Study of Mathematical Modelling of Infectious Diseases

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by

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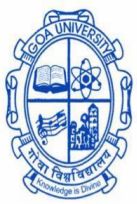

GOA UNIVERSITY

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Study of Mathematical Modelling of Infectious Diseases" is based on the study carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Mridini Gawas and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of the study given in the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Study of Mathematical Modelling of Infectious Diseases" is a bonafide work carried out by Ms. Maithili Mahadev Khandekar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: "Study of Mathematical Modelling of Infectious Diseases". This review is divided into six chapters.

FIRST CHAPTER :

This chapter provides an introduction to mathematical modelling; including steps for formulation, its types, advantages and limitations.

SECOND CHAPTER:

In this chapter we have given a brief on vector borne diseases and corona virus disease and the advances that have happened in these areas using mathematical modelling.

THIRD CHAPTER:

This chapter consists of a few definitions and concepts required for the study and analysis of mathematical modelling.

FOURTH CHAPTER:

This chapter provides an elaborate review of the paper "MODELLING AND ANALYSIS OF THE VECTOR BORNE DISEASES WITH FREE LIVING PATHOGEN GROWING IN THE ENVIRONMENT".

FIFTH CHAPTER.

This chapter contains the review of the paper "AN SIQR MATHEMATICAL MODEL TO CONTROL CORONA - VIRUS DISEASE (COVID-19) WITH SATURATED INCIDENCE RATE".

SIXTH CHAPTER.

This chapter concludes the study and analysis of mathematical modelling for both vector borne diseases as well as corona virus disease.

ACKNOWLEDGEMENTS

I would like to express my gratitude to all those who gave me the opportunity to complete this dissertation. First and foremost, I would like to express my gratitude to my guide, Dr. Mridini Gawas, whose help, suggestions and encouragement helped me in the completion of my dissertation. She pushed me to think imaginatively and was a great source of motivation. Her vast knowledge, extensive experience, and professional competence in Differential Equations enabled me to successfully complete my dissertation. This endeavour would not have been possible without her help and supervision.

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ABSTRACT

In the paper on vector borne diseases with free living pathogen in the environment we shall be studying and analysing a non-linear dynamic model. Here we find the equilibrium points and the stability is also analysed about these points. For studying the stability Sylvester criterion and Lyapunov's Stability is used.

In the paper on SIQR model with saturated incidence rate to control COVID-19 we shall be studying and analysing a non-linear dynamic model. Here we find the equilibrium points and the stability is also analysed about these points. Concepts like Sylvester criterion, Lyapunov's Stability and Lyapunov-LaSalle invariance principle is used to study the stability of the formulated model.

Keywords: Mathematical modelling; equilibrium point; basic reproduction number; local stability; global stability

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Notations and Abbreviations

$S(t)$	susceptible population at time t
$I(t)$	infected population at time t
$R(t)$	recovered population at time t
$M(t)$	susceptible vector population at time t
$V(t)$	infected vector at time t
$P(t)$	pathogen population at time t
$E(t)$	cumulative density of environmental factors
$Q_S(t)$	susceptible quarantined population at time t
$Q_I(t)$	infected quarantined population at time t
R_0	basic reproduction number

Chapter 1

INTRODUCTION

1.1 Introduction to Mathematical Modelling

1.1.1 Mathematical Modelling

When a situation or system for is described using mathematical language and concepts, the resulting model is called a mathematical model. This process is termed as mathematical modelling. This process uses mathematics to study the real world problem, analyse, make predictions and find the best suitable or optimal solution to the problem faced. In modelling, one first has to identify the problem faced in the real world and make certain assumptions as required in simplifying the problem. Once the problem has been identified we move to the modelling part of it. Mathematical modelling consists of the following steps:

1. **Formulation of the model**

   This step includes describing the context of the problem, followed by identification

of the relevant factors wherein we only consider the important factors while framing the equation. Certain assumptions are also made to be accurate and at the same time have a simplified model for the situation. The formulator has to make sure that each mathematical quantity has to be described and denoted by a suitable mathematical entity.

2. **Finding the solution of the equation**
   Once we formulate the model the solutions of the mathematical equations have to be found using the methods and concepts which are already studied.

3. **Evaluation and interpretation**
   Lastly we need to interpret the solution to the mathematical equation in the language of the real world and check whether the formulated model is good or not practically. If the model fails then we go back to the formulation step and re-frame the model with slight modifications.

1.1.2 Types of Mathematical Models

Mathematical models are mainly of two types:

1. **Empirical models**: Such models are experimental and based on observations rather than theory. These type of models before being accepted are tested against large data.

2. **Theoretical model**: These type of models are based on already existing laws and ideas

Classification of mathematical models

1. **Linear or Non-linear Model**

   Based on the modelled equation which maybe be algebraic, difference equation, differential equation etc being linear or non-linear.

2. **Static or Dynamic Model**

   In static models the describing variables and relations are in-dependant of time. In dynamic models the describing variables and relations are time dependant.

3. **Discrete or Continuous Model**

   In discrete models discrete values are assumed by the variables.

   In continuous models, continuous values are assumed by the variables.

4. **Deterministic or Stochastic Model**

   If the values assumed by the variables are predictable with certainty then the model is said to be a Deterministic Model.

   If the values assumed by the variables are not predictable with certainty then the model is said to be a Stochastic or Probabilistic Model.

1.1.3 Some characteristics of a Mathematical Model

1. The model has to be as realistic as possible.
2. The equations and inequalities involved must be consistent.
3. The model should neither be over simplified nor over complicated.
4. If parameters are estimated with the help of some data then the same data cannot be used to validate the model.

1.1.4 Advantages of Mathematical Modelling

1. Modelling is one such concept of mathematics which has helped solve problems from natural sciences, engineering disciplines, as well as social sciences.

2. Modelling forces us to think clearly as the formulator has to study the situation, eradicate irrelevant factors and think carefully before forming the model.

3. Instead of formulating a model for the entire system, one can form partial models for subsystems and then combine them.

1.1.5 Limitations of Mathematical Modelling

1. Modelling is not a one shot affair, it might require multiple attempts to formulate a mode that it closest to reality and each model is followed by a better one.

2. A model might give strange solutions if caution is not taken while framing the model.

3. The formulator has to have knowledge of a wide variety of topics.

4. One has to keep updated with the up and coming models otherwise one might use or refer to a model that has already been discarded.

Chapter 2

LITERATURE REVIEW

2.1 Infectious Diseases

Infectious diseases are caused by microorganisms such as bacteria, viruses, fungi and parasites. Most microorganisms present on and in our body are harmless but under certain conditions they cause infectious diseases. Some infectious diseases can be passed from person to person. The transmission can happen from an infected person to a susceptible via direct contact by touching, sneezing, coughing etc.

Microbes can be transmitted from a pregnant mother to unborn and newborn baby. It can also happen through blood transfusion from an infected person to non-infected one's. Some are transmitted by insects or animals when scratched or bitten by an infected insect or animal. There can be indirect transmission as well when a person comes in contact with contaminated surfaces and objects. Transmission can happen by consuming contaminated food and water which can increase the risk of spread to a larger population. For eg: Cholera, Diarrhoea, Dysentery etc.

Practising proper hygiene and sanitation are some preventive measures which can be taken. Vaccination is a measure which can be taken to create heard immunity.
Eg: Malaria, Dengue, Chickenpox, Ebola, Covid-19 etc.

The initial contributions to modern mathematical epidemiology are by P.D. En'ko dated between 1873 and 1894. Physicians like Sir R.A. Ross, W.H. Hamer, A.G. McKendrick, and W.O. Kermack between 1900 and 1935 were the ones who gave the basic ideas and foundation. H.W Hamer put forward that the spread of an infectious disease must depend upon the susceptible and infected population[8]. Further for the rate of new infections he proposed the mass action law which since has been used in various epidemic models. This gave rise to compartmental models for infectious diseases namely SIS, SIR, SIER, SIES and so on. We shall be looking at two types of infectious diseases namely, Vector borne diseases and Corona Virus Disease.

2.1.1 Mathematical Models on Vector Borne Diseases

Vectors are organisms which carry within them the pathogens which cause infectious diseases. They further transmit the pathogens between humans or from animals to humans. The most common of these vectors are bloodsucking insects. These insects transmit the pathogens when they bite a non infected person or the host. They themselves get infected by biting an infected person and are capable of further transmission for the rest of their lifespan. Vectors like flies carry the pathogen on the outside of their bodies and transmission happens via physical contact or by contamination of food and water which is then consumed by the non infected person.

Vector borne diseases are diseases caused in the human population by pathogens and parasites. These diseases are spread through insect bites as the pathogen enters the

blood stream of the human population. Apart from its dependence on the susceptible and infected populations there can be many external factors which affect the rate at which these diseases are spread, which could be climate change, change of weather, or presence of free living pathogen in the environment or no proper sanitization. In particular the environment can become a suitable place for the survival of pathogens if proper hygiene is not maintained. Activities like discharge of household and other wastes into the environment can increase the rate at which the disease is spread as the environment will play a major role in the spread via contaminated food and water, through soil or contaminated surfaces.

The first model on vector borne diseases was given by Nobel laureate R. Ross, particularly on the prevention of malaria in London[16], which was later modified by G. Macdonald[14]. This was followed by various models based on infectious diseases. Models on vector borne diseases based on other factors affecting the spread were also studied. Further effects of human movement on vector borne diseases was studied by Costner et al[2] using the spatial versions of the classical Ross-Macdonald model. In 2010 Tumwiine et al[21] studied the effect of immigration on vector borne diseases. In 2012 the effects of temperature on the transmission of dengue fever was studied by S.C. Chen et al[1] wherein they have shown how temperature affects the maturation, oviposition and death of mosquitoes. Mosquito dispersal is also a factor which affects the spread of the disease. Lutambi et al[13] studied the effect of mosquito dispersal on heterogeneous environment in 2013. Age structured model was studied based on the effect of vector biting and vector mortality on the spread of the infectious disease by K. Rock et al[17].The effect of temperature on host pathogen system and the how climate and thermal adaptability are related was studied by Waikom et al[23]. How environment affects direct and indirect spread of carrier based infectious diseases was studied by Ghosh et al[7] in 2004.

But little attention has been given to the study of effect of pathogen population growing in the environment. Here we shall be looking at the effect of environmental factors like human discharge, household waste, contaminated water bodies on the transmission of vector borne diseases

2.1.2 Mathematical Models on COVID-19

COVID-19 is caused by severe acute respiratory syndrome coronavirus 2 or SARS-CoV-2. The virus is transmitted directly; person to person contact when susceptible person comes in contact with the body fluids of infected person or indirectly through contaminated surfaces and objects.

In December of 2019 the first known case of COVID-19 was reported in the city of Wuhan, China. Transmitted via the respiratory droplets of an infected individual through cough or sneeze, it is an infectious disease caused by SARS-CoV-2 virus. Several measures were taken to control the spread of this disease one of which was quarantining the infected population. SIQ, SIQS, SIQR are some of the models with quarantine compartments. The effects of quarantine on transmission of infectious diseases were described by Z. Feng et al[5][19][20]. Hethcote et al [11] formed and studied different models with different incidence having a quarantine compartment. Here we shall be looking at a SIQR model with a quarantine compartment which is subdivided into quarantine from susceptible and quarantine from infected population, having a saturated incidence rate.

Chapter 3

PREREQUISITES

3.1 Some Definitions

1. **Positive Definite**

   A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if:

   (a) $V(0) = 0$

   (b) $V(x) > 0$ for all $x \neq 0$

2. **Negative Definite**

   A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be negative definite if:

   (a) $V(0) = 0$

   (b) $V(x) < 0$ for all $x \neq 0$

3. **Negative Semidefinite**

   A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be negative semidefinite if:

   (a) $V(0) = 0$

(b) $V(x) \leq 0$ for all $x \neq 0$

4. **Routh Hurwitz criteria**

   Routh Hurwitz criterion which says that a second degree polynomial with all positive coefficients will have negative roots.

5. **Sylvester criteria for a matrix**

   A *nxn* symmetric matrix Q is a **positive definite** iff all principal determinants are strictly greater than 0.

6. **Formation of matrix from the given quadratic equation**

   If we have an equation of the type

$$V(x) = q_{11}x_1^2 + q_{22}x_2^2 + ... + q_{nn}x_n^2 + (q_{12} + q_{21})x_1x_2 + (q_{13} + q_{31})x_1x_3 + .. \\ ... + (q_{ij} + q_{ji})x_ix_j + ...$$

   then the corresponding matrix is given by,

$$M = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

3.2 Some concepts

3.2.1 Equilibrium points

In mathematical modelling once the model has been formulated using a system of non-linear equations we want to look at its behaviour. When talking about the behaviour

of such a system mathematically we are pointing towards it's stability. To look at the stability of the system we first have to find the equilibrium points of the system and then analyse its stability.

1. **Equilibrium points** are those points where the system does not change with time. For non linear system of differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, ..., x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, ..., x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, ..., x_n)
\end{aligned}$$

The equilibrium point x_0 can be found by imposing $\frac{dx_i}{dt} = 0 \; \forall i = 1, 2, ..., n$

3.2.2 Jacobian of a matrix

Jacobian Matrix:
For a given system of differential $\frac{dx}{dt} = f(x)$ where x is an n dimensional vector and f(x) is a vector valued function and x_0 is the equilibrium point such that $f(x_0) = 0$, the Jacobian matrix which is denoted by J, defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$
i.e. for a system of ordinary differential equations;

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, ..., x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, ..., x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, ..., x_n)
\end{aligned}$$

The Jacobian matrix is given by

$$J = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}$$

3.2.3 Basic Reproduction Number

Basic Reproduction number R_0 which is a function of the parameters of the model is the average number of secondary infections arising from a single infected individual. In the cases wherein the infectious disease turns into an epidemic our main concern is how do we control or eliminate the disease. This is taken care of by the basic reproduction number as with the implementation of preventive measures, medication and vaccination drives we might see a decrease in the basic reproduction number. Thus implying that the control measures which are being taken are beneficial.

Calculation of R_0

Here we calculate R_0 using the Next Generation Matrix method. This method to calculate R_0 was given by Diekmann et al [4].

To calculate the basic reproduction number by using a next-generation matrix, the whole population is divided into n compartments in which there are $m < n$ infected compartments. Let $x_i = 1,2,3\ldots,m$ be the number of infected individuals in the i^{th} infected compartment at time t. Then the model can be written as,

$$\frac{dx_i}{dt} = F_i(x) - V_i(x)$$

where $F_i(x)$ denotes the rate of appearance of new infections in the i^{th} compartment and $V_i(x)$ is the rate of other transitions between compartment i and other infected compartments. If x_0 is the disease free equilibrium point then the jacobian of

$$F(x) = (F_1(x), F_2(x), \ldots, F_m(x))^T$$

and

$$V(x) = (V_1(x), V_2(x), \ldots, V_m(x))^T$$

are found at x_0, denoted simply by F and V.

Hence,

$$R_0 = \rho(FV^{-1})$$

where ρ is the spectral radius of FV^{-1} which is the Eigen value of FV^{-1} with largest absolute value.

If $R_0 > 1$ then it said that the number of infections will go on increasing till there are sufficient susceptibles
If $R_0 < 1$ then it is said that the disease gradually dies out.

The Basic reproduction number R_0 measures the transmission potential i.e. how fast the disease is transmitted to the susceptible population and hence can help predict the future of the disease. This helps in deciding the control measure which are to be taken and the level at which they have to be implemented. It also helps in deciding what proportion of the population has to be vaccinated to achieve herd immunity. It also gives an indication of whether the control measures are serving it's purpose.

3.2.4 Logistic growth model

This model was given by Verhults, hence also known as the Verhults model. When population is growing in a limited space the density of population gradually decreases.

Let $N(t)$ be the population at time t and N_0 be the initial population at time $t = 0$. Assume $r(N)$ to be positive and $r(N) = r_1\left(1 - \frac{N}{k}\right)$ where r_1 and k are constants.

$$\begin{aligned}
\frac{dN}{dt} &= rN = r_1\left(1 - \frac{N}{k}\right)N \\
\frac{dN}{dt} &= rN = r_1 N - r_1\frac{N^2}{k} \\
\frac{dN}{dt} &= r_1 N + r_1^{'} N^2
\end{aligned}$$

where $r_1^{'} = \frac{-r_1}{k} < 0 \quad \text{for} \quad N > 0$

Solution and Interpretation

$$\frac{dN}{dt} = r_1\left(1-\frac{N}{k}\right)N$$

$$\frac{dN}{dt} = r_1 N\left(\frac{k-N}{N}\right)$$

$$\frac{kdN}{N(k-N)} = r_1 dt$$

Using integration by partial fractions we get,

$$\left(\frac{1}{N}+\frac{1}{k-N}\right)dN = r_1 dt$$

$$\log N - \log(k-N) = r_1 t + c_1$$

At $t = 0 \quad N = N_0$

$$\log N_0 - \log(k-N_0) = c_1$$

$$\log N - \log(k-N) = r_1 t + \log N_0 - \log(k-N_0)$$

$$\implies \log\left(\frac{N}{k-N}\right) = r_1 t + \log\left(\frac{N_0}{k-N_0}\right)$$

$$\frac{N}{k-N} = e^{r_1 t}\left(\frac{N_0}{k-N_0}\right)$$
$$\frac{k-N}{N} = e^{-r_1 t}\left(\frac{k-N_0}{N_0}\right)$$
$$\frac{k}{N} = \frac{(k-N_0)}{N_0}e^{-r_1 t} + 1$$
$$\frac{1}{N} = \left(\frac{k-N_0}{N_0}e^{-r_1 t} + 1\right)\frac{1}{k}$$

Hence,

$$N(t) = \frac{k}{(ce^{-r_1 t} + 1)} \tag{3.1}$$

where $c = \frac{(k-N_0)}{N_0}$. The above equation is the size of the population at any time t. As $t \to \infty \quad N(t) \to k$ where k is the carrying capacity (maximum number of individuals that can survive in given condition) and r_1 is called the intrinsic growth rate of the population.

3.2.5 Stability

Once the equilibrium points are found we want to look at the stability of the system. The stability of a system of differential equations is checked to determine how the system behaves under small perturbations or changes in initial conditions. Stability analysis helps us understand whether the system will return to a steady state after disturbances. This helps in predicting the long-term behavior of the system and ensures it's reliability for further use.

Particularly for system differential equations:

In mathematical modeling, stability is of great importance as it ensures that the solutions

obtained from the model accurately represents the behavior of the system. Under varying conditions a stable mathematical mode will provide reliable and consistent results, thus helping researchers in studying and understanding complex systems. Hence allowing to draw the correct interpretation and predictions for the situation, which also helps in finding the best suited practical solution for the situation.

Methods for Local Stability

For non linear system of differential equations, once the equilibrium points are found

- Find the Jacobian matrix at the equilibrium point.
- Find the eigenvalues corresponding to the obtained matrix and analyze its stability.
- Repeat the same for other equilibrium points.

Methods for Global Stability

1. Lyapunov's Second Method of Stability
   (a) Choose a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ which has to be a positive definite i.e. satisfying the conditions
      - $V(0) = 0$
      - $V(x) > 0$ for all $x \neq 0$
   (b) Compute the derivative of $V(x)$ i.e $\frac{dV}{dt}$
   (c) If $\frac{dV}{dt}$ is negative definite then equilibrium point is globally asymptotically stable.

2. LaSalle invariance principle

   Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a function such that $V'(x) \leq 0$ in Ω. Let $E = \{x \in \Omega \mid V'(x) = 0\}$ and $M = \{x \in E \mid V'(x) = 0 \, \forall \, t \geq 0\}$ be the largest invariant set of E with respect to $x'(t) = f(x)$. Then every solution $x(t)$ in Ω approaches M and $t \to \infty$.

Chapter 4

VECTOR BORNE DISEASES

4.1 Introduction

Vector borne diseases are those diseases which are transmitted to humans through insect bites as the pathogen enters the blood of the human body. The first model on vector borne diseases was given by R. Ross[16], particularly on the prevention of malaria in London, which was later modified by G. Macdonald[14]. This was followed by various models based on infectious diseases. Models on vector borne diseases based on other factors affecting the spread were also studied. Further effects of human movement on vector borne diseases was studied by Costner et al[2]. Tumwiine et al[21] studied the effect of immigration on vector borne diseases. Age structured model was studied based on the effect of vector biting and vector mortality on the spread of the infectious disease. The effect of temperature on host pathogen system and the how climate and thermal adaptability are related was studied by Waikom et al[23]. How environment affects direct and indirect spread of carrier based infectious diseases was studied by Ghosh et al[7]. But little attention has been given to the study of effect of

pathogen population growing in the environment. Here we shall be looking at the effect of environmental factors like human discharge, household waste, contaminated water bodies on the transmission of vector borne diseases

4.2 Mathematical Model

This section gives an ODE model for the vector transmitted disease in host population.

4.2.1 Host Population Dynamics:

It is assumed that the host population at time t, denoted by $N_1(t)$ is partitioned into Susceptible population $S(t)$, Inflected population $I(t)$ and Recovered population $R(t)$. The following assumptions are made in the formulation process:

1. The vertical transmission in the host population is negligible, so as all the newly recruited individuals are susceptible.

2. The recovered individuals acquire permanent immunity i.e. the recovered population cannot move back to the susceptible class.

3. The Susceptible host can become infected either through direct transmission with the infected or through biting of an infectious vector.

$$\begin{aligned}
\frac{dS}{dt} &= b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S \\
\frac{dI}{dt} &= \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I \\
\frac{dR}{dt} &= \alpha I - \mu_1 R
\end{aligned}$$

$b_1 =$ constant rate at which host population is recruited

$\lambda_1 =$ rate of direct transmission from infected to susceptible host population

$\beta_1 =$ biting rate of infected vectors (pathogen-carrier)

$\mu_1 =$ natural death rate

$\alpha =$ recovery rate

4.2.2 Vector Population Dynamics:

The total vector population, denoted by $N_2(t)$ is partitioned into Susceptible vectors $M(t)$ and Infected vectors $V(t)$.The pathogen population is denoted by $P(t)$ and $E(t)$ denotes the cumulative density of environmental factors.

$$\begin{aligned}
\frac{dM}{dt} &= b_2 - \lambda_2 MI - \mu_2 M \\
\frac{dV}{dt} &= \lambda_2 MI - \mu_2 V \\
\frac{dP}{dt} &= \eta IP + \theta P\left(1 - \frac{P}{c(E)}\right) - \gamma P \\
\frac{dE}{dt} &= Q_0 - \theta_1 E + \theta_2 N
\end{aligned}$$

$b_2 =$ rate at which vector population is recruited

μ_2 = natural death rate

λ_2 = rate at which susceptible vectors become infected after biting the infected host

η = shedding rate of pathogen from infected hosts

θ = growth rate of pathogen

γ = decay rate of pathogen

Q_0 = growth rate of environmental factors which depends on the human action

θ_1 = depletion rate coefficient of environmental factors

θ_2 = growth rate coefficient of environmental factors due to human and vector population density related factors

$c(E)$ =carrying capacity of the environment

We assume that $\alpha > 0$, $\theta > 0$, $b_i > 0$, $\mu_i > 0$ for $i = 1,2$ and the initial conditions for the vector and host population are;

$S(0) = S^*, I(0) = I^*, R(0) = R^*, V(0) = V^*, M(0) = M^*, P(0) = P^*, E(0) = E^*$

Thus we have $N(t) = N_1(t) + N_2(t)$ where $N_1(t) = S(t) + I(t) + R(t)$ and

$N_2(t) = M(t) + V(t)$; $N_1(t) = \frac{b_1}{\mu_1}$ and $N_2(t) = \frac{b_2}{\mu_2}$

$\therefore\ M(t) = N_2(t) - V(t)$

Thus the model reduces to,

$$\frac{dS}{dt} = b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S \tag{4.1}$$

$$\frac{dI}{dt} = \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I \tag{4.2}$$

$$\frac{dV}{dt} = \lambda_2 \left(\frac{b_2}{\mu_2} - V \right) I - \mu_2 V \tag{4.3}$$

$$\frac{dP}{dt} = \eta IP + \theta P \left(1 - \frac{P}{c(E)} \right) - \gamma P \tag{4.4}$$

$$\frac{dE}{dt} = Q_0 - \theta_1 E + \theta_2 N \tag{4.5}$$

$\Omega = \{(S,I,V,P,E) \in \mathbb{R}^5_+, 0 \leq S+I \leq \frac{b_1}{\mu_1}; 0 \leq V \leq \frac{b_2}{\mu_2}; S,I,V \geq 0; 0 \leq P \leq m; 0 \leq E \leq E_m\}$

where $P \leq \frac{c(E)}{\theta}\left(\frac{\eta b_1 + \mu_1(\theta-\gamma)}{\mu_1}\right) = m$, $E_m = \frac{Q_0+\theta_2 N}{\theta_1}$

is the set which attracts all the solutions and the reduced system is studied in the closed set.

4.3 Existence of Equilibrium Points

We shall be looking at the following equilibrium points:

1. **Disease-Free Equilibrium:** $E_0 = (S^0, I^0, V^0, P^0, E^0) = (S^0, 0, 0, P^0, E^0)$

We impose the equilibrium condition, $\frac{dS}{dt} = 0$

$$b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S = 0$$
$$b_1 - \mu_1 S = 0$$
$$S^0 = \frac{b_1}{\mu_1}$$

Imposing $\frac{dP}{dt} = 0$

$$\eta IP + \theta P\left(1 - \frac{P}{c(E)}\right) - \gamma P = 0$$
$$\theta P - \frac{\theta P^2}{c(E)} - \gamma P = 0$$
$$P(\theta - \gamma) - \frac{\theta P^2}{c(E)} = 0$$
$$\frac{\theta P}{c(E)} = (\theta - \gamma)$$
$$P^0 = (\theta - \gamma)\frac{c(E)}{\theta}$$

for $\theta > \gamma$

Imposing $\frac{dE}{dt} = 0$

$$Q_0 - \theta_1 E + \theta_2 N = 0$$
$$E^0 = \frac{Q_0 + \theta_2 N}{\theta_1}$$

Therefore the disease free equilibrium point is $E_0 = \left(\frac{b_1}{\mu_1}, 0, 0, \frac{(\theta-\gamma)c(E)}{\theta}, \frac{Q_0+\theta_2 N}{\theta_1}\right)$

2. **Pathogen-Free Equilibrium:** $E_1 = (S^1, I^1, V^1, P^1, E^1) = (S^1, 0, 0, 0, E^1)$

Imposing $\frac{ds}{dt} = 0$ we get,

$$S^1 = \frac{b_1}{\mu_1}$$

Imposing $\frac{dE}{dt} = 0$ we get,

$$E^1 = \frac{Q_0 + \theta_2 N}{\theta_1}$$

Therefore the disease free equilibrium point is $E_1 = \left(\frac{b_1}{\mu_1}, 0, 0, 0, \frac{Q_0+\theta_2 N}{\theta_1}\right)$

3. **Endemic Equilibrium:** $E^2 = (S_2, I_2, V_2, P_2, E_2)$

Imposing $\frac{dS}{dt} = 0$ we get,

$$b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S = 0 \tag{4.6}$$

Imposing $\frac{dI}{dt} = 0$ we get,

$$\lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I = 0 \tag{4.7}$$

Solving (4.6) and (4.7) we get,

$$\begin{aligned} b_1 - \mu_1 S - (\alpha + \mu_1) I &= 0 \\ \mu_1 S &= b_1 - (\alpha + \mu_1) I \\ \implies S_2 &= \frac{b_1 - (\alpha + \mu_1) I^2}{\mu_1} \end{aligned}$$

Imposing $\frac{dV}{dt} = 0$ we get,

$$\begin{aligned} \lambda_2 \left(\frac{b_2}{\mu_2} - V \right) I - \mu_2 V &= 0 \\ \frac{\lambda_2 b_2}{\mu_2} I - \lambda_2 V I - \mu_2 V &= 0 \\ \frac{\lambda_2 b_2}{\mu_2} I - V(\lambda_2 + \mu_2) &= 0 \\ \implies V_2 &= \frac{\lambda_2 b_2 I^2}{(\lambda_2 + \mu_2)} \end{aligned}$$

Imposing $\frac{dP}{dt} = 0$ we get,

$$\begin{aligned} \eta I P + \theta P \left(1 - \frac{P}{c(E)} \right) - \gamma P &= 0 \\ \eta I P + \theta P - \frac{\theta P^2}{c(E)} - \gamma P &= 0 \\ \frac{\theta P^2}{c(E)} &= P(\eta I + \theta - \gamma) \\ \implies P_2 &= \left(\frac{(\eta I_2 + \theta - \gamma)}{\theta} \right) c(E) \end{aligned}$$

Imposing $\frac{dE}{dt} = 0$

$$Q_0 - \theta_1 E + \theta_2 N = 0$$
$$\implies E_2 = \frac{Q_0 + \theta_2 N}{\theta_1}$$

Therefore the endemic equilibrium point is,
$E^2 = \left(\frac{b_1-(\alpha+\mu_1)I_2}{\mu_1}, I_2, \frac{\lambda_2 b_2 I_2}{(\lambda_2+\mu_2)}, \left(\frac{(\eta I_2+\theta-\gamma)}{\theta}\right)c(E), \frac{Q_0+\theta_2 N}{\theta_1}\right)$

4.3.1 Basic Reproduction Number

Basic reproduction number R_0 is the average number of secondary infections arising from a single infected individual which is usually calculated about the disease free equilibrium point.

We find R_0 using the next generation matrix, FW^{-1}.

To find R_0 we consider the equations,

$$\frac{dI}{dt} = \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1)I$$
$$\frac{dV}{dt} = \lambda_2 \left(\frac{b_2}{\mu_2} - V\right) I - \mu_2 V$$

Let $f_1 = \lambda_1 SI + \beta_1 SV$, $f_2 = 0$ and $g_1 = (\alpha+\mu_1)I$, $g_2 = -\lambda_2\left(\frac{b_2}{\mu_2} - V\right)I + \mu_2 V$

Next we need to find the matrix FW^{-1} and further find its corresponding eigenvalue, one of which is the reproduction number.

where,

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial V} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \frac{\partial g_1}{\partial I} & \frac{\partial g_1}{\partial V} \\ \frac{\partial g_2}{\partial I} & \frac{\partial g_2}{\partial V} \end{bmatrix}$$

$$F(E_0) = \begin{bmatrix} \lambda_1 S^0 & \beta_1 S^0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W(E_0) = \begin{bmatrix} (\alpha+\mu_1) & 0 \\ \frac{-\lambda_2 b_2}{\mu_2} & \mu_2 \end{bmatrix}$$

$$\begin{aligned} W^{-1} &= \frac{1}{|W|} adj(W) \\ &= \frac{1}{\mu_2(\alpha+\mu_1)} \begin{bmatrix} \mu_2 & 0 \\ \frac{\lambda_2 b_2}{\mu_2} & \alpha+\mu_1 \end{bmatrix} \\ &= \begin{bmatrix} (\alpha+\mu_1) & 0 \\ \frac{\lambda_2 b_2}{\mu_2} & \mu_2 \end{bmatrix} \end{aligned}$$

Now,

$$FW^{-1} = \begin{bmatrix} \frac{\lambda_1 S^0}{(\alpha+\mu_1)} + \frac{\lambda_2 b_2 \beta_1 S^0}{(\alpha+\mu_1)\mu_1\mu_2} & \frac{\beta_1 S^0}{\mu_2} \\ 0 & 0 \end{bmatrix}$$

$$FW^{-1} - \lambda I = \begin{bmatrix} \frac{\lambda_1 S^0}{(\alpha+\mu_1)} + \frac{\lambda_2 b_2 \beta_1 S^0}{(\alpha+\mu_1)\mu_1\mu_2} - \lambda & \frac{\beta_1 S^0}{\mu_2} \\ 0 & -\lambda \end{bmatrix}$$

We know that $|FW^{-1} - \lambda I| = 0$

$$\implies 0 = \left(\frac{\lambda_1 S^0}{(\alpha+\mu_1)} + \frac{\lambda_2 b_2 \beta_1 S^0}{(\alpha+\mu_1)\mu_1\mu_2} - \lambda \right)(-\lambda)$$

This implies that, $\lambda = \frac{\lambda_1 S^0}{(\alpha+\mu_1)} + \frac{\lambda_2 b_2 \beta_1 S^0}{(\alpha+\mu_1)\mu_1\mu_2}$

Therefore, $R_0 = \frac{\lambda_1 S^0}{(\alpha+\mu_1)} + \frac{\lambda_2 b_2 \beta_1 S^0}{(\alpha+\mu_1)\mu_1\mu_2} = \frac{b_1}{\mu_1}\left(\frac{\lambda_1}{\alpha+\mu_1} + \frac{\lambda_2 b_2 \beta_1}{\mu_1\mu_2(\alpha+\mu_1)}\right)$

4.4 Stability Analysis

4.4.1 Local Stability

Disease Free equilibrium point

Theorem 4.4.1.1. *Stability of Disease Free equilibrium point:*
The disease-free equilibrium point $E_0 = (S^0, 0, 0, P^0, E^0)$ *is locally asymptotically stable if* $R_0 < 1$, $\mu_1(\mu_2 + \alpha + \mu_1) > \lambda_1 b_1$ *and* $\frac{2\theta P}{c(E)} + \gamma > \theta$ *otherwise unstable.*
Proof:

We consider

$$F_1 = b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S$$
$$F_2 = \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I$$
$$F_3 = \lambda_2 \left(\frac{b_2}{\mu_2} - V \right) I - \mu_2 V$$
$$F_4 = \eta IP + \theta P \left(1 - \frac{P}{c(E)} \right) - \gamma P$$
$$F_5 = Q_0 - \theta_1 E + \theta_2 N$$

The Jacobian of the above system of equations is given by

$$J = \begin{bmatrix} -\mu_1 & -\lambda_1 S & -\beta_1 S & 0 & 0 \\ \lambda_1 I + \beta_1 V & \lambda_1 S - (\alpha + \mu_1) & \beta_1 S^0 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} - \lambda_2 V & -\lambda_2 I - \mu_2 & 0 & 0 \\ 0 & \eta P^0 & 0 & \eta I + \theta - \gamma - \frac{2\theta P^0}{c(E)} & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 \end{bmatrix}$$

The Jacobian of linearised system around $E_0 = (S^0, 0, 0, P^0, E^0)$ *is given by:*

$$J_0 = \begin{bmatrix} -\mu_1 & -\lambda_1 S^0 & -\beta_1 S^0 & 0 & 0 \\ 0 & \lambda_1 S^0 - (\alpha + \mu_1) & \beta_1 S^0 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 & 0 & 0 \\ 0 & \eta P^0 & 0 & \theta - \gamma - \frac{2\theta P^0}{c(E)} & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 \end{bmatrix}$$

$$J_0 - \lambda I = \begin{bmatrix} -\mu_1 - \lambda & -\lambda_1 S^0 & -\beta_1 S^0 & 0 & 0 \\ 0 & \lambda_1 S^0 - (\alpha + \mu_1) - \lambda & \beta_1 S^0 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 - \lambda & 0 & 0 \\ 0 & \eta P^0 & 0 & \theta - \gamma - \frac{2\theta P^0}{c(E)} - \lambda & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 - \lambda \end{bmatrix}$$

We know that $|J_0 - \lambda I| = 0$

$$|J_0 - \lambda I| = (-\mu_1 - \lambda) \begin{vmatrix} \lambda_1 S^0 - (\alpha + \mu_1) - \lambda & \beta_1 S^0 & 0 & 0 \\ \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 - \lambda & 0 & 0 \\ \eta P^0 & 0 & \theta - \gamma - \frac{2\theta P^0}{c(E)} - \lambda & 0 \\ 0 & 0 & 0 & -\theta_1 - \lambda \end{vmatrix}$$

On finding the determinant we get an equation,

$$0 = (\lambda_1 S^0 - (\alpha + \mu_1) - \lambda) \left[(-\mu_2 - \lambda)(\theta - \gamma - \frac{2\theta P^0}{c(E)} - \lambda)(-\theta_1 - \lambda) \right] - \beta_1 S^0 \left[\frac{\lambda_2 b_2}{\mu_2} (\theta - \gamma - \frac{2\theta P^0}{c(E)} - \lambda)(-\theta_1 - \lambda) \right]$$

$$0 = (-\mu_1 - \lambda) \left[(\theta - \gamma - \frac{2\theta P^0}{c(E)} - \lambda)(-\theta_1 - \lambda) \right] \left[(\lambda_1 S^0 - (\alpha + \mu_1) - \lambda)(-\mu_2 - \lambda) - \beta_1 S^0 \frac{\lambda_2 b_2}{\mu_2} \right]$$

$$\begin{aligned}\implies \lambda_1 &= -\mu_1,\\ \lambda_2 &= -\left(\frac{2\theta P^0}{c(E)} + \gamma - \theta\right),\\ \lambda_3 &= -\theta_1\end{aligned}$$

The three corresponding eigen values are negative provided $\frac{2\theta P^0}{c(E)} + \gamma > \theta$

The nature other two eigen values are found by solving the equation

$(\lambda_1 S_1 - (\alpha+\mu_1) - \lambda)(-\mu_2 - \lambda) - \beta_1 S_1 \frac{\lambda_2 b_2}{\mu_2} = 0$

$\lambda^2 + \lambda(\alpha + \mu_1 + \mu_2 - \frac{\lambda_1 b_1}{\mu_1}) + (\mu_1(-\frac{\lambda_1 b_1}{\mu_1} + \alpha + \mu_1) - \frac{\beta_1 b_1 b_2 \lambda_2}{\mu_1 \mu_2}) = 0$

Here,

$A = 1$

$B = \alpha + \mu_1 + \mu_2 - \frac{\lambda_1 b_1}{\mu_1} > 0 \quad if \quad \mu_1(\alpha + \mu_1 + \mu_2) > \lambda_1 b_1$

$C = \dfrac{\mu_1\mu_2^2(\alpha+\mu_1) - b_1(\mu_2^2\lambda_1 + \lambda_2\beta_1 b_2)}{\mu_1\mu_2}$

$\therefore\ C = \mu_1\mu_2(\alpha+\mu_1)(1-R_0), \qquad since \quad (1-R_0) = \frac{\mu_1\mu_2^2(\alpha+\mu_1) - b_1(\mu_2^2\lambda_1 + \lambda_2\beta_1 b_2)}{\mu_1\mu_2^2(\alpha+\mu_1)}$

Now, $C > 0$ *iff* $R_0 < 1$

Using the Routh Hurwitz criterion which says that a second degree polynomial with all positive coefficients will have negative roots.

Hence the theorem.

Pathogen Free equilibrium point

Theorem 4.4.1.2. *Stability of Pathogen-free equilibrium point:*

The pathogen-free equilibrium $E_1 = (S^1, 0, 0, 0, E^1)$ *is locally asymptotically stable if*

$R_0 < 1$, $\mu_1(\mu_2+\alpha+\mu_1) > \lambda_1 b_1$ and $\gamma > \theta$ otherwise unstable.

Proof:

We consider

$$F_1 = b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S$$
$$F_2 = \lambda_1 SI + \beta_1 SV - (\alpha+\mu_1)I$$
$$F_3 = \lambda_2\left(\frac{b_2}{\mu_2} - V\right)I - \mu_2 V$$
$$F_4 = \eta IP + \theta P\left(1 - \frac{P}{c(E)}\right) - \gamma P$$
$$F_5 = Q_0 - \theta_1 E + \theta_2 N$$

The Jacobian of the above system of equations is given by

$$J = \begin{bmatrix} -\mu_1 & -\lambda_1 S & -\beta_1 S & 0 & 0 \\ \lambda_1 I + \beta_1 V & \lambda_1 S - (\alpha+\mu_1) & \beta_1 S^0 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} - \lambda_2 V & -\lambda_2 I - \mu_2 & 0 & 0 \\ 0 & \eta P^0 & 0 & \eta I + \theta - \gamma - \frac{2\theta P^0}{c(E)} & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 \end{bmatrix}$$

The Jacobian of linearised system around $E_1 = (S_1, 0, 0, 0, E_1)$ *is given by:*

$$J_{E_1} = \begin{bmatrix} -\mu_1 & -\lambda_1 S^1 & -\beta_1 S^1 & 0 & 0 \\ 0 & \lambda_1 S^1 - (\alpha + \mu_1) & \beta_1 S^1 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 & 0 & 0 \\ 0 & 0 & 0 & \theta - \gamma & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 \end{bmatrix}$$

$$J_{E_1} - \lambda I = \begin{bmatrix} -\mu_1 - \lambda & -\lambda_1 S^1 & -\beta_1 S^1 & 0 & 0 \\ 0 & \lambda_1 S^1 - (\alpha + \mu_1) - \lambda & \beta_1 S^1 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 - \lambda & 0 & 0 \\ 0 & 0 & 0 & \theta - \gamma - \lambda & 0 \\ 0 & 0 & 0 & 0 & -\theta_1 - \lambda \end{bmatrix}$$

We know that $|J_{E_1} - \lambda I| = 0$

$$|J_{E_1} - \lambda I| = (-\mu_1 - \lambda) \begin{vmatrix} \lambda_1 S^1 - (\alpha + \mu_1) - \lambda & \beta_1 S^1 & 0 & 0 \\ \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 - \lambda & 0 & 0 \\ 0 & 0 & \theta - \gamma - \lambda & 0 \\ 0 & 0 & 0 & -\theta_1 - \lambda \end{vmatrix}$$

On finding the determinant we get an equation

$$0 = (-\mu_1 - \lambda)(\lambda_1 S^1 - (\alpha + \mu_1) - \lambda) \begin{vmatrix} -\mu_2 - \lambda & 0 & 0 \\ 0 & \theta - \gamma - \lambda & 0 \\ 0 & 0 & -\theta_1 - \lambda \end{vmatrix}$$

$$- (-\mu_1 - \lambda)\beta_1 S^1 \begin{vmatrix} \frac{\lambda_2 b_2}{\mu_2} & 0 & 0 \\ 0 & \theta - \gamma - \lambda & 0 \\ 0 & 0 & -\theta_1 - \lambda \end{vmatrix}$$

$$0 = (-\mu_1 - \lambda)\left[(\lambda_1 S^1 - (\alpha + \mu_1)(-\mu_2 - \lambda)(\theta - \gamma - \lambda)(-\theta_1 - \lambda) - \beta_1 S^1 \frac{\lambda_2 b_2}{\mu_2}(\theta - \gamma - \lambda)(-\theta_1 - \lambda)\right]$$

$$0 = (-\mu_1 - \lambda)[(\theta - \gamma - \lambda)(-\theta_1 - \lambda)]\left[(\lambda_1 S^1 - (\alpha + \mu_1) - \lambda)(-\mu_2 - \lambda) - \beta_1 S^1 \frac{\lambda_2 b_2}{\mu_2}\right]$$

$$\begin{aligned} \implies \lambda_1 &= -\mu_1, \\ \lambda_2 &= -(\gamma - \theta), \\ \lambda_3 &= -\theta_1 \end{aligned}$$

The three corresponding eigen values are negative provided $\gamma > \theta$.

The nature other two eigen values are found by solving the equation

$(\lambda_1 S^1 - (\alpha + \mu_1) - \lambda)(-\mu_2 - \lambda) - \beta_1 S^1 \frac{\lambda_2 b_2}{\mu_2} = 0$

$\lambda^2 + \lambda(\alpha + \mu_1 + \mu_2 - \frac{\lambda_1 b_1}{\mu_1}) + (\mu_1(-\frac{\lambda_1 b_1}{\mu_1} + \alpha + \mu_1) - \frac{\beta_1 b_1 b_2 \lambda_2}{\mu_1 \mu_2}) = 0$

Here,

$A = 1$

$B = \alpha + \mu_1 + \mu_2 - \frac{\lambda_1 b_1}{\mu_1} > 0 \quad if \quad \mu_1(\alpha + \mu_1 + \mu_2) > \lambda_1 b_1$

$C = \dfrac{\mu_1\mu_2^2(\alpha+\mu_1) - b_1(\mu_2^2\lambda_1 + \lambda_2\beta_1 b_2)}{\mu_1\mu_2}$

$\therefore\ C = \mu_2(\alpha+\mu_1)(1-R_0), \qquad since \quad (1-R_0) = \frac{\mu_1\mu_2^2(\alpha+\mu_1) - b_1(\mu_2^2\lambda_1+\lambda_2\beta_1 b_2)}{\mu_1\mu_2^2(\alpha+\mu_1)}$

Now, $C > 0$ iff $R_0 < 1$

Using the Routh Hurwitz criterion which says that a second degree polynomial with all positive coefficients will have negative roots.
Hence the theorem.

Endemic Equilibrium

Theorem 4.4.1.3. *The endemic equilibrium $E_2 = (S^2, I^2, V^2, P^2, E^2)$ is locally asymptotically stable for $R_0 > 1$.*

4.4.2 Global Stability

Disease Free equilibrium point

Theorem 4.4.2.1. *The disease-free equilibrium $(E_0 = S^0, 0, 0, P^0, E^0)$ is non-linearly asymptotically stable in the region Ω provided the following conditions are satisfied:*

1. $\mu_1 > \frac{\beta_1}{2\mu_1}(\lambda_1 + \beta_1)$
2. $2\mu_1(\alpha_1 + \mu_1) > b_1(3\lambda_1 + \beta_1) + \eta m \mu_1$
3. $\mu_2 > \beta_1 \frac{b_1}{\mu_1} + \lambda_2(\frac{b_2}{\mu_2} + \frac{b_1}{\mu_1})$
4. $\frac{\theta m}{c(E)} + \gamma > \frac{\eta m}{2} + \theta$

Proof: We transform the system using $S = S^0 + x_1$, $I = I^0 + x_2$, $V = V^0 + x_3$, $P = P^0 = x_4$, $E = E^0 + x_5$

On substituting in the system (4.1)-(4.5) as shown in the appendix we get,

$$\begin{aligned}
\frac{dx_1}{dt} &= -(\lambda_1 x_2 + \beta_1 x_3)S^0 - x_1(\lambda_1 x_2 + \beta_1 x_3) - \mu_1 x_1 \\
\frac{dx_2}{dt} &= (\lambda_1 x_2 + \beta_1 x_3)S^0 + x_1(\lambda_1 x_2 + \beta_1 x_3)(\alpha + \mu_1)x_2 \\
\frac{dx_3}{dt} &= \lambda_2 \frac{b_2}{\mu_2} - \lambda_2 x_2 x_3 - x_3(\lambda_2 I^0 + \mu_2) \\
\frac{dx_4}{dt} &= \eta(x_2 P^0 + x_2 x_4) + x_4\left(\theta\left(1 - \frac{1}{c(E)}(2P^0 + x_4)\right) - \gamma\right) \\
\frac{dx_5}{dt} &= -\theta_1 x_5
\end{aligned}$$

Consider a positive definite

$V_1(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)$, $V_1 : \mathbb{R}^5 \to \mathbb{R}$

Next step is to compute $\frac{dV_1}{dt}$,

$\frac{dV_1}{dt} = x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} + x_3\frac{dx_3}{dt} + x_4\frac{dx_4}{dt} + x_5\frac{dx_5}{dt}$

$$\begin{aligned}
\frac{dV_1}{dt} = {} & [-\lambda_1 x_1 x_2 S^0 - \lambda_1 x_2 x_1^2 - \beta_1 x_1 x_3 S^0 - \beta_1 x_3 x_1^2 - \mu_1 x_1^2] \\
& + [\lambda_1 x_2^2 S^0 + \lambda_1 x_1 x_2^2 + \beta_1 x_2 x_3 S^0 + \beta_1 x_1 x_2 x_3 - (\alpha + \mu_1)x_2^2] \\
& + [\lambda_2\left(\frac{b_2}{\mu_2}x_3^2 + I^0 x_3^2 + x_2 x_3^2\right) - \mu_2 x_3^2] \\
& + [\eta x_2 x_4^2 + \eta P^0 x_2 x_4 + \theta x_4^2 - \frac{1}{c(E)}\theta(2P^0 + x_4)x_4^2 - \gamma x_4^2] + [-\theta_1 x_5]
\end{aligned}$$

$$\frac{dV_1}{dt} = -\lambda_1 x_1 x_2 (S^0 + x_1) - \beta_1 x_1 x_3 (S^0 + x_1) - \mu_1 x_1^2 + \lambda_1 x_2^2 (S^0 + x_1) + \beta_1 x_2 x_3 (S^0 + x_1)$$
$$- (\alpha + \mu_1) x_2^2 + \lambda_2 \frac{b_2}{\mu_2} x_3^2 + \lambda_2 x_3^2 (I^0 + x_1) - \mu_2 x_3^2 + \eta x_2 x_4 (P^0 + x_4)$$
$$+ \theta x_4^2 - \frac{\theta}{c(E)} (2P^0 + x_4) - \gamma x_4^2 - \theta_1 x_5$$

Using the region Ω *and the inequality* $\pm 2ab \leq (a^2 + b^2)$ *on the right side of the equation.*

$$\frac{dV_1}{dt} \leq \frac{\lambda_1 b_1}{2\mu_1} x_1^2 + \frac{\lambda_1 b_1}{2\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_1^2 + \frac{\beta_1 b_1}{2\mu_1} x_3^2 - \mu_1 x_1^2 + \frac{\lambda_1 b_1}{\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_3^2$$
$$- (\alpha + \mu_1) x_2^2 + \frac{\lambda_2 b_2}{2\mu_2} x_3^2 + \frac{\lambda_2 b_1}{\mu_1} x_3^2 - \mu_2 x_3^2 + \frac{\eta x_2^2 m}{2} + \frac{\eta x_4^2 m}{2}$$
$$+ \theta x_4^2 - \frac{\theta m}{c(E)} x_4^2 - \gamma x_4^2 - \theta_1 x_5$$

$$\frac{dV_1}{dt} = -[b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + b_5 x_5^2]$$

$$b_1 = \mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1} \qquad b_2 = (\alpha + \mu_1) - \frac{3}{2}\frac{\lambda_1 b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1} - \frac{\eta m}{2}$$
$$b_3 = \mu_2 - \frac{\beta_1 b_1}{\mu_1} - \lambda_2 \left(\frac{b_2}{\mu_2} + \frac{b_1}{\mu_1}\right) \quad b_4 = \frac{\theta m}{c(E)} + \gamma - \frac{\eta m}{2} - \theta \qquad b_5 = \theta$$

Now, $\frac{dV_1}{dt}$ *is negative definite if each* $b_i > 0 \; \forall i = 1, 2..5$ *and* $\theta > 0$ *(assumed)*

Hence by Lyapunov's second method of stability the required conditions

1. $2\mu_1^2 > \lambda_1 b_1 + \beta_1 b_1$

2. $2\mu_1(\alpha+\mu_1) > b_1(3\lambda_1+\beta_1)+\eta m\mu_1$

3. $\mu_2 > \beta_1\frac{b_1}{\mu_1}+\lambda_2(\frac{b_2}{\mu_2}+\frac{b_1}{\mu_1})$

4. $\frac{\theta m}{c(E)}+\gamma > \frac{\eta m}{2}+\theta$

can be obtained.

Therefore E^0 is globally asymptotically stable if the above conditions hold.

Pathogen Free equilibrium point

Theorem 4.4.2.2. *The pathogen-free equilibrium $(E_1 = S^1, 0, 0, 0, E^1)$ is non-linearly asymptotically stable in the region Ω provided the following conditions are satisfied*

1. $2\mu_1(\alpha+\mu_1) > b_1(2\lambda_1+\beta_1)$

2. $\mu_1\mu_2^2 > (\lambda_2\beta_2\mu_1+b_1\mu_2)$

3. $2\mu_1^2 > b_1(\lambda_1+\beta_1)$

4. $m > 0$

Proof: We transform the system using
$S = S^1+y_1$, $I = I^1+y_2$, $V = V^1+y_3$, $P = P^1+y_4$, $E = E^1+y_5$
Consider a positive definite
$V_2(x) = \frac{1}{2}(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_4^2+B_5y_5^2)$, $V_2 : \mathbb{R}^5 \to \mathbb{R}$ *where B_i are positive reals for* $i = 1, 2, ..., 5$
Next step is to compute $\frac{dV_2}{dt}$,

On substituting in the system (4.1)-(4.5) we get,

$$\frac{dy_2}{dt} = -(\lambda_1 y_2 + \beta_1 y_3)S^1 - y_1(\lambda_1 y_2 + \beta_1 y_3) - \mu_1 y_1 \tag{4.8}$$

$$\frac{dy_2}{dt} = (\lambda_1 y_2 + \beta_1 y_3)S^1 + y_1(\lambda_1 y_2 + \beta_1 y_3)(\alpha + \mu_1)y_2 \tag{4.9}$$

$$\frac{dy_3}{dt} = \lambda_2 \frac{b_2}{\mu - 2} - \lambda_2 y_2 y_3 - y_3(\lambda_2 I^1 + \mu_2) \tag{4.10}$$

$$\frac{dy_4}{dt} = \eta(y_2 P^1 + y_2 y_4) + y_4\left(\theta\left(1 - \frac{1}{c(E)}(2P^1 + y_4)\right) - \gamma\right) \tag{4.11}$$

$$\frac{dy_5}{dt} = -\theta_1 y_5 \tag{4.12}$$

$$\frac{dV_2}{dt} = B_1[-\lambda_1 y_1 y_2 S^1 - \lambda_1 y_2 y_1^2 - \beta_1 y_1 y_3 S^1 - \beta_1 y_3 y_1^2 - \mu_1 y_1^2] + B_2[\lambda_1 y_2^2 S^1 + \lambda_1 y_1 y_2^2$$
$$+ \beta_1 y_2 y_3 S^1 + \beta_1 y_1 y_2 y_3 - (\alpha + \mu_1)y_2^2] + B_3[\lambda_2\left(\frac{b_2}{\mu_2} y_3^2 + I^1 y_3^2 + y_2 y_3^2\right) - \mu_2 y_3^2]$$
$$+ B_4[\eta y_2 y_4^2 + \eta P^1 y_2 y_4 + \theta y_4^2 - \frac{1}{c(E)}\theta(2P^1 + y_4)y_4^2 - \gamma y_4^2] + B_5[-\theta_1 y_5]$$

$$\frac{dV_2}{dt} = B_1[-\lambda_1 y_1 y_2 (S^1 + y_1) - \beta_1 y_1 y_3 (S^1 + y_1) - \mu_1 y_1^2] + B_2[\lambda_1 y_2^2 (S^1 + y_1) + \beta_1 y_2 y_3 (S^1 + x_1)$$
$$- (\alpha + \mu_1)x_2^2] + B_3[\lambda_2 \frac{b_2}{\mu_2} x_3^2 + \lambda_2 y_3^2 (I^1 + y_1) - \mu_2 x_3^2] + B_4[\eta y_2 y_4 (P^1 + x_4)$$
$$+ \theta y_4^2 - \frac{1}{c(E)}(2P^1 + x_4) - \gamma y_4^2] + B_5[-\theta_1 x_5]$$

Using the region Ω and the inequality $\pm 2ab \leq (a^2 + b^2)$ on the right side of the equation.

$$\frac{dV_2}{dt} \leq B_1\Big[\frac{\lambda_1 b_1}{2\mu_1}y_1^2+\frac{\lambda_1 b_1}{2\mu_1}y_2^2+\frac{\beta_1 b_1}{2\mu_1}y_1^2+\frac{\beta_1 b_1}{2\mu_1}y_3^2-\mu_1 y_1^2\Big]+B_2\Big[\frac{\lambda_1 b_1}{\mu_1}y_2^2+\frac{\beta_1 b_1}{2\mu_1}y_2^2+\frac{\beta_1 b_1}{2\mu_1}y_3^2$$
$$-(\alpha+\mu_1)y_2^2\Big]+B_3\Big[\frac{\lambda_2 b_2}{2\mu_2}y_3^2+\frac{\lambda_2 b_1}{\mu_1}y_3^2-\mu_2 y_3^2\Big]$$
$$+B_4\Big[\eta m y_2 y_4-\frac{1}{c(E)}\theta m y_4^2-\gamma y_4^2\Big]+B_5[-\theta_1 x_5]$$

$$\frac{dV_2}{dt} = -[b_{11}y_1^2+b_{33}y_3^2+(b_{22}y_2^2-b_{24}y_2y_4+b_{44}y_4^2)+b_{55}y_5^2]$$

$$b_{11} = B_1\Big(\mu_1-\frac{\lambda_1 b_1}{2\mu_1}-\frac{\beta_1 b_1}{2\mu_1}\Big)$$
$$b_{22} = B_2\Big((\alpha+\mu_1)-\lambda_1\tfrac{b_1}{\mu_1}-\tfrac{\beta_1 b_1}{2\mu_1}\Big)-B_1\frac{\lambda_1 b_1}{2\mu_1}$$
$$b_{33} = B_3\Big(\mu_2-\lambda_2\tfrac{b_2}{\mu_2}-\tfrac{b_1}{\mu_1}\Big)-\tfrac{\beta_1 b_1}{2\mu_1}(B_1+B_2)$$
$$b_{44} = B_4\Big(\tfrac{\theta m}{c(E)}+\gamma\Big)$$
$$b_{55} = B_5\theta_1$$
$$b_{24} = B_4\eta m$$

Using **Sylvester criteria**: A *nxn* symmetric matrix M is **positive definite** iff all principal determinants are strictly greater than 0, we show that $\frac{dV_2}{dt}$ is negative definite. From the above equation we form the matrix

$$D = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & \frac{-b_{24}}{2} & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & \frac{-b_{24}}{2} & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{bmatrix}$$

$\frac{dV_2}{dt}$ is negative definite if the following conditions are satisfied:

$|A_1| = b_{11} > 0 \quad \text{if} \quad B_1(\mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1}) > 0$

$|A_2| = b_{11}b_{22} > 0 \; if \; \left(\mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right)\left((\alpha+\mu_1) - \frac{3}{2}\lambda_1 b_1 \mu_1\right) - B_1 \frac{\lambda_1 b_1}{2\mu_1} > 0$

$|A_3| = b_{11}b_{22}b_{33} > 0 \quad \text{if} \quad b_{33} > 0$

$|A_4| = b_{11}\left[b_{22}b_{33}b_{44} - \frac{(b_{24})^2}{4}b_{33}\right] > 0 \text{ if } b_{22}b_{44} > \frac{(b_{24})^2}{4}$

$|A_5| > 0 \quad \text{if} \quad b_{11}[b_{22}(b_{33}b_{44}b_{55}) - \frac{b_{24}}{2}(\frac{b_{24}}{2}b_{33}b_{55})] > 0$

$$\Longrightarrow b_{11}b_{33}b_{55}\left(b_{22}b_{44} - \frac{(b_{24})^2}{4}\right) > 0 \; if \; b_{55} > 0$$

i.e.,

$$b_{11} = B_1\left(\mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right) > 0$$

$$b_{33} = B_3\left(\mu_2 - \lambda_2\frac{b_2}{\mu_2} - \frac{b_1}{\mu_1}\right) - \frac{\beta_1 b_1}{2\mu_1}(B_1 + B_2) > 0$$

$$\left(B_2\left((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right) - B_1\frac{\lambda_1 b_1}{2\mu_1}\right)\left(\frac{\theta m}{c(E)} + \gamma\right) > \frac{1}{4}B_4(\eta m)^2$$

$$B_5\theta_1 > 0$$

$$B_4\eta m > 0$$

Now suppose $B_1 = B_4 = B_5 = 1$, we have

$2\mu_1^2 > b_1(\lambda_1 + \beta_1)$

For B_2, we have

$$\Big(B_2\Big((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\Big) - \frac{\lambda_1 b_1}{2\mu_1}\Big)\Big(\frac{\theta m}{c(E)} + \gamma\Big) > \frac{1}{2}(\eta m)^2$$

$$\Big(B_2\Big((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\Big) - \frac{\lambda_1 b_1}{2\mu_1}\Big) > \frac{1}{2}\Big(\frac{(\eta m)^2 c(E)}{2(\theta m + \gamma c(E))} + \frac{\lambda_1 b_1}{\mu_1}\Big)$$

$$B_2\Big(\frac{(2\mu_1(\alpha+\mu_1) - 2\lambda_1 b_1 - \beta_1 b_1)}{2\mu_1}\Big) > \frac{1}{2}\Big(\frac{1}{2}\frac{(\eta m)^2 c(E)}{(\theta m + \gamma c(E))} + \frac{\lambda_1 b_1}{\mu_1}\Big)$$

So we have,

$$B_2 > \frac{\mu_1}{(2\mu_1(\alpha+\mu_1) - b_1(2\lambda_1+\beta_1))}\Big(\frac{(\eta m)^2 c(E)}{2(\theta m + \gamma c(E))} + \frac{\lambda_1 b_1}{\mu_1}\Big)$$

For $B_3\Big(\mu_2 - \lambda_2\frac{b_2}{\mu_2} - \frac{b_1}{\mu_1}\Big) - \frac{\beta_1 b_1}{2\mu_1}(B_1 + B_2) > 0$, we have

$$B_3\Big(\mu_2 - \lambda_2\frac{b_2}{\mu_2} - \frac{b_1}{\mu_1}\Big) > \frac{\beta_1 b_1}{2\mu_1}(1 + B_2)$$

$$B_3\Big(\frac{\mu_1\mu_2^2 - (\mu_1\lambda_2 b_2 + b_1\mu_2)}{\mu_1\mu_2}\Big) > \frac{\beta_1 b_1}{2\mu_1} + \frac{B_2\beta_2 b_1}{2\mu_1}$$

$$B_3\Big(\tfrac{\mu_1\mu_2^2 - (\mu_1\lambda_2 b_2 + b_1\mu_2)}{\mu_1\mu_2}\Big) > \tfrac{\beta_1 b_1}{2\mu_1} + \tfrac{\mu_1}{(2\mu_1(\alpha+\mu_1) - b_1(2\lambda_1+\beta_1))}\Big(\tfrac{(\eta m)^2 c(E)}{2(\theta m+\gamma c(E))} + \tfrac{\lambda_1 b_1}{\mu_1}\Big)\tfrac{\beta_1 b_1}{2\mu_1}$$

$$B_3\Big(\tfrac{\mu_1\mu_2^2 - (\mu_1\lambda_2 b_2 + b_1\mu_2)}{\mu_1\mu_2}\Big) > \tfrac{\beta_1 b_1}{2\mu_1}\Big(1 + \tfrac{\mu_1}{(2\mu_1(\alpha+\mu_1) - b_1(2\lambda_1+\beta_1))}\Big(\tfrac{(\eta m)^2 c(E)}{2(\theta m+\gamma c(E))} + \tfrac{\lambda_1 b_1}{\mu_1}\Big)\Big)$$

So we have,

$$B_3 > \frac{\mu_1\mu_2}{\mu_1\mu_2^2-(\mu_1\lambda_2 b_2+b_1\mu_2)}\frac{\beta_1 b_1}{2\mu_1}\left(1+\frac{\mu_1}{(2\mu_1(\alpha+\mu_1)-b_1(2\lambda_1+\beta_1))}\left(\frac{(\eta m)^2 c(E)}{2(\theta m+\gamma c(E))}+\frac{\lambda_1 b_1}{\mu_1}\right)\right)$$

The above inequality holds provided,

$\mu_1\mu_2^2 > (\mu_1\lambda_2 b_2 + b_1\mu_2)$, and

$2\mu_1(\alpha+\mu_1) > b_1(2\lambda_1+\beta_1)$

Therefore the required conditions are:

1. $2\mu_1(\alpha+\mu_1) > b_1(2\lambda_1+\beta_1)$
2. $\mu_1\mu_2^2 > (\mu_1\lambda_2 b_2 + b_1\mu_2)$
3. $2\mu_1^2 > b_1(\lambda_1+\beta_1)$
4. $m > 0$

If the above conditions are satisfied then the coefficients are positive and hence $\frac{dV_2}{dt}$ is a negative definite.

Hence proving the theorem.

Endemic Equilibrium point

Theorem 4.4.2.3. *The disease-free equilibrium $E^2 = (S_2, V_2, I_2, P_2, E_2)$ is non-linearly asymptotically stable in the region W provided the following conditions are satisfied:*

1. $(\mu_1+\lambda_1 b_1\mu_1+\beta_1 b_2\mu_2)\left((\alpha+\mu_1)-\frac{\lambda_1 b_1}{\mu_1}\right) > \frac{3}{2}[(\lambda_1 I_2+\beta_1 V_2)-\lambda_1 S_2]^2$
2. $(\alpha+\mu_1) > \lambda_1 b_1$

3. $c(E)\theta + \eta c(E)I_2 > \gamma c(E) + \theta m$

Proof. We transform the system using

$S = S_2 + z_1$, $I = I_2 + z_2$, $V = V_2 + z_3$, $P = P_2 + z_4$, $E = E_2 + z_5$

On substituting in the system

$$\begin{aligned}
\frac{dS}{dt} &= b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S \\
\frac{dI}{dt} &= \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I \\
\frac{dV}{dt} &= \lambda_2 \Big(\frac{b_2}{\mu_2} - V \Big) I - \mu_2 V \\
\frac{dP}{dt} &= \eta IP + \theta P \Big(1 - \frac{P}{c(E)} \Big) - \gamma P \\
\frac{dE}{dt} &= Q_0 - \theta_1 E + \theta_2 N
\end{aligned}$$

We have,

$$\frac{dS_2}{dt} + \frac{dz_1}{dt} = b_1 - \lambda_1 (S_2 + z_1)(I_2 + z_2) - \beta_1 (S_2 + z_1)(V_2 + z_3) - \mu_1 (S_2 + z_1)$$

$$\frac{dS_2}{dt} + \frac{dz_1}{dt} = b_1 - \lambda_1 (S_2 I_2 + S_2 z_2 + I_2 z_1 + z_1 z_2) - \beta_1 (S_2 V_2 + S_2 z_3 + V_2 z_1 + z_1 z_3) - \mu_1 S_2 - \mu_1 z_1$$

$$\frac{dS_2}{dt} + \frac{dz_1}{dt} = b_1 - \lambda_1 S_2 I_2 - \lambda_1 S_2 z_2 - \lambda_1 I_2 z_1 - \lambda_1 z_1 z_2 - \beta_1 S_2 V_2 - \beta_1 S_2 z_3 - \beta_1 V_2 z_1 - \beta_1 z_1 z_3 - \mu_1 S_2 - \mu_1 z_1$$

$$\frac{dS_2}{dt} + \frac{dz_1}{dt} = -(\mu_1 + \lambda_1 S_2 + \beta_1 V_2) z_1 - (\lambda_1 z_2 + \beta_1 z_3) S_2 - (\lambda_1 z_2 + \beta_1 z_3) + b_1 - \lambda_1 S_2 I_2 - \beta_1 S_2 V_2 - \mu_1 S_2$$

$$\frac{dz_1}{dt} = -(\mu_1 + \lambda_1 S_2 + \beta_1 V_2) z_1 - (\lambda_1 z_2 + \beta_1 z_3) S_2 - (\lambda_1 z_2 + \beta_1 z_3)$$

$$\frac{dI_2}{dt}+\frac{dz_2}{dt}=\lambda_1(S_2+z_1)(I_2+z_2)+\beta_1(S_2+z_1)(V_2+z_3)-(\alpha+\mu_1)(I_2+z_2)$$

$$\frac{dI_2}{dt}+\frac{dz_2}{dt}=\lambda_1(S_2I_2+S_2z_2+I_2z_1+z_1z_2)+\beta_1(S_2V_2+S_2z_3+V_2z_1+z_1z_3)-\alpha I_2-\alpha z_2$$
$$-\mu_1I_2-\mu_1z_2$$

$$\frac{dI_2}{dt}+\frac{dz_2}{dt}=\lambda_1S_2I_2+\lambda_1S_2z_2-\lambda_1I_2z_1+\lambda_1z_1z_2+\beta_1S_2V_2+\beta_1S_2z_3-\beta_1V_2z_1+\beta_1z_1z_3$$
$$-\alpha I_2-\alpha z_2-\mu_1I_2-\mu_1z_2$$

$$\frac{dI_2}{dt}+\frac{dz_2}{dt}=(\lambda_1z_2+\beta_1z_3)S_2+(\lambda_1I_2+\beta_1V_2)z_1+(\lambda_1z_2+\beta_1z_3)z_1-(\alpha+\mu_1)z_1$$
$$+\lambda_1S_2I_2+\beta_1S_2V_2-\alpha I_2-\mu_1I_2$$

$$\frac{dz_2}{dt}=(\lambda_1z_2+\beta_1z_3)S_2+(\lambda_1I_2+\beta_1V_2)z_1+(\lambda_1z_2+\beta_1z_3)z_1-(\alpha+\mu_1)z_1$$

$$\frac{dV_2}{dt}+\frac{dz_3}{dt}=\frac{\lambda_2b_2}{\mu_2}(I_2+z_2)-\lambda_2(v_2+z_2)(I_2+z_2)-\mu_2(V_2+z_3)$$

$$\frac{dV_2}{dt}+\frac{dz_3}{dt}=\frac{\lambda_2b_2}{\mu_2}I_2+\frac{\lambda_2b_2}{\mu_2}z_2-\lambda_2V_2I_2-\lambda_2V_2z_2-\lambda_2I_2z_3-\lambda_2z_2z_3-\mu_2V_2-\mu_2z_3$$

$$\frac{dV_2}{dt}+\frac{dz_3}{dt}=\left[\frac{\lambda_2b_2}{\mu_2}z_2-\lambda_2(V_2z_2+z_2z_3)-z_3(\lambda_2I_2+\mu_2)\right]+\frac{\lambda_2b_2}{\mu_2}I_2-\lambda_2V_2I_2-\mu_2V_2$$

$$\frac{dz_3}{dt}=\frac{\lambda_2b_2}{\mu_2}z_2-\lambda_2(V_2z_2+z_2z_3)-z_3(\lambda_2I_2+\mu_2)$$

$$\frac{dP_2}{dt}+\frac{dz_4}{dt}=\eta(I_2+z_2)(P_2+z_4)+\theta(P_2+z_4)-\theta\frac{(P_2+z_4)^2}{c(E)}-\gamma(P_2+z_4)$$

$$\frac{dP_2}{dt}+\frac{dz_4}{dt}=\eta(I_2P_2+I_2z_4+P_2z_2+z_2z_4)+\theta P_2+\theta z_4+\frac{\theta P_2^2}{c(E)}-\frac{2\theta P_2}{c(E)}z_4-\frac{\theta z_4^2}{c(E)}-\gamma P_2-\gamma z_4$$

$$\frac{dP_2}{dt}+\frac{dz_4}{dt}=\eta I_2P_2+\eta I_2z_4+\eta P_2z_2+\eta z_2z_4+\theta P_2+\theta z_4+\frac{\theta P_2^2}{c(E)}-\frac{2\theta_2}{c(E)}-\frac{\theta z_4^2}{c(E)}-\gamma P_2-\gamma z_4$$

$$\frac{dP_2}{dt}+\frac{dz_4}{dt}=\eta(I_2z_4+P_2z_2+z_2z_4)+z_4\left(\theta-\frac{2\theta P_2}{c(E)}-\frac{\theta z_4}{c(E)}\right)+\eta I_2P_2+\theta P_2-\frac{\theta P_2^2}{c(E)}-\gamma P_2$$

$$\frac{dP_2}{dt}+\frac{dz_4}{dt}=\eta(I_2z_4+P_2z_2+z_2z_4)+z_4\left(\theta\left(1-\frac{1}{c(E)}(2P_2+z_4)\right)-\gamma\right)+\eta I_2P_2+\theta P_2-\frac{\theta P_2^2}{c(E)}-\gamma P_2$$

$$\frac{dz_4}{dt}=\eta(I_2z_4+P_2z_2+z_2z_4)+z_4\left(\theta\left(1-\frac{1}{c(E)}(2P_2+z_4)\right)\right)$$

$$\frac{dE_2}{dt}+\frac{dz_5}{dt}=Q_0-\theta_1(E_2+z_5)+\theta_2N$$

$$\frac{dE_2}{dt}+\frac{dz_5}{dt}=Q_0-\theta_1E_2-\theta_1z_5+\theta_2N$$

$$\frac{dz_5}{dt}=-\theta_1z_5$$

Hence we have the system;

$$\begin{aligned}
\frac{dz_1}{dt} &= -(\mu_1+\lambda_1 S_2+\beta_1 V_2)z_1-(\lambda_1 z_2+\beta_1 z_3)S_2-(\lambda_1 z_2+\beta_1 z_3)\\
\frac{dz_2}{dt} &= (\lambda_1 z_2+\beta_1 z_3)S_2+(\lambda_1 I_2+\beta_1 V_2)z_1+(\lambda_1 z_2+\beta_1 z_3)z_1-(\alpha+\mu_1)z_1\\
\frac{dz_3}{dt} &= \frac{\lambda_2 b_2}{\mu_2}z_2-\lambda_2(V_2 z_2+z_2 z_3)-z_3(\lambda_2 I_2+\mu_2)\\
\frac{dz_4}{dt} &= \eta(I_2 z_4+P_2 z_2+z_2 z_4)+z_4\left(\theta\left(1-\frac{1}{c(E)}(2P_2+z_4)\right)-\gamma\right)\\
\frac{dz_5}{dt} &= -\theta_1 z_5
\end{aligned}$$

Next we consider the positive definite $V_3(z)=\frac{1}{2}(C_1z_1^2+C_2z_2^2+C_3z_3^2+C_4z_4^2+C_5z_5^2)$

$V_3:\mathbb{R}^5\to\mathbb{R}$ *where* C_i *are positive reals for* $i=1,2,...,5$

$$\begin{aligned}
\frac{dV_3}{dt} &= C_1\frac{dz_1}{dt}+C_2\frac{dz_2}{dt}+C_3\frac{dz_3}{dt}+C_5\frac{dz_5}{dt}+C_5\frac{dz_5}{dt}\\
\frac{dV_3}{dt} &= C_1[-\mu_1 z_1^2-\lambda_1 I_2 z_1^2-\beta_1 V_2 z_1^2-\lambda_1 z_1 z_2 S_2-\beta_1 z_1 z_3 S_2-\lambda_1 z_2 z_1^2-\beta_1 z_3 z_1^2]\\
&\quad +C_2[\lambda_1 z_2^2 S_2+\beta_1 z_2 z_3 S_2+\lambda_1 z_1 z_2 I_2+\beta_1 z_1 z_2 V_2+\lambda_1 z_1 z_2^2+\beta_1 z_1 z_2 z_3\\
&\quad -(\alpha+\mu_1)z_2^2]+C_3\left[\frac{\lambda_2 b_2}{\mu_2}z_2 z_3-\lambda_2 z_2 z_3 V_2-\lambda_2 z_2 z_3^2-\lambda_2 z_3^2 I_2-\mu_2 z_3^2\right]\\
&\quad +C_4\left[\eta I_2 z_4^2+\eta z_2 z_4(P_2+z_4)+\left(\theta\left(1-\frac{1}{c(E)}(2P_2+z_4)\right)-\gamma\right)z_4^2\right]\\
&\quad +C_5[-\theta_1 z_5]
\end{aligned}$$

$$\frac{dV_3}{dt} = C_1[-\mu_1 z_1^2 - \lambda_1(I_2+z_2) - \beta_1(V_2+z_3) - \lambda_1 z_1 z_2 S_2 - \beta_1 z_1 z_3 S_2]$$
$$+ C_2[\lambda_1(S_2+z_1)z_2^2 + \beta_1 z_2 z_3(S_2+z_1) + (\lambda_1 I_2 + \beta_1 V_2) - (\alpha+\mu_1)z_2^2]$$
$$+ C_3\left[\frac{\lambda_2 b_2}{\mu_2} z_2 z_3 - \lambda_2 z_2 z_3(V_2+z_3) - \lambda_2 z_3^2 I_2 - \mu_2 z_3^2\right]$$
$$+ C_4\left[\eta I_2 z_4^2 + \eta z_2 z_4(P_2+z_4) + \left(\theta\left(1 - \frac{1}{c(E)}(2P_2+z_4)\right) - \gamma\right)z_4^2\right]$$
$$+ C_5[-\theta_1 z_5]$$

Using the region Ω and the inequality $\pm 2ab \leq (a^2+b^2)$ on the right side of the above equation.

$$\frac{dV_3}{dt} \leq C_1[-\mu_1 z_1^2 - \frac{\lambda_1 b_1}{\mu_1} z_1^2 - \frac{\beta_1 b_2}{\mu_2} z_1^2 - \lambda_1 z_1 z_2 S_2 - \beta_1 z_1 z_3 S_2]$$
$$+ C_2\left[\frac{\lambda_1 b_1}{\mu_1} z_2^2 + \frac{\beta_1 b_2}{\mu_2} z_2 z_3 + (\lambda_1 I_2 + \beta_1 V_2 z_1 z_2) - (\alpha+\mu_1)z_2^2\right]$$
$$+ C_3\left[\frac{\lambda_2 b_2}{\mu_2} z_2 z_3 - \frac{\lambda_2 b_2}{\mu_2} z_2 z_3 - \lambda_2 z_3^2 I_2 - \mu_2 z_3^2\right]$$
$$+ C_4\left[\eta I_2 z_4^2 + + \eta m z_2 z_4 + \left(\theta\left(1 - \frac{m}{c(E)}\right) - \gamma\right)z_4^2\right]$$
$$+ C_5[-\theta_1 z_5]$$

$$\frac{dV_3}{dt} = -\Big[\left(\frac{1}{2}c_{11}z_1^2 - c_{12}z_1 z_2 + \frac{1}{3}c_{22}z_2^2\right) + \left(\frac{1}{2}c_{11}z_1^2 - c_{13}z_1 z_3 + \frac{1}{3}c_{33}z_3^2\right)$$
$$+ \left(\frac{1}{3}c_{22}z_2^2 - c_{12}z_2 z_3 + \frac{1}{2}c_{33}z_3^2\right) + \left(\frac{1}{3}c_{22}z_2^2 - c_{12}z_2 z_4 + c_{44}z_4^2\right) + c_{55}z_5^2\Big]$$

where,

$c_{11} = C_1\left(\mu_1 + \frac{\lambda_1 b_1}{\mu_1} + \frac{\beta_1 b_2}{\mu_2}\right)$

$c_{22} = C_2\left((\alpha+\mu_1) - \frac{\lambda_1 b_1}{\mu_1}\right)$

$c_{33} = C_3(\lambda_2 I_2 + \mu_2)$

$c_{44} = C_4\left(\gamma + \frac{\theta m}{c(E)} - \theta - \eta I_2\right)$

$c_{55} = C_5\theta_1$

$c_{12} = C_2(\lambda_1 I_2 + \beta_1 V_2) - C_1\lambda_1 S_2$

$c_{13} = -C_1\beta_1 S_2$

$c_{23} = C_2\frac{\beta_1 b_1}{\mu_1}$

$c_{24} = C_4\eta m$

To show that $\frac{dV_3}{dt}$ *is a negative definite we use the Sylvester criteria; A nxn symmetric matrix M is* ***positive definite*** *iff all principal determinants are strictly greater than* 0.

From the above equation we form the matrix

$$D = \begin{bmatrix} \frac{c_{11}}{2} & -\frac{c_{12}}{2} & -\frac{c_{13}}{2} & 0 & 0 \\ -\frac{c_{12}}{2} & \frac{c_{22}}{3} & -\frac{c_{23}}{2} & -\frac{c_{24}}{2} & 0 \\ -\frac{c_{13}}{2} & -\frac{c_{23}}{2} & \frac{c_{33}}{2} & 0 & 0 \\ 0 & -\frac{c_{24}}{2} & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}$$

$\frac{dV_3}{dt}$ *is negative definite if the following conditions are satisfied:*

$|D_1| = \frac{c_{11}}{2} > 0$

$|D_2| = c_{11}c_{22} - \frac{3c_{24}^2}{2} > 0$ *if* $c_{11}c_{22} > \frac{3c_{24}^2}{2}$

$$i.e.\ C_1(\mu_1+\frac{\lambda_1 b_1}{\mu_1}+\frac{\beta_1 b_2}{\mu_2})C_2\Big((\alpha+\mu_1)-\frac{\lambda_1 b_1}{\mu_1}\Big)>\frac{3}{2}[C_2(\lambda_1 I_2+\beta_1 V_2)-C_1\lambda_1 S_2]^2$$

$|D_3|=\frac{c_{11}}{2}\Big(\frac{c_{22}}{3}\frac{c_{33}}{2}-\frac{c_{23}^2}{4}\Big)-\frac{c_{13}}{2}\Big(\frac{c_{12}}{2}\frac{c_{23}}{2}+\frac{c_{13}}{2}\frac{c_{22}}{3}\Big)>0$ *if*
$\frac{c_{11}}{2}\Big(\frac{c_{22}}{3}\frac{c_{33}}{2}-\frac{c_{23}^2}{4}\Big)>\frac{c_{13}}{2}\Big(\frac{c_{12}}{2}\frac{c_{23}}{2}+\frac{c_{13}}{2}\frac{c_{22}}{3}\Big)$

$$i.e.\ \frac{c_{11}}{2}\frac{c_{22}}{3}\frac{c_{33}}{2}>\frac{c_{11}}{2}\frac{c_{23}^2}{4}+\frac{c_{13}}{2}\frac{c_{12}}{2}\frac{c_{23}}{2}+\frac{c_{13}^2}{2}\frac{c_{22}}{3}$$
$$\frac{c_{11}}{2}\frac{c_{22}}{3}\frac{c_{33}}{2}>\frac{c_{13}^2}{2}\frac{c_{22}}{3}$$
$$c_{11}c_{33}>c_{13}^2$$
$$\Longrightarrow C_3\Big(\mu_1+\frac{\lambda_1 b_1}{\mu_1}+\frac{\beta_1 b_2}{\mu_2}\Big)(\lambda_2 I_2+\mu_2)>C_1(\beta_1 S_2)^2$$

$|D_4|>0$ *if* $\frac{c_{11}}{2}\Big[\frac{c_{22}}{3}\frac{c_{33}}{2}c_{44}-\frac{c_{23}^2}{4}c_{44}+\frac{c_{24}^2}{4}\frac{c_{33}}{2}\Big]>0$

$$i.e.\ \frac{c_{22}}{3}\frac{c_{33}}{2}c_{44}>\frac{c_{23}^2}{4}c_{44}-\frac{c_{24}^2}{4}\frac{c_{33}}{2}$$
$$c_{22}c_{33}>\frac{3}{2}c_{23}^2\ \ and\ \ c_{22}c_{44}>-\frac{3}{4}c_{24}^2$$

$$\Longrightarrow\ C_3\Big((\alpha+\mu_1)-\frac{\lambda_1 b_1}{\mu_1}\Big)(\lambda_2 I_2+\mu_2)>C_2\frac{3}{2}(\beta_1 b_1\mu_1)^2\quad and$$
$$C_2\Big((\alpha+\mu_1)-\frac{\lambda_1 b_1}{\mu_1}\Big)\left(\gamma+\frac{\theta m}{c(E)}-\theta-\eta I_2\right)>-C_4\frac{3}{4}(\eta m)^2$$

$|D_5|>0\quad if\quad C_5\theta_1>0$

Suppose $C_1 = C_2 = C_5 = 1$, *we get*

$(\mu_1 + \lambda_1 b_1 \mu_1 + \beta_1 b_2 \mu_2)\Big((\alpha + \mu_1) - \frac{\lambda_1 b_1}{\mu_1}\Big) > \frac{3}{2}[(\lambda_1 I_2 + \beta_1 V_2) - \lambda_1 S_2]^2$

For C_3

$$L_1 > \frac{\mu_1 \mu_2 (\beta_1 S_2)^2}{(\mu_2 \mu_2^2 + \lambda_1 \mu_2 b_1 + \beta_1 \mu_1 b_2)(\lambda_2 I_2 + \mu_2)} \ \ also \ \ L_2 > \frac{3(\beta_1 b_1)^2}{2\mu_1\big(\mu_1(\alpha + \mu_1) - \lambda_1 b_1\big)(\lambda_2 I_2 + \mu_2)}$$

$\therefore C_3 = max\{L_1, L_3\}$

From the condition on $|D_4|$, *we get*

$$C_4 \frac{3}{4}(\eta m)^2 > \Big(\frac{\lambda_1 b_1}{\mu_1} - (\alpha + \mu_1)\Big)\Big(\theta + \eta I_2 - \gamma - \frac{\theta m}{c(E)}\Big)$$
$$\implies C_4 > \frac{4(\lambda_1 b_1 - \mu_1(\alpha + \mu_1))\big(c(E)\theta + \eta c(E) I_2 - \gamma c(E) - \theta m\big)}{3\mu_1 c(E)(\eta m)^2}$$

Therefore the conditions required for the matrix to be a positive definite are;

1. $(\mu_1 + \lambda_1 b_1 \mu_1 + \beta_1 b_2 \mu_2)\Big((\alpha + \mu_1) - \frac{\lambda_1 b_1}{\mu_1}\Big) > \frac{3}{2}[(\lambda_1 I_2 + \beta_1 V_2) - \lambda_1 S_2]^2$
2. $(\alpha + \mu_1) > \lambda_1 b_1$
3. $c(E)\theta + \eta c(E) I_2 > \gamma c(E) + \theta m$

If these conditions hold then $\frac{dV_3}{dt}$ *is a negative definite and hence using Lyapunov's stability method we can show that the Endemic equilibrium point* E^2 *is globally asymptotically stable.*

4.5 Conclusion

Here we have seen the formulation of the model by forming two models, one each for the host and vector population separately and then combining them. Further the equilibrium points are found, namely Disease free equilibrium, Pathogen free equilibrium and Endemic equilibrium. The reproduction number R_0 is also estimated using the Next generation matrix method about the disease free equilibrium point. On analysing the stability one can see that the disease free equilibrium and pathogen free equilibrium are locally asymptotically stable if $R_0 < 1$. The endemic equilibrium is locally asymptotically stable if $R_0 > 1$. Lastly one can see that the equilibrium points are globally asymptotically stable only if certain conditions hold.

Chapter 5

SIQR MODEL FOR COVID-19

5.1 Introduction

In December of 2019 the first known case of **COVID-19** was reported in the city of Wuhan, China. Transmitted via the respiratory droplets of an infected individual through cough or sneeze, it is an infectious disease caused by SARS-CoV-2 virus. Several measures were taken to control the spread of this disease one of which was quarantining the infected population. SIQ, SIQS, SIQR are some of the models with quarantine compartments. The effects of quarantine on transmission of infectious diseases were described by Z. Feng et al[5][19][20]. Hethcote et al[11] formed and studied different models with different incidence rates having a quarantine compartment. Here we shall be looking at a SIQR model with a quarantine compartment which is subdivided into quarantine from susceptible and quarantine from infected population, having a saturated incidence rate.

Incidence rate is the rate at which new cases of a particular disease occur in a specified population. Saturated incidence rate is when the occurrence of new cases reach a

maximum(saturation) level as the susceptible goes on increasing. This rate is introduced referring to Anderson et al[15] and Gao et al[6], which indicates that a large proportion of the population has already been infected and further any control or preventive measures might not give required results.

5.2 Formulation of Mathematical model

The total population denoted by N comprises of four compartments namely; the susceptible compartment S, the infected compartment I and the quarantined compartment Q which is further divided into quarantine from susceptible Q_S and quarantine from infected Q_I.

$$\frac{dS}{dt} = b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S \tag{5.1}$$

$$\frac{dI}{dt} = \frac{\lambda SI}{1+\alpha S} - (d_n + d_d + q_I + r_I)I \tag{5.2}$$

$$\frac{dQ_S}{dt} = q_S S - d_n Q_S \tag{5.3}$$

$$\frac{dQ_I}{dt} = q_I I - (d_n + d_d + r_q)Q_I \tag{5.4}$$

$$\frac{dR}{dt} = r_I I + r_q Q_I - d_n R \tag{5.5}$$

where, $S(0) \geq 0$, $I(0) \geq 0$, $Q_S(0) \geq 0$, $Q_S(0) \geq 0$ and $R(0) \geq 0$

$b =$ recruitment constant

$\lambda =$ transmission rate of susceptible to infected individuals

$\alpha =$ positive prohibition constant taken by the susceptibles

$d_n =$ natural death rate

$d_d =$ disease related death rate

$q_S =$ quarantine rate of susceptible

$q_I =$ quarantine rate of infectives

$r_I =$ recovery rate of infective

$r_q =$ recovery rate of quarantine infective

The incidence rate $\frac{\lambda SI}{1+\alpha S}$ tends to $\frac{\lambda I}{\alpha}$ as $S \to \infty$ i.e. incidence rate coverges to a saturated level.

The measures which should be taken by the susceptible to control the spread are given by α.

5.3 Positivity and Boundedness of the formulated model

Lemma: The set $\Omega = \{(S,I,Q_S,Q_I,R) \in \mathbb{R}^5_+ : 0 \leq S+I+Q_S+Q_I+R \leq \frac{b}{d_n}\}$ is positively invariant region of the formulated model and all solution of model which starts in Ω remains in Ω for all $t \geq 0$.

Proof. We know that the total population is the sum of all the compartments i.e.

$$N(t) = S(t) + I(t) + Q_S(t) + Q_I(t) + R(t)$$

By adding the equations (5.1)-(5.5), we get

$$\begin{aligned}\frac{dN(t)}{dt} &= b - d_n(S + I + Q_S + Q_I + R) - d_d(q_I + I) \\ &= b - d_n N(t) - d_d(q_I + I)\end{aligned}$$

If the disease does not exist then,

$$\begin{aligned}\frac{dN(t)}{dt} &= b - d_n N(t) \\ \int \frac{dN(t)}{b - d_n N(t)} &= \int dt \\ \frac{\log\left(b - d_n N(t)\right)}{-d_n} &= t \\ b - d_n N(t) &= e^{-d_n t}\end{aligned}$$

as $t \to \infty$,

$$\begin{aligned}b - d_n N(t) &= 0 \\ N(t) &= \frac{b}{d_n}\end{aligned}$$

This shows that $N(t)$ tends to the carrying capacity $\frac{b}{d_n}$ as $t \to \infty$.

This proves that the solution of the system exist and remains in Ω.

Now, by initial conditions it is observed that,

$$\left[\frac{dS}{dt}\right]_{S=0} = b > 0$$
$$\left[\frac{dI}{dt}\right]_{I=0} > 0 \; for \; I(0) \geq 0$$
$$\left[\frac{dQ_S}{dt}\right]_{Q_S=0} = q_s S \geq 0 \; for \; S(0) \geq 0$$
$$\left[\frac{dQ_I}{dt}\right]_{Q_I=0} = q_I I \geq 0 \; for \; I(0) \geq 0$$
$$\left[\frac{dR}{dt}\right]_{R=0} = r_I I + r_q Q_I \geq 0$$

This shows that model is mathematically and epidemiologically well posed.

5.4 Existence of Equilibrium points

The recovered population has been assumed to attain a permanent immunity to the disease and further does not play an active role in the spread of the disease. Hence the model is reduced to;

$$\frac{dS}{dt} = b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S \tag{5.6}$$
$$\frac{dI}{dt} = \frac{\lambda SI}{1+\alpha S} - (d_n + d_d + q_I + r_I)I \tag{5.7}$$
$$\frac{dQ_S}{dt} = q_S S - d_n Q_S \tag{5.8}$$
$$\frac{dQ_I}{dt} = q_I I - (d_n + d_d + r_q)Q_I \tag{5.9}$$

which is used to study the stability of the model.

5.4.1 Disease-Free Equilibrium

1. **Disease-Free Equilibrium:** $E_0 = (S^0, I^0, Q_S^0, Q_I^0) = (S^0, 0, Q_S^0, 0)$

We impose the equilibrium condition, $\frac{dS}{dt} = 0$

$$b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S = 0$$
$$b = (d_n + q_S)S$$
$$S^0 = \frac{b}{(d_n + q_S)}$$

Imposing $\frac{dQ_S}{dt} = 0$

$$q_S S^0 - d_n Q_S^0 = 0$$
$$\frac{q_S b}{(d_n + q_S)} = d_n Q_S^0$$
$$Q_S^0 = \frac{q_S b}{d_n(d_n + q_S)}$$

$\therefore E_0 = (S^0, I^0, Q_S^0, Q_I^0) = (S^0, 0, Q_S^0, 0) = \left(\frac{b}{(d_n+q_S)}, 0, \frac{q_S b}{d_n(d_n+q_S)}, 0\right)$

5.4.2 Basic Reproduction Number

We find the reproduction number R_0 using the Next Generation matrix method i.e. by finding the spectral radius of FV^{-1}

Let $X = (I, Q_S, Q_I, S)^T$, hence the system $(5.6) - (5.9)$ becomes,

$$\frac{dX}{dt} = F(X) - V(X)$$

$$F(X) = \begin{bmatrix} \frac{\lambda SI}{1+\alpha S} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad V(X) = \begin{bmatrix} (d_n + d_d + q_I + r_I)I \\ -q_S S + d_n Q_S \\ -q_I I + (d_n + d_d + r_q)Q_I \\ -b + \frac{\lambda SI}{1+\alpha S} + (d_n + q_S)S \end{bmatrix}$$

Therefore the Jacobian matrix of $F(X)$ and $V(X)$ at the Disease free equilibrium point are;

$$F_* = \begin{bmatrix} 0 & \frac{\lambda b}{d_n + q_S + \alpha b} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V_* = \begin{bmatrix} 0 & d_n + d_d + q_I + r_I & 0 & 0 \\ -q_S & 0 & d_n & 0 \\ 0 & q_I & 0 & d_n + d_d + r_q \\ d_n + q_S & \frac{\lambda b}{d_n + q_S + \alpha b} & 0 & 0 \end{bmatrix}$$

To find $V_*^{-1} = \frac{1}{|V_*|} adj(V_*)$

$$|V_*| = (d_n + d_d + q_I + r_I) \begin{vmatrix} -q_S & d_n & 0 \\ 0 & 0 & d_n + d_d + r_q \\ d_n + q_S & 0 & 0 \end{vmatrix}$$

$$= (d_n + d_d + q_I + r_I)[-q_S(0) + d_n(d_n + q_S)(d_n + d_d + r_q)]$$

$$= d_n(d_n + q_S)(d_n + d_d + r_q)(d_n + d_d + q_I + r_I)$$

To find the $adj(V_*)$, we first find the co-factor of V_*,

$C_{11} = d_n \frac{\lambda b}{d_n + q_S + \alpha b}(d_n + d_d + r_q)$

$C_{12} = d_n(d_n+q_S)(d_n+d_d+r_q)$

$C_{13} = -q_S(d_n+d_d+r_q)\frac{\lambda b}{d_n+q_S+\alpha b}$

$C_{14} = d_n q_S(d_n+q_S)\ C_{23} = (d_n+q_S)(d_n+d_d+r_q)(d_n+d_d+q_I+r_I)$

$C_{34} = d_n(d_n+q_S)(d_n+d_d+q_I+r_I)$

$C_{41} = -d_n(d_n+d_d+r_q)(d_n+d_d+q_I+r_I)$

$C_{43} = q_S(d_n+d_d+r_q)(d_n+d_d+q_I+r_I)$

Rest of the entrees are 0.

$\therefore V_*^{-1}$ is given by the matrix

$$\begin{bmatrix} \frac{\lambda b}{(d_n+q_S+\alpha b)(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & 0 & \frac{-1}{d_n+d_q} \\ \frac{1}{(d_n+d_d+q_I+r_I)} & 0 & 0 & 0 \\ \frac{-q_S\lambda b}{d_n(d_n+q_S+\alpha b)(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & 0 & \frac{q_S}{d_n(d_n+q_S)} \\ \frac{q_I}{(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & \frac{1}{(d_n+d_d+r_q)} & 0 \end{bmatrix}$$

$$F_*V_*^{-1} = \begin{bmatrix} 0 & \frac{\lambda b}{d_n+q_S+\alpha b} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\lambda b}{(d_n+q_S+\alpha b)(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & 0 & \frac{-1}{d_n+d_q} \\ \frac{1}{(d_n+d_d+q_I+r_I)} & 0 & 0 & 0 \\ \frac{-q_S\lambda b}{d_n(d_n+q_S+\alpha b)(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & 0 & \frac{q_S}{d_n(d_n+q_S)} \\ \frac{q_I}{(d_n+q_S)(d_n+d_d+q_I+r_I)} & 0 & \frac{1}{(d_n+d_d+r_q)} & 0 \end{bmatrix}$$

$$F_*V_*^{-1} = \begin{bmatrix} \frac{\lambda b}{(d_n+q_S+\alpha b)(d_n+d_d+q_I+r_I)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_0 = \frac{\lambda b}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)} \tag{5.10}$$

is the maximum eigen value of the above matrix.

$$\begin{aligned} R_0 &= \frac{\lambda b}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)} \\ R_0 - 1 &= \frac{\lambda b}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)} - 1 \\ R_0 - 1 &= \frac{\lambda b - (d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)} \end{aligned}$$

$$(R_0 - 1)(d_n + q_S + \alpha b) = \frac{\lambda b - (d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)}{(d_n + d_d + q_I + r_I)} \tag{5.11}$$

5.4.3 Endemic Equilibrium

2. **Endemic Equilibrium:** $E_1 = (S^1, I^1, Q_S^1, Q_I^1)$

We impose the equilibrium condition, $\frac{dI}{dt} = 0$

$$\frac{\lambda SI}{1 + \alpha S} - (d_n + d_d + q_I + r_I)I = 0 \tag{5.12}$$

$$\begin{aligned}
\lambda SI - (d_n + d_d + q_I + r_I)I(1+\alpha S) &= 0 \\
I\Big(\lambda S - (d_n + d_d + q_I + r_I)(1+\alpha S)\Big) &= 0 \\
\implies (\lambda S - (d_n + d_d + q_I + r_I)(1+\alpha S)) &= 0 \\
\lambda S - (d_n + d_d + q_I + r_I) - (d_n + d_d + q_I + r_I)\alpha S &= 0 \\
S(\lambda - \alpha(d_n + d_d + q_I + r_I)) &= (d_n + d_d + q_I + r_I) \\
\implies S^1 &= \frac{(d_n + d_d + q_I + r_I)}{\lambda - \alpha(d_n + d_d + q_I + r_I)}
\end{aligned}$$

Imposing $\frac{dS}{dt} = 0$, we get

$$b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S = 0 \tag{5.13}$$

Using (5.12) & (5.13) we get,

$$\begin{aligned}
b - (d_n + q_S)S^1 - (d_n + d_d + q_I + r_I)I^1 &= 0 \\
b - (d_n + q_S)S^1 &= (d_n + d_d + q_I + r_I)I^1
\end{aligned}$$

Substituting the value of S^1,

$$b - (d_n + q_S)\frac{(d_n + d_d + q_I + r_I)}{\lambda - \alpha(d_n + d_d + q_I + r_I)} = (d_n + d_d + q_I + r_I)I^1$$

$$(d_n + d_d + q_I + r_I)I^1 = \frac{b\lambda - b\alpha(d_n + d_d + q_I + r_I) - (d_n + q_S)(d_n + d_d + q_I + r_I)}{\lambda - \alpha(d_n + d_d + q_I + r_I)}$$

$$(d_n + d_d + q_I + r_I)I^1 = \frac{\lambda b - (d_n + d_d + q_I + r_I)(d_n + q_S + \alpha b)}{\lambda - \alpha(d_n + d_d + q_I + r_I)}$$

$$I^1 = \frac{\lambda b - (d_n + d_d + q_I + r_I)(d_n + q_S + \alpha b)}{(d_n + d_d + q_I + r_I)(\lambda - \alpha(d_n + d_d + q_I + r_I))}$$

$$I^1 = \frac{(R_0 - 1)(d_n + q_S + \alpha b)}{\lambda - \alpha(d_n + d_d + q_I + r_I)} \qquad ,using \quad (5.11)$$

Imposing $\frac{dQ_S}{dt} = 0$, we get

$$q_S S - d_n Q_S = 0$$
$$\implies Q_S^1 = \frac{q_S S}{d_n}$$

Imposing $\frac{dQ_I}{dt} = 0$, we get

$$q_I I - (d_n + d_d + r_q) Q_I = 0$$
$$\implies Q_I^1 = \frac{q_I I}{(d_n + d_d + r_q)}$$

Therefore the Endemic equilibrium point is,

$$E_1 = (S^1, I^1, Q_S^1, Q_I^1) = \left(\frac{(d_n+d_d+q_I+r_I)}{\lambda-\alpha(d_n+d_d+q_I+r_I)}, \frac{(R_0-1)(d_n+q_S+\alpha b)}{\lambda-\alpha(d_n+d_d+q_I+r_I)}, \frac{q_S S}{d_n}, \frac{q_I I}{(d_n+d_d+r_q)} \right)$$

5.5 Stability Analysis

5.5.1 Local Stability of Diseases-Free Equilibrium Point

Theorem 5.5.1.1. *The disease free equilibrium point* $E_0 = (S^0, I^0, Q_S^0, Q_I^0)$ *is locally asymptotically stable if* $R_0 < 1$ *otherwise unstable.*

Proof.

To show the local stability of the system (5.6)-(5.9), at the disease free equilibrium point we consider

$$
\begin{aligned}
F_1 &= b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S \\
F_2 &= \frac{\lambda SI}{1+\alpha S} - (d_n + d_d + q_I + r_I)I \\
F_3 &= q_S S - d_n Q_S \\
F_4 &= q_I I - (d_n + d_d + r_q)Q_I
\end{aligned}
$$

The Jacobian of the above system of equations is given by

$$
J = \begin{bmatrix}
\frac{[-(1+\alpha S)\lambda I + \alpha\lambda SI]}{(1+\alpha S)^2} - (d_n + q_s) & \frac{-\lambda S}{(1+\alpha S)} & 0 & 0 \\
\frac{[(1+\alpha S)\lambda I - \alpha\lambda SI]}{(1+\alpha S)^2} & \frac{\lambda S}{(1+\alpha S)} - (d_n + d_d + q_I + r_I) & 0 & 0 \\
q_S & 0 & -d_n & 0 \\
0 & q_I & 0 & -(d_n + d_d + r_q)
\end{bmatrix}
$$

The Jacobian matrix at the disease free equilibrium point is given by;

$$J(E_0) = \begin{bmatrix} -(d_n+q_s) & \frac{-\lambda b}{(d_n+q_S+\alpha b)} & 0 & 0 \\ 0 & \frac{-\lambda b}{(d_n+q_S+\alpha b)} - (d_n+d_d+q_I+r_I) & 0 & 0 \\ q_S & 0 & -d_n & 0 \\ 0 & q_I & 0 & -(d_n+d_d+r_q) \end{bmatrix}$$

Now, $|J(E_0) - \lambda_1 I| = 0$

$$0 = \begin{bmatrix} -(d_n+q_s)-\lambda_1 & \frac{-\lambda b}{(d_n+q_S+\alpha b)} & 0 & 0 \\ 0 & \frac{-\lambda b}{(d_n+q_S+\alpha b)} - (d_n+d_d+q_I+r_I)-\lambda_1 & 0 & 0 \\ r_I & 0 & -d_n-\lambda_1 & 0 \\ 0 & q_I & 0 & -(d_n+d_d+r_q)-\lambda_1 \end{bmatrix}$$

$$0 = (-(d_n+q_s)-\lambda_1) \begin{vmatrix} \frac{-\lambda b}{(d_n+q_S+\alpha b)} - (d_n+d_d+q_I+r_I)-\lambda_1 & 0 & 0 \\ 0 & -d_n-\lambda_1 & 0 \\ q_I & 0 & -(d_n+d_d+r_q)-\lambda_1 \end{vmatrix}$$

$$0 = (-(d_n+q_s)-\lambda_1)\Big(\frac{-\lambda b}{(d_n+q_S+\alpha b)} - (d_n+d_d+q_I+r_I)-\lambda_1\Big)(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)$$

This implies that;

$\lambda_1 = -(d_n+q_s)$

$\lambda_2 = -d_n$

$\lambda_3 = -(d_n+d_d+r_q)$

$\lambda_4 = \frac{\lambda b}{(d_n+q_S+\alpha b)} + (d_n + d_d + q_I + r_I)$

But,

$$\lambda_4 = \frac{\lambda b}{(d_n + q_S + \alpha b)} + (d_n + d_d + q_I + r_I)$$
$$\lambda_4 = \frac{\lambda b - (d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)}{(d_n + q_S + \alpha b)}$$
$$\lambda_4 = (R_0 - 1)(d_n + d_d + q_I + r_I)$$

λ_4 is negative only when $R_0 < 1$

All eigen values are negative if $R_0 < 1$

Hence $E_0 = (S^0, I^0, Q_S^0, Q_I^0)$ is locally asymptotically stable if $R_0 < 1$ otherwise unstable.

5.5.2 Local Stability of Endemic Equilibrium Point

Theorem 5.5.2.1. *The endemic equilibrium $E_1 = (S^1, I^1, Q_S^1, Q_I^1)$is locally asymptotically stable when $\frac{\lambda S}{(1+\alpha S)} < (d_n + d_d + q_S + r_q)$.*

Proof.

To show the local stability of the system (5.6)-(5.9), at the endemic equilibrium point we consider

$$F_1 = b - \frac{\lambda SI}{1+\alpha S} - (d_n + q_S)S$$
$$F_2 = \frac{\lambda SI}{1+\alpha S} - (d_n + d_d + q_I + r_I)I$$
$$F_3 = q_S S - d_n Q_S$$
$$F_4 = q_I I - (d_n + d_d + r_q)Q_I$$

The Jacobian of the above system of equations is given by

$$J=\begin{bmatrix} \frac{[-(1+\alpha S)\lambda I+\alpha\lambda SI]}{(1+\alpha S)^2}-(d_n+q_s) & \frac{-\lambda S}{(1+\alpha S)} & 0 & 0 \\ \frac{[(1+\alpha S)\lambda I-\alpha\lambda SI]}{(1+\alpha S)^2} & \frac{\lambda S}{(1+\alpha S)}-(d_n+d_d+q_I+r_I) & 0 & 0 \\ q_S & 0 & -d_n & 0 \\ 0 & q_I & 0 & -(d_n+d_d+r_q) \end{bmatrix}$$

The Jacobian matrix at the endemic equilibrium point is given by;

$$J(E_1)=\begin{bmatrix} \frac{-\lambda I}{(1+\alpha S)^2}-(d_n+q_s) & \frac{-\lambda S}{(1+\alpha S)} & 0 & 0 \\ \frac{\lambda I}{(1+\alpha S)^2} & \frac{\lambda S}{(1+\alpha S)}-(d_n+d_d+q_I+r_I) & 0 & 0 \\ q_S & 0 & -d_n & 0 \\ 0 & q_I & 0 & -(d_n+d_d+r_q) \end{bmatrix}$$

We know that, $|J(E_1)-\lambda_1 I|=0$

$$0=\begin{bmatrix} \frac{-\lambda I}{(1+\alpha S)^2}-(d_n+q_s)-\lambda_1 & \frac{-\lambda S}{(1+\alpha S)} & 0 & 0 \\ \frac{\lambda I}{(1+\alpha S)^2} & \frac{\lambda S}{(1+\alpha S)}-(d_n+d_d+q_I+r_I)-\lambda_1 & 0 & 0 \\ q_S & 0 & -d_n-\lambda_1 & 0 \\ 0 & q_I & 0 & -(d_n+d_d+r_q)-\lambda_1 \end{bmatrix}$$

$$0=\left(\frac{-\lambda I}{(1+\alpha S)^2}-(d_n+q_s)-\lambda_1\right)\left[\left(\frac{\lambda S}{(1+\alpha S)}-(d_n+d_d+q_I+r_I)-\lambda_1\right)(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\right]+\frac{\lambda S}{(1+\alpha S)}\left[\frac{\lambda I}{(1+\alpha S)}(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\right]$$

$$0=(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\left[\left(\frac{-\lambda I}{(1+\alpha S)^2}-(d_n+q_s)-\lambda_1\right)\left(\frac{\lambda S}{(1+\alpha S)}-(d_n+d_d+q_I+r_I)-\lambda_1\right)+\frac{\lambda S}{(1+\alpha S)}\frac{\lambda I}{(1+\alpha S)}\right]$$

$$0=(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\left[-\frac{\lambda I}{(1+\alpha S)^2}\frac{\lambda S}{(1+\alpha S)}+(d_n+d_d+q_I+r_I)\frac{\lambda I}{(1+\alpha S)^2}+\lambda_1\frac{\lambda I}{(1+\alpha S)^2}-(d_n+q_s)\frac{\lambda S}{(1+\alpha S)}+(d_n+q_s)(d_n+d_d+q_I+r_I)+(d_n+q_s)\lambda_1-\frac{\lambda S}{(1+\alpha S)}\lambda_1+(d_n+d_d+q_I+r_I)\lambda_1+\lambda_1^2+\frac{\lambda^2 SI}{(1+\alpha S)^2}\right]$$

$$0=(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\left[\lambda_1^2+\lambda_1\left(\frac{\lambda I}{(1+\alpha S)^2}+(d_n+q_s)-\frac{\lambda S}{(1+\alpha S)}+(d_n+d_d+q_I+r_I)\right)-\frac{\lambda^2 SI}{(1+\alpha S)^3}+(d_n+d_d+q_I+r_I)\frac{\lambda I}{(1+\alpha S)^2}-(d_n+q_s)\frac{\lambda S}{(1+\alpha S)}+(d_n+q_s)(d_n+d_d+q_I+r_I)\right]$$

$$0=(-d_n-\lambda_1)(-(d_n+d_d+r_q)-\lambda_1)\left[\lambda_1^2+\lambda_1\left(2d_n+d_d+q_s+q_I+r_I+\frac{\lambda I}{(1+\alpha S)^2}-\frac{\lambda S}{(1+\alpha S)}\right)+(d_n+q_s)\left((d_n+d_d+q_I+r_I)-\frac{\lambda S}{(1+\alpha S)}\right)+\frac{\lambda I}{(1+\alpha S)^2}\left((d_n+d_d+q_I+r_I)-\frac{\lambda S}{(1+\alpha S)}\right)\right]$$

From the above equation we have,

$\lambda_1=-d_n$

$\lambda_2=-(d_n+d_d+r_q)$

The corresponding eigen values are negative.

We also have the quadratic equation,

$$0=\lambda_1^2+\lambda_1\left(2d_n+d_d+q_s+q_I+r_I+\frac{\lambda I}{(1+\alpha S)^2}-\frac{\lambda S}{(1+\alpha S)}\right)+(d_n+q_s)\left((d_n+d_d+q_I+r_I)-\frac{\lambda S}{(1+\alpha S)}\right)+\frac{\lambda I}{(1+\alpha S)^2}\left((d_n+d_d+q_I+r_I)-\frac{\lambda S}{(1+\alpha S)}\right)$$

The coefficients of this equation are;

$C_1=1$

$C_2=2d_n+d_d+q_s+q_I+r_I+\frac{\lambda I}{(1+\alpha S)^2}-\frac{\lambda S}{(1+\alpha S)}$

$C_3=\left((d_n+q_s)+\frac{\lambda I}{(1+\alpha S)^2}\right)\left((d_n+d_d+q_I+r_I)-\frac{\lambda S}{(1+\alpha S)}\right)$

The above coefficients C_2 and C_3 are positive when $(d_n+d_d+q_I+r_I)>\frac{\lambda S}{(1+\alpha S)}$

Using the Routh Hurwitz criterion which says that a second degree polynomial with all positive coefficients will have negative roots, one can show that the endemic equilibrium is locally asymptotically stable under the condition $(d_n+d_d+q_I+r_I)>\frac{\lambda S}{(1+\alpha S)}$, otherwise unstable.

5.5.3 Global Stability of Diseases-Free Equilibrium Point

Theorem 5.5.3.1. *If $R_0<1$, then the disease free equilibrium is globally asymptotically stable.*

Proof.

We consider the Lyapunov function to be $L=I$ where $I(t)\in\mathbb{R}_+\cup\{0\}\quad\forall\, t\geq 0$

$\implies \frac{dL}{dt}=\frac{dI}{dt}$ *Therefore we have,*

$$\begin{aligned}\frac{dL}{dt} &= \frac{\lambda_1 SI}{1+\alpha S} - (d_n + d_d + q_I + r_I)I \\ &= \left(\frac{\lambda_1 S_0}{1+\alpha S_0} - (d_n + d_d + q_I + r_I)\right)I \\ &= \left(\frac{\frac{\lambda b}{d_n+q_S}}{1+\frac{\alpha b}{d_n+q_S}} - (d_n + d_d + q_I + r_I)\right)I \\ &= \left(\frac{\lambda b}{d_n + q_S + \alpha b} - (d_n + d_d + q_I + r_I)\right)I\end{aligned}$$

$$\begin{aligned}\frac{dL}{dt} &= I\left[(d_n + d_d + q_I + r_I)\left(\frac{\lambda b}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)} - 1\right)\right] \\ \frac{dL}{dt} &= (R_0 - 1)(d_n + d_d + q_I + r_I)I \quad since \quad R_0 = \frac{\lambda b}{(d_n + q_S + \alpha b)(d_n + d_d + q_I + r_I)}\end{aligned}$$

$\implies \frac{dL}{dt} = 0 \; iff \; I = 0 \; and \; \frac{dL}{dt} < 0 \; if \; R_0 < 1$

Hence E_0 is the largest invariant set in $\{(S, I, Q_S, Q_I) : L = 0\}$ as with $\frac{dL}{dt} = 0$ we have shown that the function is constant along the trajectories whose points are in Ω

So by Lyapunov-Lasalle invariance principle disease free equilibrium E_0 is globally asymptotically stable.

5.6 Conclusion

Here the model has been formulated with saturated incidence rate. Further the equilibrium points are found, namely Disease free equilibrium and Endemic equilibrium. The

reproduction number R_0 is also estimated using the Next generation matrix method. On analysing the stability one can see that the disease free equilibrium is locally asymptotically stable if $R_0 < 1$ and the endemic equilibrium is locally asymptotically stable if $R_0 > 1$. Lastly one can see that the disease free equilibrium is globally asymptotically stable if $R_0 < 1$.

Chapter 6

CONCLUSION

In this dissertation we have seen how a free living pathogen in the environment affects the spread of the disease. The pathogen population is assumed to be directly proportional to the infected population. Using numerical simulation one can see that how the modelled system behaves when the parameters are given a certain value. When the rate of shedding of the pathogen η is increased the pathogen population in the environment increases. Also as the biting rate of the vector population β_1 increases the infected population increases. It can be noted that the pathogen population also increases with increase in the carrying capacity of the environment and hence measures have to be taken to make the environment less feasible for the growth and survival of the pathogen population. Similarly controlling environmental factors like household discharge and human waste can reduce the cumulative density Q_0 which in turn reduces the pathogen and vector population. All these variations help us better understand how the system will behave on the implementation of preventive and control measures and which one of them is the optimal solution for our situation.

This dissertation also contains a study on mathematical modelling of an SIQR model with saturated incidence rate wherein we have seen how considering the incidence rate to reach a saturated level affects the formulation of the model; consisting a system of non-linear differential equations. Here the quarantined compartment is divided into quarantine from susceptibles and quarantine from the infected population. This is done so that the population does not spread the disease actively thus helping to reduce the average infectious period. Using numerical simulation one can see that how the modelled system behaves when the parameters are given a certain value. Here with the increase in the prohibition constant α the infected population decreases. This indicates that the spread of this infection can be reduced by implementing preventive measures like wearing of mask, social distancing, maintaining proper sanitisation and hygiene and also by quarantining the populations separately. This variation helps us better understand how the system will behave on the implementation of preventive and control measures and which one of them is the optimal solution for our situation.

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Appendices

APPENDIX

Chapter 4

For the disease free equilibrium point $E_0 = (S^0, 0, 0, P^0, E^0)$ we have,

$$\begin{aligned}
\frac{dS}{dt} &= b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S \\
\frac{dI}{dt} &= \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I \\
\frac{dV}{dt} &= \lambda_2 \left(\frac{b_2}{\mu_2} - V \right) I - \mu_2 V \\
\frac{dP}{dt} &= \eta IP + \theta P \left(1 - \frac{P}{c(E)} \right) - \gamma P \\
\frac{dE}{dt} &= Q_0 - \theta_1 E + \theta_2 N
\end{aligned}$$

We transform the system using

$S = S^0 + x_1$, $I = I^0 + x_2$, $V = V^0 + x_3$, $P = P^0 + x_4$, $E = E^0 + x_5$

We have,

$$\frac{dS^0}{dt} + \frac{dx_1}{dt} = b_1 - \lambda_1 (S^0 + x_1)(I^0 + x_2) - \beta_1 (S^0 + x_1)(V^0 + x_3) - \mu_1 (S^0 + x_1)$$

$$\frac{dS^0}{dt} + \frac{dx_1}{dt} = b_1 - \lambda_1 S^0 I^0 - \lambda_1 S^0 x_2 - \lambda_1 I^0 x_1 - \lambda_1 x_1 x_2 - \beta_1 S^0 V^0 - \beta_1 S^0 x_3 - \beta_1 V^0 x_1 - \beta_1 x_1 x_3 - \mu_1 S^0 - \mu_1 x_1$$

$$\frac{dS^0}{dt} + \frac{dx_1}{dt} = [-(\lambda_1 x_2 + \beta_1 x_3) S^0 - (\lambda_1 x_2 + \beta_1 x_3) x_1 - \mu_1 x_1] + b_1 - \lambda_1 S^0 I^0 - \beta_1 S^0 V^0 - \mu_1 S^0 - \lambda_1 I^0 x_1 - \beta_1 V^0 x_1$$

$$\frac{dx_1}{dt} = -(\lambda_1 x_2 + \beta_1 x_3) S^0 - (\lambda_1 x_2 + \beta_1 x_3) x_1 - \mu_1 x_1$$

$$\frac{dI^0}{dt}+\frac{dx_2}{dt}=\lambda_1(S^0+x_1)(I^0+x_2)+\beta_1(S^0+x_1)(V^0+x_3)-(\alpha+\mu_1)(I^0+x_2)$$
$$\frac{dx_2}{dt}=\lambda_1(S^0+x_1)(x_2)+\beta_1(S^0+x_1)(x_3)-(\alpha+\mu_1)(x_2)$$
$$\frac{dx_2}{dt}=\lambda_1 S^0x_2+\lambda_1x_1x_2+\beta_1S^0x_3+\beta_1x_1x_3-(\alpha+\mu_1)(x_2)$$
$$\frac{dx_2}{dt}=(\lambda_1x_2+\beta_1x_3)S^0+(\lambda_1x_2+\beta_1x_3)x_1-(\alpha+\mu_1)(x_2)$$

$$\frac{dV^0}{dt}+\frac{dx_3}{dt}=\frac{\lambda_2b_2}{\mu_2}(I^0+x_2)-\lambda_2(V^0+x_3)(I^0+x_2)-\mu_2(V^0+x_3)$$
$$\frac{dx_3}{dt}=\frac{\lambda_2b_2}{\mu_2}I^0+\frac{\lambda_2b_2}{\mu_2}x_2-\lambda_2V^0I^0--\lambda_2V^0x_2-\lambda_2I^0x_3-\lambda_2x_2x_3-\mu_2V^0-\mu_2x_3$$
$$\frac{dx_3}{dt}=\frac{\lambda_2b_2}{\mu_2}x_2-\lambda_2I^0x_3-\lambda_2x_2x_3-\mu_2x_3$$
$$\frac{dx_3}{dt}=\frac{\lambda_2b_2}{\mu_2}x_2-\lambda_2x_2x_3-x_3(\lambda_2I^0+\mu_2)$$

$$\frac{dP^0}{dt}+\frac{dx_4}{dt}=\eta(I^0+x_2)(P^0+x_4)+\theta(P^0+z_4)-\theta\frac{(P^0+x_4)^2}{c(E)}-\gamma(P^0+x_4)$$
$$\frac{dP^0}{dt}+\frac{dx_4}{dt}=\eta(P^0x_2+x_2x_4)+\theta P^0+\theta x_4+\frac{\theta {P^0}^2}{c(E)}-\frac{2\theta P^0}{c(E)}x_4-\frac{\theta x_4^2}{c(E)}-\gamma P^0-\gamma x_4$$
$$\frac{dP^0}{dt}+\frac{dx_4}{dt}=\eta P^0x_2+\eta x_2x_4+\theta P^0+\theta x_4+\frac{\theta {P^0}^2}{c(E)}-\frac{2\theta_2}{c(E)}-\frac{\theta x_4^2}{c(E)}-\gamma P^0-\gamma x_4$$
$$\frac{dP^0}{dt}+\frac{dx_4}{dt}=\eta(P^0x_2+x_2x_4)+x_4\left(\theta-\frac{2\theta P^0}{c(E)}-\frac{\theta x_4}{c(E)}\right)+\theta P^0-\frac{\theta {P^0}^2}{c(E)}-\gamma P^0$$
$$\frac{dP^0}{dt}+\frac{dx_4}{dt}=\eta(P^0x_2+x_2x_4)+x_4\left(\theta\left(1-\frac{1}{c(E)}(2P^0+x_4)\right)-\gamma\right)+\theta P^0-\frac{\theta {P^0}^2}{c(E)}-\gamma P^0$$

$$\frac{dx_4}{dt} = \eta(P^0 x_2 + x_2 x_4) + x_4\left(\theta\left(1 - \frac{1}{c(E)}(2P^0 + x_4)\right) - \gamma\right)$$

$$\frac{dE^0}{dt} + \frac{dz_5}{dt} = Q_0 - \theta_1(E_2 + x_5) + \theta_2 N$$

$$\frac{dE^0}{dt} + \frac{dz_5}{dt} = Q_0 - \theta_1 E_2 - \theta_1 x_5 + \theta_2 N$$

$$\frac{dz_5}{dt} = -\theta_1 x_5$$

Hence we have the system;

$$\begin{aligned}
\frac{dx_1}{dt} &= -(\lambda_1 x_2 + \beta_1 x_3)S^0 - (\lambda_1 x_2 + \beta_1 x_3)x_1 - \mu_1 x_1 \\
\frac{dx_2}{dt} &= (\lambda_1 x_2 + \beta_1 x_3)S^0 + (\lambda_1 x_2 + \beta_1 x_3)x_1 - (\alpha + \mu_1)(x_2) \\
\frac{dx_3}{dt} &= \frac{\lambda_2 b_2}{\mu_2} x_2 - \lambda_2 x_2 x_3 - x_3(\lambda_2 I^0 + \mu_2) \\
\frac{dx_4}{dt} &= \eta(P^0 x_2 + x_2 x_4) + x_4\left(\theta\left(1 - \frac{1}{c(E)}(2P^0 + x_4)\right) - \gamma\right) \\
\frac{dx_5}{dt} &= -\theta_1 x_5
\end{aligned}$$

In the same manner system (4.8)-(4.12) can be obtained.