q^{14})_\infty (-q; q)_\infty \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, n \not\equiv 0,3,4(mod 7) \qquad (3.6.2)$$

Again setting $n = 2, a = 1 \; and \; b = -\frac{1}{q^2}$ in (3.6.1) we get on some simplification, the following equation:

$$\frac{(-q^2; q^2)_\infty}{(q^2; q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2}}{(-q^2; q^2)_{2m} (q^4; q^4)_m} = \sum_{m=0}^{\infty} \frac{q^{m(m+1)} (-q^2; q^2)_m}{(q^2; q^2)_{2m}}$$

This equation for $q \to q^{\frac{1}{2}}$ and m replace with n gives the following identity:

$$\frac{(-q; q)_\infty}{(q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(-q; q)_{2n} (q^2; q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q; q)_n}{(q; q)_{2n}}$$

$$= \frac{Q(q^7; -q)}{\varphi(-q)} \qquad from \; (v)$$

$$= \frac{(q^6; q; q^7; q^7)_\infty (q^5, q^9; q^{14})_\infty (-q; q)_\infty}{(q; q)_\infty}$$

$$= (q^5, q^9; q^{14})_\infty (-q; q)_\infty \frac{(q^6; q^7)_\infty (q; q^7)_\infty (q^7; q^7)_\infty}{(q; q)_\infty}$$

$$= (q^5, q^9; q^{14})_\infty (-q; q)_\infty \frac{\prod_{j=1}^{\infty}(1 - q^{6+7j})(1 - q^{1+7j})(1 - q^{7+7j})}{\prod_{j=1}^{\infty}(1 - q^{1+j})}$$

$$= (q^5, q^9; q^{14})_\infty (-q; q)_\infty \prod_{n=1}^{\infty} \frac{1}{1 - q^n}, n \not\equiv 0,1,6(mod 7) \qquad (3.6.3)$$

Moreover, the equation (3.6.1) for $n = 2, a = 1 \; and \; b = -q^2$ gives

$$\frac{(-q^2; q^2)_\infty}{(q^2; q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2(m^2+2m)}}{(-q^2; q^2)_{2m} (q^4; q^4)_m} = \sum_{m=0}^{\infty} \frac{q^{m(m+1)} (-q^2; q^2)_m}{(q^2; q^2)_{2m+1}} \qquad (3.6.4)$$

Taking $q \to q^{\frac{1}{2}}$ and replacing m by n in (3.6.4), it yields

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n^2+2n)}}{(-q;q)_{2n}(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(-q;q)_n}{(q;q)_{2n+1}}$$

$$=\frac{Q(q^7;-q^2)}{\varphi(-q)}\quad from\ (iv)$$

$$=\frac{(q^5;q^2;q^7;q^7)_\infty(q^3,q^{11};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}$$

$$=(q^3,q^{11};q^{14})_\infty(-q;q)_\infty\frac{(q^5;q^7)_\infty(q^2;q^7)_\infty(q^7;q^7)_\infty}{(q;q)_\infty}$$

$$=(q^3,q^{11};q^{14})_\infty(-q;q)_\infty\frac{\prod_{j=1}^{\infty}(1-q^{5+7j})(1-q^{2+7j})(1-q^{7+7j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$=(q^3,q^{11};q^{14})_\infty(-q;q)_\infty\prod_{n=1}^{\infty}\frac{1}{1-q^n},n\not\equiv 0,2,5(mod7)\qquad(3.6.5)$$

Lastly, taking $q\rightarrow q^{1/2}$ in the transformation (4) and then setting

$a=-q^{1/2},b=-q$, it gives

$$\frac{(-q;q)_\infty}{(q^{1/2};q)_\infty}\sum_{n=0}^{\infty}\frac{(-q^{1/2};q)_n q^{\frac{(n^2+2n)}{2}}}{(q;q)_n(-q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+n}{2}}(-q^{1/2};q^{1/2})_n}{(q^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n}$$

Now using the identity (i) after replacement of q by q^2, it gives the following identity:

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n^2+2n)}}{(q^4;q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{(n^2+n)}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_n}$$

$$=\frac{f(-q;-q^5)}{\varphi(-q)}$$

$$=\frac{(q;q^5;q^6;q^6)_\infty(-q;q)_\infty}{(q;q)_\infty}$$

That is,

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n^2+2n)}}{(q^4;q^4)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},n\not\equiv 0,1,5(mod7)\qquad(3.6.6)$$

3.7. Some Identities of Rogers-Ramanujan Type Related to Modulo 8:

Replacing q by q^2 in (5) we get

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n}{({q^2}/{b};q^2)_n(q^2;q^2)_n}=(-\gamma q^4;q^4)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n(-\frac{q^2}{b};q^2)_{2n}}{(q^4;q^4)_n(\frac{q^4}{b^2};q^4)_n(-\gamma q^4;q^4)_n} \quad (3.7.1)$$

Setting $b=\frac{1}{q}$ and $\gamma=q^2$ in (3.7.1) we have

$$(-q^6;q^4)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^4)_n(q^6;q^4)_n(-q^6;q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^3;q^2)_n(q^2;q^2)_n}$$

Which on some reduction, yields

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \quad (3.7.2)$$

Now using (xiii) in (3.7.2) we get the following identity

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n}=\frac{f(q;q^7)}{f(-q^2)}$$

$$=\frac{(-q;-q^7;q^8;q^8)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(-q;q)_\infty(q;q)_\infty}$$

$$=\frac{\prod_{j=1}^{\infty}(1+q^{1+8j})(1+q^{7+8j})(1-q^{8+8j})}{\prod_{j=1}^{\infty}(1+q^{j+1})(1-q^{j+1})}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1+q^n}.\prod_{m=1}^{\infty}\frac{1}{1-q^m},\ \ where\ n\not\equiv 1,7(mod 8)\ \&\ m\not\equiv 0(mod 8) \quad (3.7.3)$$

Again, placing $q^{1/2}$ in place of q in transformation (5) we have

$$\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}}\gamma^n}{({q^{1/2}}/{b};q^{1/2})_n(q^{1/2};q^{1/2})_n}=(\gamma q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}}\gamma^n(-\frac{q^{1/2}}{b};q^{1/2})_{2n}}{(q;q)_n(\frac{q}{b^2};q)_n(-\gamma q;q)_n} \quad (3.7.4)$$

Which for $b=q^{1/4}$ and $\gamma=1$ gives

$$\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q^{1/4};q^{1/2})_n(q^{1/2};q^{1/2})_n}=(-q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2/2}(-q^{1/4};q^{1/2})_{2n}}{(q;q)_n(q^{7/8};q)_n(-q;q)_n}$$

Now, taking $q\to q^4$ we get

$$(-q^4;q^4)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q;q^2)_{2n}}{(q^8;q^8)_n(q^{7/2};q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n(q^2;q^2)_n}$$

$$=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}$$

$$=\frac{f(q^3;q^5)}{f(-q^2)}\ \ from\ (xiv)$$

$$=\frac{(-q^3;-q^5;q^8;q^8)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(-q;q)_\infty(q;q)_\infty}$$

$$=\frac{\prod_{j=1}^{\infty}(1+q^{3+8j})(1+q^{5+8j})(1-q^{8+8j})}{\prod_{j=1}^{\infty}(1+q^{j+1})(1-q^{j+1})}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1+q^n}.\prod_{m=1}^{\infty}\frac{1}{1-q^m},\ \ where\ n\not\equiv 3,5(mod8)\ \&\ m\not\equiv 0(mod8)\quad (3.7.5)$$

3.8. Some identities of Rogers-Ramanujan Type Related to modulo 10:

Replacing $q\ by\ q^2$ in (2) we get

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(b;q^2)_n(q^2;q^2)_n}=(\gamma;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2-3n)}(-b\gamma)^n}{(q^2;q^2)_n(b;q^2)_n(\gamma;q^2)_n}\qquad(3.8.1)$$

Setting $b=q^3\ \gamma=-q^3$ in (3.8.1) we get

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q^3;q^2)_n(q^2;q^2)_n}=(1-q^2)(-q^3;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{3n^2+3n}}{(q^2;q^4)_{n+1}(q^2;q^2)_n}$$

Which for $q\to q^{1/2}$ gives

$$\frac{1}{(1-q)(-q^{3/2};q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+2)}{2}}}{(q^{3/2};q)_n(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^{\frac{(3n^2+3n)}{2}}}{(q;q^2)_{n+1}(q;q)_n}$$

$$=\frac{f(-q^2;-q^8)}{f(-q)}\ \ from\ (ii)$$

$$= \frac{(q^2; q^8; q^{10}; q^{10})_\infty}{(q;q)_\infty}$$

$$= \frac{(q^2; q^{10})_\infty (q^8; q^{10})_\infty (q^{10}; q^{10})_\infty}{(q;q)_\infty}$$

$$= \frac{\prod_{j=1}^{\infty}(1-q^{2+10j})(1-q^{8+10j})(1-q^{10+10j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,2,8 (mod 10) \qquad (3.8.2)$$

The equation (3.8.1) for $b = q$, $\gamma = -q$ gives

$$\frac{1}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n (q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{3n^2-n}}{(q^2;q^2)_n (q^2;q^4)_n}$$

Which for $q \to q^{1/2}$ gives :

$$\frac{1}{(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}}}{(q^{1/2};q)_n (q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2-n}{2}}}{(q;q^2)_n (q;q)_n}$$

$$= \frac{f(-q^4;-q^6)}{f(-q)} \; from (iii)$$

$$= \frac{(q^4; q^6; q^{10}; q^{10})_\infty}{(q;q)_\infty}$$

$$= \frac{(q^4; q^{10})_\infty (q^6; q^{10})_\infty (q^{10}; q^{10})_\infty}{(q;q)_\infty}$$

$$= \frac{\prod_{j=1}^{\infty}(1-q^{4+10j})(1-q^{6+10j})(1-q^{10+10j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,4,6 (mod 10) \qquad (3.8.3)$$

Also the equation (3.8.1) for $b = q^3, \gamma = -q^2$ gives

$$\frac{(-q^2;q^2)_\infty}{(1-q^2)} \sum_{n=0}^{\infty} \frac{q^{3n^2+2n}}{(q^4;q^4)_n (q^3;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}}$$

$$= \frac{Q(q^{10};-q^3)}{f(-q)} \quad from\ (vii)$$

$$= \frac{(q^7; q^3; q^{10}; q^{10})_\infty (q^4; q^{16}; q^{20})_\infty}{(q; q)_\infty}$$

$$= \frac{(q^4; q^{16}; q^{20})_\infty (q^7; q^{10})_\infty (q^3; q^{10})_\infty (q^{10}; q^{10})_\infty}{(q; q)_\infty}$$

Thus we have

$$\frac{(-q^2; q^2)_\infty}{(q^4; q^{16}; q^{20})_\infty (1-q^2)} \sum_{n=0}^{\infty} \frac{q^{3n^2+2n}}{(q^4; q^4)_n (q^3; q^2)_n} = \frac{\prod_{j=1}^{\infty}(1-q^{4+10j})(1-q^{6+10j})(1-q^{10+10j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,3,7 (mod 10) \qquad (3.8.4)$$

Moreover the equation(3.8.1) for $b = q^3, \gamma = -q^3$ gives

$$\frac{(-q^3; q^2)_\infty}{(1-q)} \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}}{(q^2; q^2)_n (q^6; q^4)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q; q)_{2n+1}}$$

$$= \frac{Q(q^{10}; -q^4)}{f(-q)} \quad from\ (viii)$$

$$= \frac{(q^6; q^4; q^{10}; q^{10})_\infty (q^2; q^{18}; q^{20})_\infty}{(q; q)_\infty}$$

$$= \frac{(q^2; q^{18}; q^{20})_\infty (q^6; q^{10})_\infty (q^4; q^{10})_\infty (q^{10}; q^{10})_\infty}{(q; q)_\infty}$$

Thus we have

$$\frac{(-q^3; q^2)_\infty}{(q^2; q^{18}; q^{20})_\infty (1-q)} \sum_{n=0}^{\infty} \frac{q^{3n^2+3n}}{(q^2; q^2)_n (q^6; q^4)_n} = \frac{\prod_{j=1}^{\infty}(1-q^{6+10j})(1-q^{4+10j})(1-q^{10+10j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,4,6 (mod 10) \qquad (3.8.5)$$

And finally setting $b = q^3, \gamma = -q^2$ in (3.8.1) it yields

$$(-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q; q^2)_n (q^4; q^4)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q^2)_n (q^2; q^2)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}}$$

$$= \frac{Q(q^{10};-q)}{f(-q)}\ from\ (ix)$$

$$\frac{(q^9;q;q^{10};q^{10})_\infty (q^8;q^{12};q^{20})_\infty}{(q;q)_\infty}$$

$$= \frac{(q^8;q^{12};q^{20})_\infty (q^9;q^{10})_\infty (q;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}$$

Thus we have the following identity

$$\frac{(-q^2;q^2)_\infty}{(q^8;q^{12};q^{20})_\infty} \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n (q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,1,9 (mod 10) \qquad (3.8.6)$$

3.8. Some identities of Rogers-Ramanujan Type Related to modulo 12:

Placing q^2 in place of q in transformation (9), we get

$$\sum_{n=0}^{\infty} \frac{(a^2;q^2)_n q^{3n^2+n} (a)^{2n}}{(q^2;q^2)_n} = \frac{(a^2q^2;q^2)_\infty}{(-aq^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-a;q^2)_n q^{n^2-n} (aq^2)^n}{(aq^2,q^2;q^2)_n} \qquad (3.9.1)$$

Setting $a = q$ in (3.9.1), it reduces to:

$$\sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^{3n^2+3n}}{(q^2;q^2)_n} = \frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^3,q^2;q^2)_n}$$

Which reduces to the following identity after some reduction:

$$\frac{(-q^3;q^2)_\infty}{(q^4;q^2)_\infty (1-q)} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^{3n^2+3n}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q;q)_{2n+1}}$$

$$= \frac{f(-q^2;-q^{10})}{f(-q)}\ from\ (xii)$$

$$= \frac{(q^2;q^{10};q^{12};q^{12})_\infty}{(q;q)_\infty}$$

$$= \frac{(q^2;q^{12})_\infty (q^{10};q^{12})_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty}$$

Thus we have

$$= \frac{\prod_{j=1}^{\infty}(1-q^{2+12j})(1-q^{10+12j})(1-q^{12+12j})}{\prod_{j=1}^{\infty}(1-q^{1+j})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,2,10(mod 12) \qquad (3.9.2)$$

Chapter 4: Further Theorems of the Rogers-Ramanujan Type Theorems

4.1. Introduction

In the theory of partitions, we find a number of identities which state that for each positive integers n the partitions of n with parts restricted to certain residue classes are equinumerous with the partitions of n on which certain difference conditions are imposed. Among the most striking result of this type are Rogers-Ramanujan identities. These were stated combinatorically by P. A. McMahon as follows

1.1. The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are congruent to $\pm 1 (mod\ 5)$.

1.2. The number of partitions of n with minimal part 2 and minimal difference equals the number of partitions of n into parts which are congruent to $\pm 2 (mod\ 5)$.

Recently, Hirschhorn using some of the Slater's identities [8] proved four theorems of the Rogers-Ramanujan type. Later, using the same identities of Slater's, Subbarao established entirely different combinatorial results. Subbarao's results bear striking resemblance with the Rogers-Ramanujan identities.

1.3. Let A(n) denote the number of partition of n into parts congruent to $\pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 (mod 20)$. Let B(n) denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_t, where\ b_i \geq b_{i+1}$, and, if

$$1 \leq i \leq \left[\frac{t-2}{2}\right], b_i - b_{i+1} \geq 2$$

Then A(n)=B(n) for all n.

The object of this chapter is to prove the following theorem:(see [9])

Theorem 1.

Let C(n) denote the number of partition of n into parts congruent to $\pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{4}, \pm\mathbf{5}, \pm\mathbf{6}, \pm\mathbf{7} (\boldsymbol{mod 20})$. Let D(n) denote the number of partitions of n of the form $\boldsymbol{n = b_1 + b_2 + \cdots + b_t, where\ b_t \geq 2, b_i \geq b_{i+1}}$, and, if

$$\boldsymbol{1 \leq i \leq \left[\frac{t-2}{2}\right], b_i - b_{i+1} \geq 2}$$

Then C(n)=D(n) for all n.

Proof:

let $\pi_t(n)$ be a partition enumerated by D(n).

Then for some $s \geq 1, t = 2s - 1 \; or \; t = 2s$.

First for $t = 2s$

then,

$$\pi_t(n) = b_1 + b_2 + \cdots + b_{2s}$$

$$\pi_{2s}(n) = b_1 + b_2 + \cdots + b_{2s}$$

With

$$b_s \geq 2, b_{s-1} \geq 4, \dots b_1 \geq 2s$$

And $$b_{s+1} \geq b_{s+2} \geq, \dots \geq b_{2s} \geq 2$$

We subtract $2,4,6, \dots ,2s$ from $b_s, b_{s-1}, \dots , b_1, b_2$ respectively and 2 from each of $b_{s+1}, b_{s+2}, \dots , b_{2s}$. This produce a partition of

$$(q;q)_{2s} = \prod_{i=0}^{\infty} \frac{(1 - qq^i)}{(1 - qq^{2s+i})}$$

$$= \prod_{i=0}^{\infty} \frac{(1 - q^{i+1})}{(1 - q^{2s+i+1})}$$

$$= \frac{(1-q)}{(1-q^{2s+1})} \cdot \frac{(1-q^2)}{(1-q^{2s+2})} \cdots \frac{(1-q^{2s})}{(1-q^{2s+2s})} \cdot \frac{(1-q^{2s+1})}{(1-q^{2s+2s+1})} \cdots$$

$$= (1-q)(1-q^2) \dots (1-q^{2s})$$

$$\frac{q^{s^2+3s}}{(q;q)_{2s}} = \frac{q^{s^2+3s}}{(1-q)(1-q^2) \dots (1-q^{2s})}$$

$\Longrightarrow n - [(2 + 4 + 6 + \cdots + 2s) + 2s] = n - (s^2 + 3s)$ into at most 2s parts.

Thus the partition of the type $\pi_{2s}(n)$ are generated by

$$\frac{q^{s^2+3s}}{(q;q)_{2s}} . (s = 1,2, \dots)$$

Similarly if $t = 2s - 1$ then

$$\pi_{2s-1}(n) = b_1 + b_2 + \cdots + b_{2s-1}$$

With $b_{s-1} \geq 2, b_{s-2} \geq 4, \dots b_1 \geq 2s - 2$,

And $b_s \geq b_{s+1} \geq b_{s+2} \geq, \dots \geq b_{2s-1} \geq 2$

We subtract $2,4,6, \dots ,2(s-1)$ from $b_{s-1}, \dots , b_1$ respectively and 2 from each of $b_s, b_{s+1}, b_{s+2}, \dots , b_{2s-1}$ we are left with a partition of

$n-[2(1+2+3+\cdots+(s-1))-2s]=n-(s^2+s)$

Into at most 2s-1 parts.

This shows that the partition of the type$\pi_{2s-1}(n)$ are generated by

$$\frac{q^{s^2+s}}{(q;q)_{2s-1}}.\,(s=1,2,\dots)$$

Thus

$$\sum_{n=0}^{\infty} D(n)q^n=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n}}$$

Now an appeal to slater's identity [8, (99), p.162)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}(1-q^{10n-1})(1-q^{10n-9})(1-q^{20n-8})\times(1-q^{20n-12})(1-q^{10n})$$

Hence the theorem.

We shall also prove two more identities stated below:

Theorem 2:
let $P_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1,\pm 4,\pm 6,\pm 7 (mod\ 16)$. $P_2(n)$ denote the number of partitions of n of the form $n=b_1+b_2+\cdots+b_{2s+1}$, $where\ b_i\geq b_{i+1}, b_{s+1}\geq s, b_s\neq b_{s+1}$ and if $1\leq i\leq s-1, b_i-b_{i+1}\geq 2$. Then $P_1(n)=P_2(n)$

Proof:

$$\text{let } \pi_{2s+1}(n)=b_1+b_2+\cdots+b_s+b_{s+1}+\cdots+b_{2s+1}$$

With $b_{s+1}\geq s, b_s\geq s+1, b_{s-1}\geq s+3,\dots,b_1\geq s+(2s-1)$

And $b_{s+2}\geq b_{s+3}\geq b_{s+4}\geq\cdots\geq b_{2s+1}\geq 1$

Subtract $s, s+1, s+3,\dots,s+(2s-1)$ from $b_{s+1}, b_s,\dots,b_1$respectively and 1 from each $b_{s+2}, b_{s+3},\dots,b_{2s+1}$. This produce a partition of n-2s(s+1) into at most 2s+1 parts. This shows that the partitions of the type $\pi_{2s+1}(n)$ are generated by

$$\frac{q^{2s(s+1)}}{(q;q)_{2s+1}}.\,(s=1,2,\dots)$$

The theorem follows immediately once we recall the following identity of slater [8, (86), p.161]

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-3})(1-q^{8n-5}) \times (1-q^{16n-14})(1-q^{16n-2})(1-q^{8n})$$

Hence the theorem.

Theorem 3:

The number of partitions of n into odd parts equals the number of partition of n into an odd number, say 2s+1, of parts, satisfying the conditions that the middle part is at least s and the first s parts have minimal difference 1.

Proof:

Let $\mu(n)$ denote the number of partitions of n of the type described in the second part of the theorem.

By the usual argument it can be shown that

$$\sum_{n=0}^{\infty} \mu(n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q;q)_{2n+1}} \quad (a)$$

The theorem follows immediately once we note that the right-hand side of (a) equals $(-q;q)_\infty$ in view of the following identity due Slater

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n})}{1-q^{2n}}.$$

Hence Proved.

Chapter 5: Some New Partition Theorems

5.1 Introduction:

It appears that P. A. MacMahon was the first to recognize the combinatorial significance of the Rogers-Ramanujan identities. The second of these identities was stated thusly:

The partitions of any positive integer, n, into parts of the forms 5m++2, 5m+3 are equinumerous with those partition of n into parts ≥ 2 which involve neither sequence nor repetitions.

In the next chapter of MacMahon's book, the following striking (and little know) theorem which resembles the second Rogers-Ramanujan identity is proved, although not explicitly stated:

The partitions of any integer, n, into parts of the forms 6m, 6m+2, 6m+3, 6m+4 are equinumerous with those partitions of n into parts ≥ 2 which do not involve sequences.

Thus with n=12, the partitions of the first type are

$$12, 10+2, 9+3, 8+4, 8+2+2, 6+6, 6+4+2, 6+3+3, 6+2+2+2,$$

$$4+4+4, 4+4+2+2, 4+3+3+2, 4+2+2+2+2, 3+3+3+3,$$

$$3+3+2+2+2, 2+2+2+2+2+2$$

While the 16 partitions of the second kind are

$$12, 10+2, 9+3, 8+4, 8+2+2, 7+5, 6+6, 6+4+2, 6+3+3, 6+2+2+2,$$

$$5+5+2, 4+4+4, 4+4+2+2, 4+2+2+2+2, 3+3+3+3, 2+2+2+2+2+2$$

The following theorem in certain sense appears to be intermediate between the second Rogers-Ramanujan identity and MacMahon's Theorem.

Theorem 1:

The partitions of any positive integer, n, into parts of the forms 6m+2, 6m+3, 6m+4 are equinumerous with those partitions of n into parts ≥ 2 which neither involve sequences nor allow any part to appear more than twice.

Thus again with n=12, the 11 partitions of the first type are

$$10+2, 9+3, 8+4, 8+2+2, 4+4+4, 4+4+2+2, 4+3+3+2,$$

$$4+2+2+2+2, 3+3+3+3, 3+3+2+2+2, 2+2+2+2+2+2;$$

While the 11 partitions of the second type are

$$12, 10+2, 9+3, 8+4, 8+2+2, 7+5, 6+6, 6+4+2, 6+3+3$$

$$5+5+2,4+4+2+2.$$

Actually it is a special case of the following theorem with k=1, a=0.

Theorem 2:

Let $\mathbf{0} \leq \boldsymbol{a} < \boldsymbol{k}$ be integers. Let $\boldsymbol{A}_{\boldsymbol{k},\boldsymbol{a}}(\boldsymbol{N})$ denote the number of partitions of N into parts $\not\equiv \mathbf{0}, \pm(\mathbf{2}\boldsymbol{a}+\mathbf{1})(\boldsymbol{mod}\mathbf{4}\boldsymbol{k}+\mathbf{2})$.Let $\boldsymbol{B}_{\boldsymbol{k},\boldsymbol{a}}(\boldsymbol{N})$ denote the number of partitions of N of the form $\sum_{i=1}^{\infty} \boldsymbol{f}_{\boldsymbol{i}}.\, \boldsymbol{i}$, where

1. $\boldsymbol{f}_{\mathbf{1}} \leq \mathbf{2}\boldsymbol{a}$
2. $\left[\frac{1}{2}(\boldsymbol{f}_{\boldsymbol{i}}+\mathbf{1})\right] + \left[\frac{1}{2}(\boldsymbol{f}_{\boldsymbol{i}+\mathbf{1}}+\mathbf{1})\right] \leq \boldsymbol{k}$

Then

$$\boldsymbol{A}_{\boldsymbol{k},\boldsymbol{a}}(\boldsymbol{N}) = \boldsymbol{B}_{\boldsymbol{k},\boldsymbol{a}}(\boldsymbol{N})\,.$$

Proof:

We first discuss the second condition on the partitions enumerated by $B_{k,a}(N)$; it states that, if i appears $2j-1$ *or* $2j$ times as a summand, then $i+1$ appears at most $2(k-j)$ times. Thus we see (with j=k) that no part appears more than 2k times.

We now proceed by the technique developed in [1] and [2]. If

$$C_{k,i}(x;q) = \sum_{\mu=0}^{\infty}(-1)^{\mu}x^{k\mu}q^{1/2(2k+1)\mu(\mu+1)-i\mu}\left(1-x^{i}q^{(2\mu+1)i}\right) \times \frac{(1-xq)\ldots(1-xq^{\mu})}{(1-q)\ldots(1-q^{\mu})}, \qquad (1)$$

Then [12, p.4]

$$C_{k,i}(x;q) - C_{k,i-1}(x;q) = x^{i-1}q^{i-1}(1-xq)C_{k,k-i+1}(xq;q), \qquad (2)$$

And [12, p.4]

$$C_{k,-i}(x;q) = -x^{-i}q^{-i}C_{k,i}(x;q) \qquad (3)$$

Consequently, if we define

$$R_{k,i}(x) = C_{k,i+\frac{1}{2}}(x^{2};q^{2})\prod_{j=1}^{\infty}\frac{1}{(1-xq^{j})},$$

Then for $0 \leq i \leq k$, (2) implies

$$R_{k,i}(x) - R_{k,i-1}(x) = x^{2i-1}q^{2i-1}(1+xq)R_{k,k-i}(xq) \qquad (4)$$

And (2) and (3) imply

$$R_{k,0}(x) = R_{k,k}(xq) \qquad (5)$$

We may expand $R_{k,i}(x)$ as follows

$$R_{k,i}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} C_{k,i}(M,N)\, x^M q^N, \qquad |x| \leq 1, |q| < 1. \quad (6)$$

Then by mean of (4), (5), (6), and the definition of $R_{k,i}(x)$ we easily verify that

$$c_{k,i}(M,N) = \begin{cases} 1 & if\ M = N = 0 \\ 0\ if\ either\ M \leq 0\ or\ N \leq 0\ and\ M^2 + N^2 \neq 0, \end{cases} \qquad (7)$$

$$C_{k,0}(M,N) = C_{k,k}(M, N-M), \qquad (8)$$

$$c_{k,i}(M,N) - c_{k,i-1}(M,N) = C_{k,k-i}(M-2i+1, N-M) + C_{k,k-i}(M-2i, N-M) \quad 0 < i \leq k \qquad (9)$$

Let $p_{k,i}(M,N)$ denote the number of partitions of N into M parts of the form N=$\sum_{j=1}^{\infty} f_j . j$ with $f_1 \leq 2i$ and

$$\left[\frac{1}{2}(f_j + 1)\right] + \left[\frac{1}{2}(f_{j+1} + 1)\right] \leq k.$$

We wish to show that $p_{k,i}(M,N)$ satisfies (7), (8), and (9). Now (7) is by definition.

As for (8), let us consider any partition enumerated by $p_{k,0}(M,N)$.

Since 1 does not appear, every summand is ≥ 2. Subtracting 1 from every summand, we obtain a partition of N-M into M parts with 1 appearing at most 2k times and again

$$\left[\frac{1}{2}(f_j + 1)\right] + \left[\frac{1}{2}(f_{j+1} + 1)\right] \leq k.$$

Thus we have a partition of the type enumerated by $p_{k,k}(M, N-M)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,k}(M, N-M)$ and the partitions enumerated by $p_{k,0}(M,N)$. Hence

$$p_{k,0}(M,N) = p_{k,k}(M, N-M).$$

Finally, we treat (9). We note that $p_{k,i}(M,N) - p_{k,i-1}(M,N)$ enumerates the number of partitions of N into M parts of the form N=$\sum_{j=1}^{\infty} f_j . j$ with $f_1 = 2i - 1\ or\ 2i$

And

$$\left[\frac{1}{2}(f_j + 1)\right] + \left[\frac{1}{2}(f_{j+1} + 1)\right] \leq k.$$

In case $f_1 = 2i - 1$, we see that $f_2 \leq 2(k-i)$; subtracting 1 from every summand, we obtain a partition of N-M into M-2i+1 parts with 1 appearing at most 2(k-i) times and

$$\left[\frac{1}{2}(f_j + 1)\right] + \left[\frac{1}{2}(f_{j+1} + 1)\right] \leq k.$$

Thus we have a partition of the type enumerated by $p_{k,k-i}(M-2i+1, N-M)$.

In case $f_1 = 2i$, we see that $f_2 \leq 2(k-i)$; subtracting 1 from every summand, we obtain a partition of N-M into M-2i parts with 1 appearing at most 2(k-i) times and

$$\left[\frac{1}{2}(f_j+1)\right] + \left[\frac{1}{2}(f_{j+1}+1)\right] \leq k.$$

Thus we have a partition of the type enumerated by $p_{k,k-i}(M-2i, N-M)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by

$$p_{k,i}(M,N) - p_{k,i-1}(M,N)$$

And the partitions enumerated by

$$p_{k,k-i}(M-2i+1, N-M) + p_{k,k-i}(M-2i, N-M).$$

Hence

$$p_{k,i}(M,N) - p_{k,i-1}(M,N) = p_{k,k-i}(M-2i+1, N-M) + p_{k,k-i}(M-2i, N-M).$$

Thus by the comment following (9),

$$C_{k,i}(M,N) = p_{k,i}(M,N) \qquad (10)$$

Thus for $0 \leq a \leq k$

$$\begin{aligned}\sum_{N=0}^{\infty} A_{k,a}(N)q^N &= \prod_{\substack{n=1 \\ n \equiv 0, \pm(2a+1)(mod\ 4k+2)}}^{\infty} \frac{1}{1-q^n} \\ &= R_{k,a}(1) \\ &= \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} p_{k,a}(M,N)q^N \\ &= \sum_{N=0}^{\infty} B_{k,a}(N)q^N,\end{aligned}$$

Where the second equation follows from Jacobi's identity

Therefore

$$A_{k,a}(N) = B_{k,a}(N).$$

Hence proved.

Chapter 6: Conclusion

We have only touched on a small part of the Rogers-Ramanujan story in this survey. The main goal has been to present an expanded version of Slater's list with the earliest known reference to each identity in the literature. Slater's list contained only a few references to the earlier literature, and of course, Ramanujan's lost notebook was unknown to the mathematical community in 1952. Accordingly, we believe it was a useful endeavor to bring together Slater's list with Ramanujan's lost notebook, and the dozens of additional identities of similar type which have been scattered throughout the literature over the years. Since Slater's main tool was Bailey's lemma and Bailey pairs, was included an exposition of this material in the introduction.

The transformations can also be used for searching further identities of Rogers-Ramanujan type list.

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