CROSSING NUMBER AND GRACEFULNESS OF GRAPHS

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, “Crossing Number and Gracefulness of Graphs” is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical and Applied Sciences, Goa University under the Supervision of Dr. Jessica Fernandes e Pereira and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University or its authorities will be not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Crossing Number and Gracefulness of Graphs" is a bonafide work carried out by Ms. Megan Darlene Mascarenhas under my supervision in partial fulfilment of the requirements for the award of the degree of Masters of Science in the Mathematics Discipline at the School of Physical and Applied Sciences, Goa University.

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PREFACE

In the present times Mathematics occupies an important place in curriculum. Crossing number and gracefulness of graphs are important fields for research in modern times. In this dissertation an attempt has been made to cover up some of the research topics related to crossing number, gracefulness and k-hypergraceful labeling of graphs. This dissertation has been written in a simple and lucid manner and is up-to-date in its contents. To illustrate theory some examples have been given.

It is hoped that this dissertation will be appreciated by teachers and students alike. While preparing the dissertation, material has been used from works of different authors, periodicals and journals and I'm extremely grateful to all such persons and their publishers.

All suggestions for improvement of the dissertation shall be thankfully accepted.

ACKNOWLEDGEMENT

First and foremost, I would like to express my sincere gratitude to my guide, Dr. Jessica Fernandes e Pereira, her guidance and encouragement helped me in the completion of my dissertation. She pushed me to think imaginatively and urged me to do this dissertation without hesitation. Her vast knowledge, extensive experience, and professional competence in Mathematics enabled me to successfully accomplish this dissertation. This endeavour would not have been possible without her help and supervision. She was always there to cheer me on, and that is what kept me going until the end.

ABSTRACT

The crossing numbers of the Cartesian products of given three graphs on five vertices with paths is determined. The new measure $m(G)$ determines how close G is to being graceful. Here $m(G)$ for a few families of nongraceful graphs is determined. Also the characterization of k-hypergraceful complete graphs K_p when $p - 4 \leq k \leq p - 1$ is determined. Lastly the cycle C_n is 3-hypergraceful if $n \equiv 1 (mod\ 4)$ and 2-hypergraceful if $n \equiv 2 (mod\ 4)$ is determined.

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1 INTRODUCTION

Let G be a simple graph with the vertex set V and the edge set E. A drawing is a mapping of a graph into a surface. The vertices go into distinct points, nodes. An edge and its incident vertices map into a homeomorphic image of the closed interval $[0, 1]$ with the relevant nodes as endpoints and the interior, an arc, containing no node. For graph theoretic terminology and notations refer to Chartrand and Lesniak [6], also West [21]. A *good drawing* is one in which no two arcs incident to a common node have a common point; and no two arcs have more than one point in common.

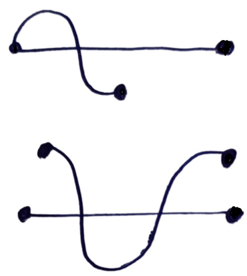


Figure 1: Examples of not a good drawing.

A common point of two arcs is a *crossing*. The *crossing number* $cr(G)$ of a graph G is the minimum number of crossings in any good drawing of G in the plane.

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has a vertex set

$V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} : u_i = u_h \text{ and } \{v_j, v_k\} \in E(G_2)$

or $\{u_i, u_h\} \in E(G_1)$ and $v_j = v_k\}$.

Let C_n be the cycle, P_n the path of length n and S_n the star $K_{1,n}$. The crossing numbers of the Cartesian products of all 4–vertex graphs with cycles are determined in [4] and [9] and with paths and stars in [11] and [12]. The precise values of the crossing numbers of some products $G \times P_n$ where G is 5–vertex graph is determined in [10].

Most of the graph labeling methods trace their origin to the one introduced by Rosa [17].

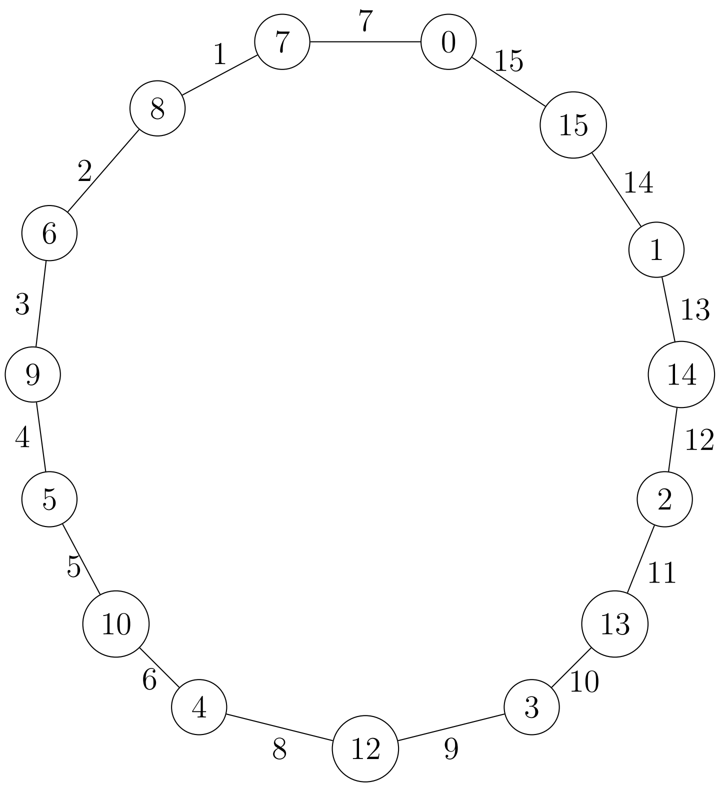


Figure 2: Graceful labelling of C_{15}

Let $G = (V, E)$ be a (p, q)-graph without isolated vertices. By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are dentoed by p and q respectively. An injection $f : V \rightarrow \{0, 1, ..., q\}$ is said to be graceful if the induced edge function g_f defined by $g_f(uv) = |f(u) - f(v)|$ is a bijection from E to $\{1, 2, ..., q\}$. Any graph which admits such a labeling is called a *graceful graph* and nongraceful otherwise (see, [5], [7], [8], [17]).

In [6] the *gracefulness* $grac(G)$ of a graph G with $V(G) = \{v_1, v_2, ..., v_p\}$ without isolates is defined to be the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, ..., k\}$ in such a way that distinct edges receive distinct labels. Obviously $grac(G) \geq q$ and $grac(G) = q$ if and only if G is graceful. Thus $grac(G)$ gives a measure of gracefulness of G.

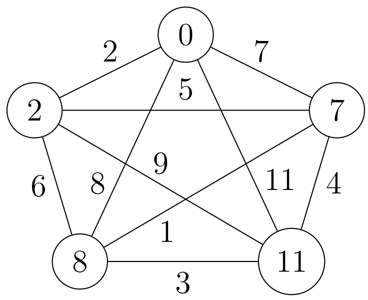


Figure 3: $grac(K_5) = 11$

A new measure of gracefulness of graphs and the same for some families of non-graceful graphs is determined in [14].

For standard terminology and notations in signed graphs refer [22], [23]. The notion of graceful labeling has been extended to signed graphs by Acharya and Singh [2], [3] and Singh [18].

A *signed graph* (or simply *sigraph*), denoted as S, is defined as a graph $G = (V, E)$ along with a function $s : E(G) \rightarrow \{+, -\}$ called its signing function. This function assigns a sign to each edge in the graph. The graph G itself, without considering the signs, is referred to as the underlying graph of the signed graph S. It comprises the vertices and the edges of S. The set of positive edges of S is denoted by $E^+(S)$, while the set of negative edges is denoted by $E^-(S)$. Together, these sets cover all edges of the signed graph, satisfying the condition that the union of positive and negative edges equals the edge set of S, i.e., $E^+(S) \cup E^-(S) = E(S)$.

If the number of positive edges is denoted as m and the number of negative edges as n, such that $m + n = q$, then the signed graph S is termed a (p, m, n)-sigraph. An all-positive sigraph S is one where all edges have a positive sign, meaning $E^+(S) = E(S)$. Similarly, an all-negative sigraph has all edges with a negative sign, indicated by $E^-(S) = E(S)$. A sigraph is considered homogeneous if it consists entirely of positive or negative edges. Similarly, if it contains both positive and negative edges, it is termed heterogeneous.

Consider a signed graph S with edge set $E(S)$ and a signing function $s(uv)$ that assigns a sign to each edge uv. Let $f : V(S) \rightarrow \{0, 1, \ldots, q = m + n\}$ be an injection, where q is the total number of edges in S (i.e., the sum of positive and negative edges).

The induced edge labeling g_f is defined as follows:

$g_f(uv) = s(uv)|f(u) - f(v)|$

for all edges $uv \in E(S)$. Here, $s(uv)$ denotes the sign of the edge uv. The function f is said to be a graceful labeling of the signed graph S if the induced edge labeling g_f satisfies two conditions:

$g_f(E^+(S)) = \{1, 2, \ldots, m\}$: The set of labels received by positive edges is a consecutive sequence starting from 1 up to the number of positive edges.

$g_f(E^-(S)) = \{-1, -2, \ldots, -n\}$: The set of labels received by negative edges is a consecutive sequence starting from -1 down to the negative number of edges. A signed graph that admits a graceful labeling is called a *graceful signed graph.* In other words, if there exists an injection f such that the induced edge labeling g_f satisfies the conditions mentioned above, then the signed graph S is graceful.

The *negation of a signed graph* S, denoted as $\eta(S)$, is obtained by changing the sign of every edge to its opposite. In other words, if an edge in S has a positive sign (+), it will have a negative sign ($-$) in $\eta(S)$, and vice versa. If a signed graph S is graceful with a graceful labeling f, then the negation of the signed graph S is also graceful under the same f.

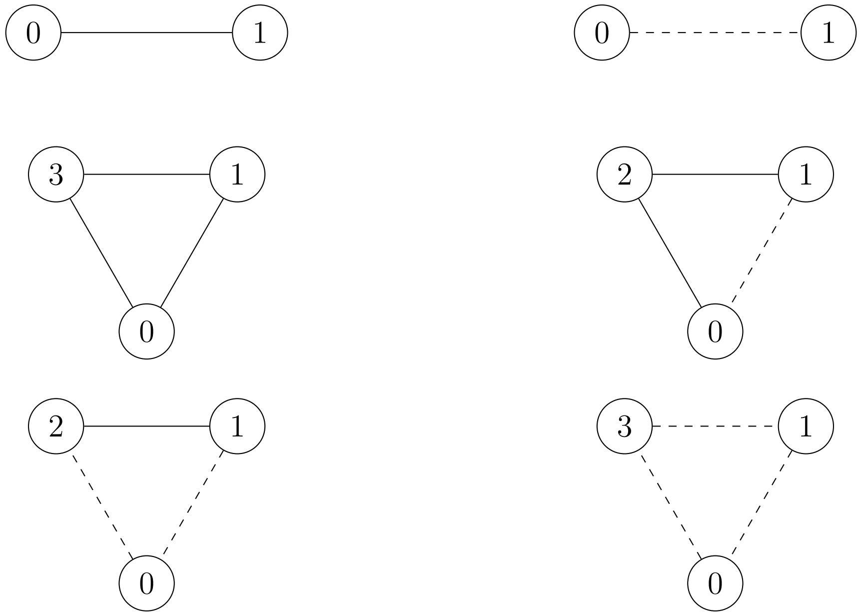


Figure 4: Graceful sigraphs on K_p when $p \leq 3$

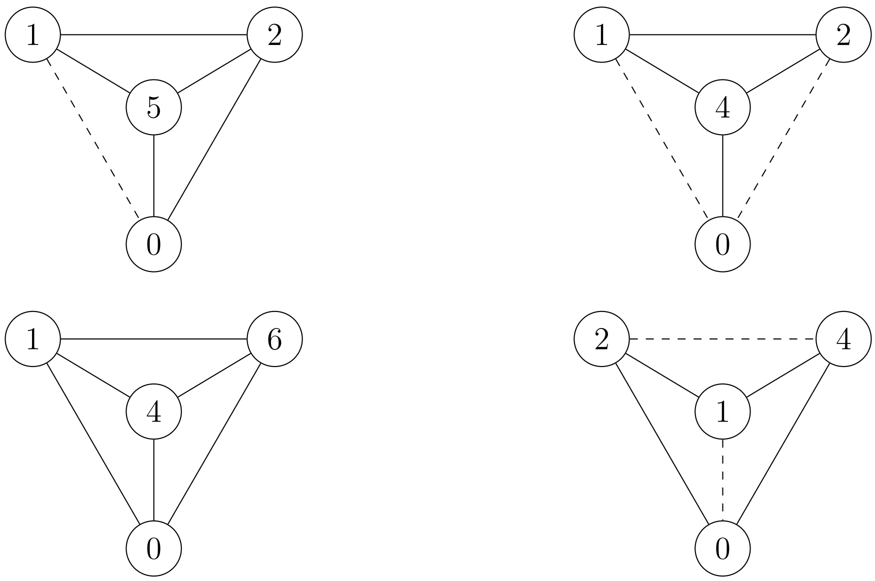


Figure 5: Graceful sigraphs on K_4

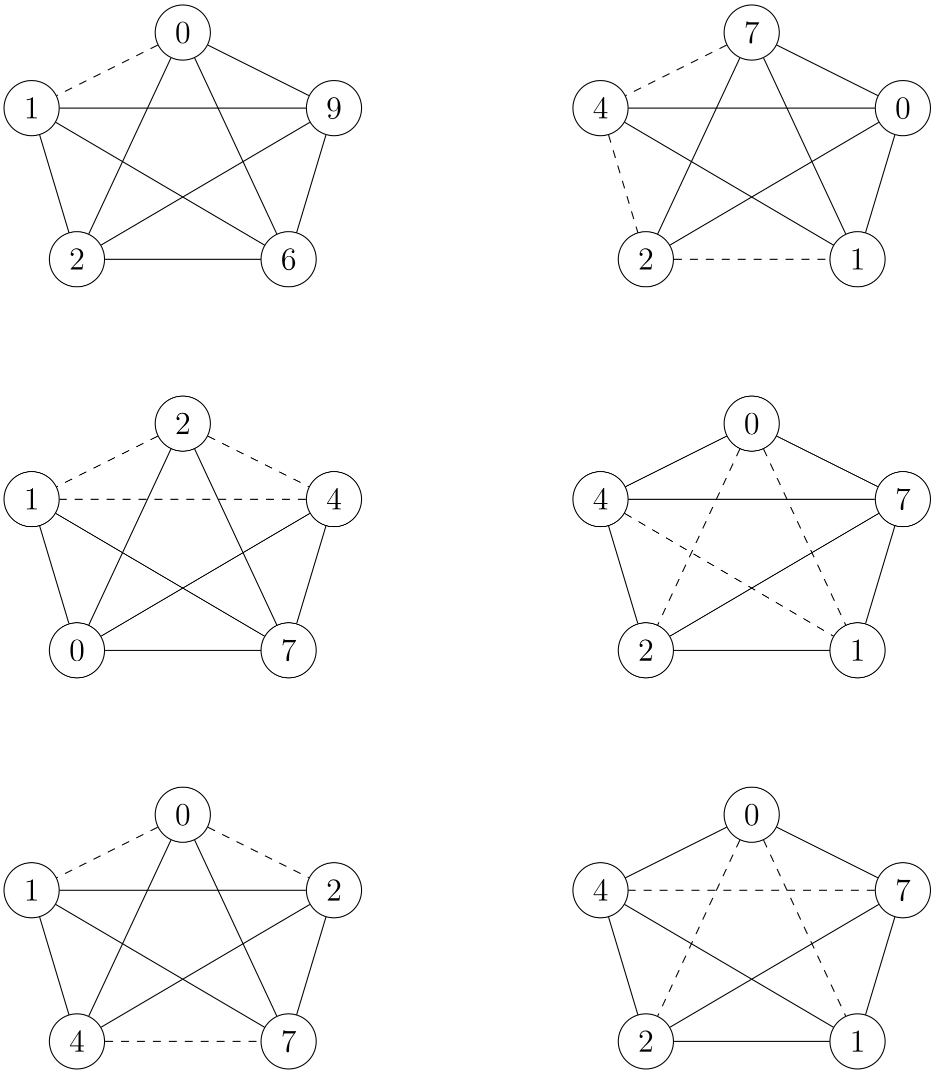


Figure 6: Illustration of graceful sigraphs on K_5

The notion of hypergraceful decomposition of graphs was first introduced by Acharya [1], which is a generalization of graceful graphs and graceful signed graphs [18], [19].

A (p, q)-graph $G = (V, E)$, that is, $|V(G)| = p$ and $|E(G)| = q$, is said to be *k-hypergraceful* if there exists a decomposition of G into edge induced subgraphs $G_1, G_2, \ldots, G_k$ having sizes $m_1, m_2, \ldots, m_k$ respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$, the set of labels received by the edges of G_i is precisely $\{1, 2, \ldots, m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *k-hypergraceful decomposition* of G and f is called a *k-hypergraceful labeling* of G. Further, G is said to be *hypergraceful* if it possesses a k-hypergraceful decomposition for some k.

The k-hypergraceful labeling for a graph G for $k = 1, 2, 3$ and 4 is given in Figure 7.

The friendship graph F_3 is known to be non-graceful [13]. It is also known that no signed graph on F_3 is graceful [20]. Therefore, F_3 is neither a 1-hypergraceful nor a 2-hypergraceful graph. In Figure 8, the 3-hypergraceful and 4-hypergraceful labelings of F_3 are shown.

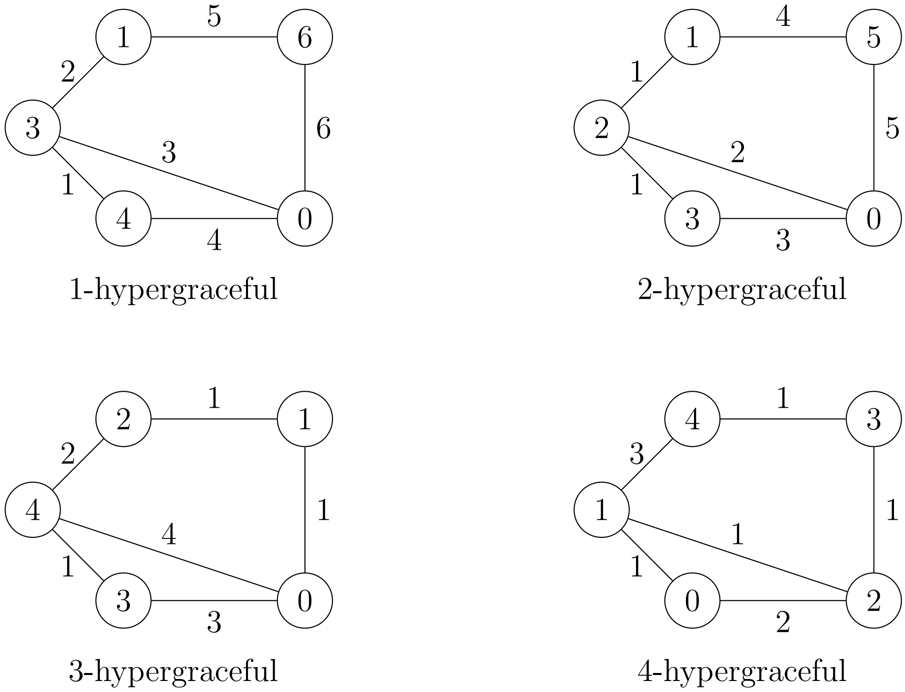


Figure 7: k-hypergraceful labelings of a graph G.

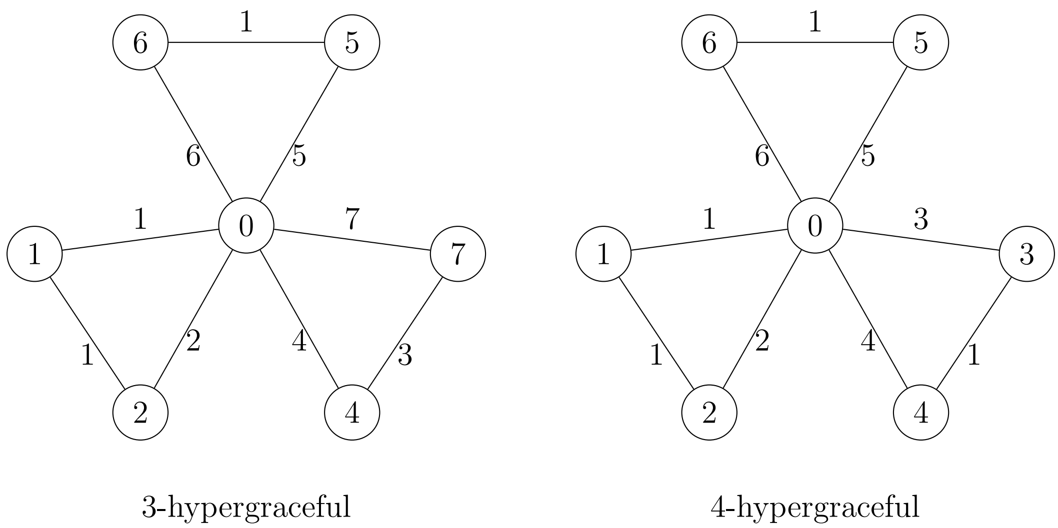


Figure 8: k-hypergraceful labeling of F_3 for $k = 3, 4$.

The following results are essential for [15].

Theorem 1.0.1 ([8]). *A complete graph K_p is graceful if and only if $p \leq 4$.*

Theorem 1.0.2 ([16]). *A necessary condition for a (p, q)-graph $G = (V, E)$ to be k-hypergraceful with decomposition $G_1, G_2, \ldots, G_k$ is that it is possible to partition its vertex set V into two subsets V_o and V_e such that for each integer $i \in \{1, 2, \ldots, k\}$ there are exactly $\left\lfloor \frac{m_i+1}{2} \right\rfloor$ edges of G_i each of which joins a vertex of V_o with one of V_e.*

Lemma 1.0.3 ([16]). *If for no integer j, $0 \leq j \leq k$, $p - 2j$ is a perfect square, then K_p is not k-hypergraceful with respect to any decomposition of K_p.*

Remark 1.0.4. *If for some integer j, there exists a k-hypergraceful decomposition of K_p for which $p - 2j$ is a perfect square, then j represents the number of G_i's with odd size.*

Lemma 1.0.5 ([16]). *If any integer p is such that none of p, $p - 2$, $p - 4$ is a perfect square, then no signed graph on K_p is graceful.*

Theorem 1.0.6 ([16]).

1. *No signed graph on K_p, $p \geq 6$, is graceful.*

2. *Every signed graph on K_p, $p \leq 3$, is graceful.*

3. *A signed graph on K_4 is graceful if and only if the number of negative edges in it is not three.*

4. *A signed graph S on K_5 with n negative edges is graceful if and only if either $n = 1$ or $n = 3$ and the three negative edges in S are not incident at the same vertex or $\eta(S)$ satisfies similar conditions.*

2 MAIN RESULTS

2.1 The Crossing Numbers of Certain Cartesian Products

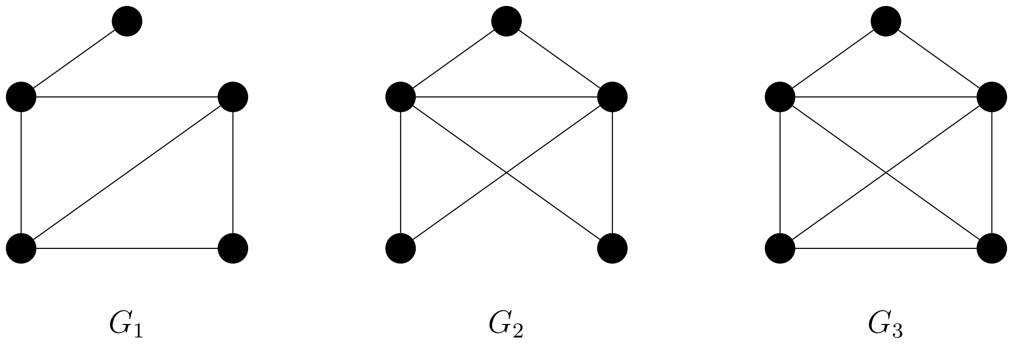


Figure 9: Graphs G_k, $k \in \{1, 2, 3\}$, each of order five.

Three graphs of order five are shown in Figure 9. Assume $n \geq 1$ and find it convenient to consider the graph $G_k \times P_n$, $k \in \{1, 2, 3\}$, in the following way. It has $5(n+1)$ vertices and edges that are the edges in the $n+1$ copies G_k^i, $i = 0, 1, ..., n$, and five paths of length n. Furthermore, we call the former edges red and the latter ones blue.

Let $H^{i,j}$ be a subgraph of $G_k \times P_n$, $k \in \{1, 2, 3\}$, induced by the vertices of G_k^i, G_k^{i+1}, ..., G_k^j for $0 \leq i < j \leq n$. The subgraph $H^{i,j} - G_k^i$ is obtained by the removal of all edges of G_k^i from the graph $H^{i,j}$.

Figure 10: Good drawing of G_1

Lemma 2.1.1. *If D is a good drawing of $G_1 \times P_n$, $n \geq 2$, in which every G_1^i, $i = 0, 1, ..., n$, has at most one crossing, then D has at least $2(n-1)$ crossings.*

Proof. Show that in every drawing $D^{0,i}$ of $H^{0,i}$, $i = 2, 3, ..., n$, induced by D there are at least two crossings more than the number of crossings in the drawing $D^{0,i-1}$ induced by $D^{0,i}$. Consider the drawing $D^{0,i}$ of $H^{0,i}$ induced by D.

By the assumption of Lemma 2.1.1 in the drawing $D^{0,i-1}$ induced by $D^{0,i}$ there is no region with 5 vertices and at most one region with 4 vertices of G_1^{i-1} on its boundary. (The crossings are considered to be vertices of the map.) Suppose that in $D^{0,i-1}$ there is one region with four vertices of G_1^{i-1} on its boundary. In this case G_1^{i-1} has one crossing with a blue edge joining G_1^{i-2} to G_1^{i-1} and in $D^{0,i}$ all vertices of G_1^i must lie outside this region.

Therefore, in the drawing $D^{0,i}$ there are at least two crossings between the edges of $H^{0,i-1}$ and the edges of $H^{i-1,i} - G_1^{i-1}$. Otherwise, $D^{0,i-1}$ induces the map with at most three vertices of G_1^{i-1} on the boundary of every region and the edges of $H^{i-1,i} - G_1^{i-1}$ have at least two crossings in $D^{0,i}$.

Since i runs through $2, 3, ..., n$, the drawing D has at least $2(n-1)$ crossings.

□

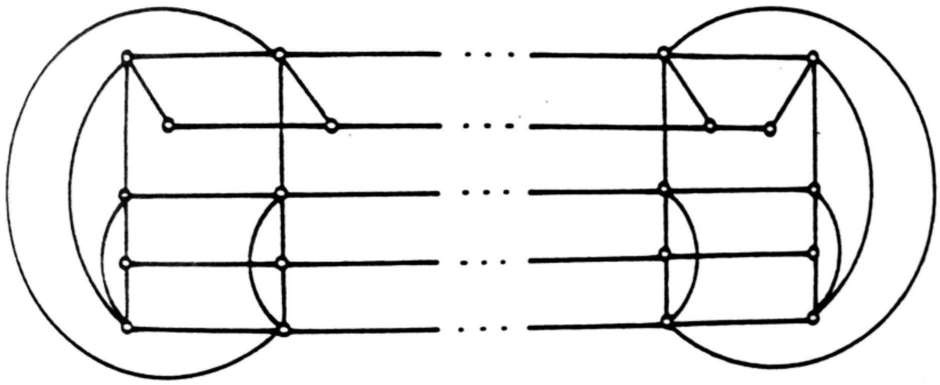


Figure 11: $G_1 \times P_n$, $n \geq 1$

Theorem 2.1.2. *$cr(G_1 \times P_n) = 2(n-1)$ for $n \geq 1$.*

Proof. The drawing in Figure 11 shows that $cr(G_1 \times P_n) \leq 2(n-1)$ for $n \geq 1$. To prove the reverse inequality by induction on n. The case $n = 1$ is trivial.

Assume that the result is true for $n = k$, $k \geq 1$, and suppose that there is a good drawing of $G_1 \times P_{k+1}$ with fewer than $2k$ crossings. By Lemma 2.1.1, some of G_1^i must then be crossed at least twice. By the removal of all edges of this G_1^i we obtain a graph, which is homeomorphic to $G_1 \times P_k$ or which contains the subgraph $G_1 \times P_k$, and has a drawing with fewer than $2(k-1)$ crossings. This contradicts the induction hypothesis. □

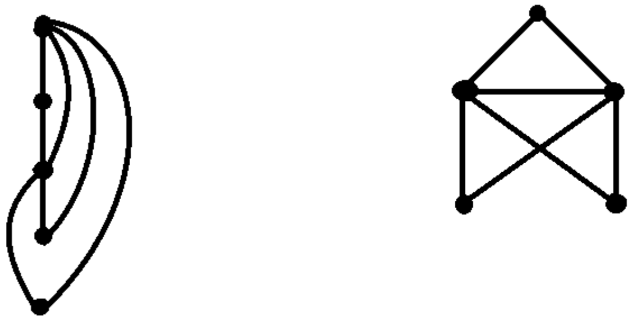


Figure 12: Good drawing of G_2

If all the vertices of the graph G_2 are joined to a vertex x different from the vertices of G_2, we obtain a new graph which cannot be drawn without having a G_2-edge crossed because it contains a subgraph $K_{3,3}$.

Similarly, if all the vertices of the graph G_2 are joined to vertices of a connected graph G, we again obtain a graph which cannot be drawn without having a G_2-edge crossed.

Lemma 2.1.3. *If D is a good drawing of $G_2 \times P_n$, $n \geq 1$, in which every G_2^i, $i = 0, 1, ..., n$, has at most two crossings, then D has at least $3n - 1$ crossings.*

Proof. By the assumption of Lemma 2.1.3 the red edges of two different G_2^i and G_2^j cannot cross. Otherwise, G_2^i $\left(G_2^j\right)$ has at least three crossings (at least two crossings with the red edges of G_2^j (G_2^i) and at least one crossing with the blue edges joining G_2^i to G_2^{i-1} or G_2^{i+1} (G_2^j to G_2^{j-1} or G_2^{j+1})).

Consider the drawing $D^{i,i+1}$ of $H^{i,i+1}$, $i \in \{0, 1, ..., n-2\}$, induced by D.

Case 1. Let no edges of G_2^{i+1} cross each other in $D^{i,i+1}$. Then the drawing D^{i+1}

of G_2^{i+1} induced by $D^{i,i+1}$ induces the map with two quadrangular regions and two triangular regions. By the assumption of Lemma 2.1.3 in the drawing D all copies G_2^0, G_2^1, ..., G_2^i, G_2^{i+2}, ..., G_2^n must lie in the quadrangular region of D^{i+1}. In $D^{i,i+1}$ there is exactly one crossing between the red edges of G_2^{i+1} and the blue edges of $H^{i,i+1}$ (Figure 13).

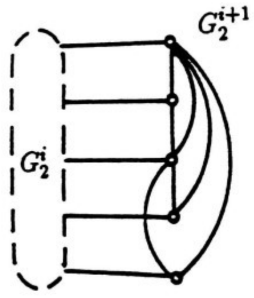


Figure 13: Case 1.

The drawing $D^{i,i+1}$ divides the quadrangular region of D^{i+1} into new regions with at most two vertices of G_2^{i+1} on the boundary of every region. (The crossings are again considered to be vertices of the map.) Consider now the drawing $D^{i,i+2}$ of $H^{i,i+2}$, $i \in \{0, 1, ..., n-2\}$, induced by D. In the drawing $D^{i,i+2}$ there are at least three crossings between the edges of $H^{i,i+1}$ and the edges of $H^{i+1,i+2} - G_2^{i+1}$.

Case 2. Let in the drawing D^{i+1} of G_2^{i+1} induced by $D^{i,i+1}$ there be a region with all vertices of G_2^{i+1} on its boundary (G_2^{i+1} has an internal crossing). Then the drawing $D^{i,i+1}$ divides this region of D^{i+1} into new regions with at most two vertices (Figure 14(a)) or with at most three vertices (Figure 14(b)) on the boundary of every region.

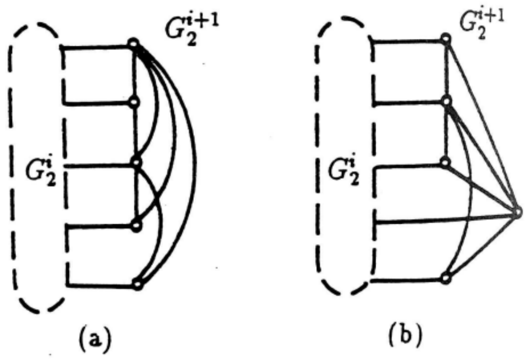


Figure 14: Case 2.

Consider now the drawing $D^{i,i+2}$ of $H^{i,i+2}$, $i \in \{0, 1, ..., n-2\}$, induced by D. In the drawing $D^{i,i+2}$ there are at least three crossings between the edges of $H^{i,i+1}$ and the edges of $H^{i+1,i+2} - G_2^{i+1}$.

Since $H^{0,1}$ has at least two crossings and i runs through $0, 1, ..., n-2$, the drawing D has at least $3(n-1)+2$ crossings. □

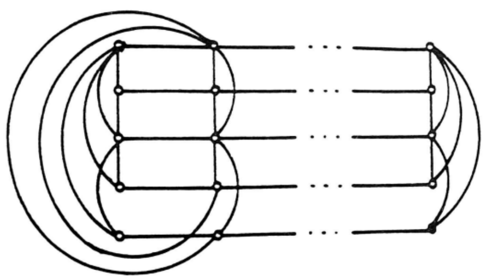


Figure 15: $G_2 \times P_n$, $n \geq 1$

Theorem 2.1.4. *$cr(G_2 \times P_n) = 3n - 1$ for $n \geq 1$.*

Proof. The drawing in Figure 15, shows that $cr(G_2 \times P_n) \leq 3n - 1$ for $n \geq 1$. The proof of the reverse inequality proceeds by induction on n in the same way as in Theorem 2.1.2 using Lemma 2.1.3. □

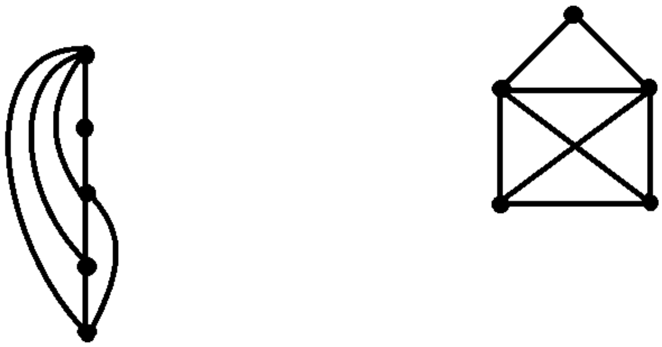


Figure 16: Good drawing of G_3

Theorem 2.1.5. *$cr(G_3 \times P_n) = 3n - 1$ for $n \geq 1$.*

Proof. Into drawing of $G_2 \times P_n$ in Figure 15, draw edges so that we obtain a good drawing of $G_3 \times P_n$ with at most $3n-1$ crossings. As $G_2 \times P_n$ is a subgraph of $G_3 \times P_n$ and $cr(G_2 \times P_n) = 3n - 1$, then $cr(G_3 \times P_n) \geq 3n - 1$. □

2.2 A New Measure for Gracefulness of Graphs

Definition 2.2.1. *Let G be a (p,q) graph. Let $f : V(G) \to N \cup \{0\}$ be an injection such that the edge induced function g_f defined on E by $g_f(uv) = |f(u) - f(v)|$ is also injective. Let $c(f) = max\{i : 1, 2, ..., i$ are edge labels under $f\}$. Then $m(G) = max_f c(f)$ is called the m-gracefulness of G.*

Let $M(G)$ denote the maximum vertex label received by G under f. If G is a graceful graph, then $m(G) = q$ and $M(G) = q$. There are exactly three connected nongraceful graphs of order five and for each of them $m(G) = q-1$ and it is known that $grac(G) = q+1$. These three graphs with appropriate labelings for $grac(G)$ and $m(G)$ are given in Figure 17. If the label of a vertex is (a,b), then a is the label corresponding to $m(G)$ and b is the label corresponding to $grac(G)$.

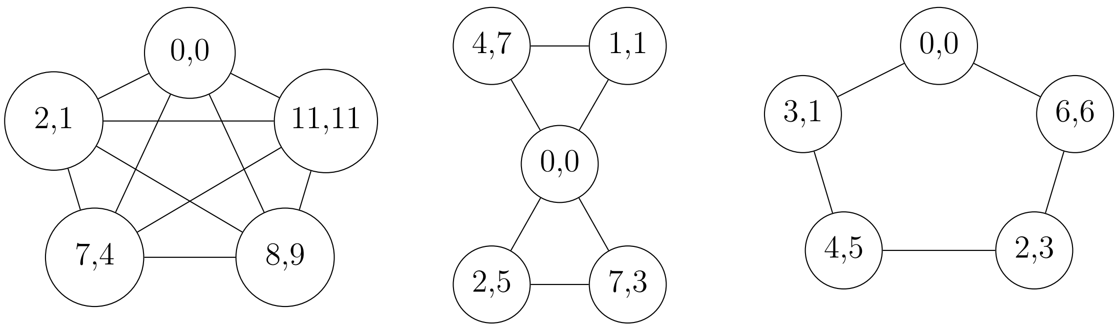


Figure 17: The three connected graphs of order five that are not graceful.

If $n \equiv 1$ or $2(mod\ 4)$, then the cycle C_n is not graceful, the following theorem determines $m(C_n)$ and $M(C_n)$.

Theorem 2.2.2. *Let $n \equiv 1$ or $2(mod\ 4)$. Then $m(C_n) = n-1$ and $M(C_n) = n+1$.*

Proof. Let $n = 4x + 2$ or $4x + 1$ according as $n \equiv 2(mod\ 4)$ or $n \equiv 1(mod\ 4)$. Let $C_n \equiv (a_1, b_1, a_2, b_2, ..., a_{2x+1}, b_{2x+1})$ if $n \equiv 2(mod\ 4)$ and let $C_n = (a_1, b_1, a_2, b_2, ..., a_{2x}, b_{2x}, a_{2x+1})$ if $n \equiv 1(mod\ 4)$.

Let $f : V(C_n) \to \{0, 1, ..., n+1\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0 & \text{if } i = 1 \\ i & \text{if } 2 \leq i \leq 2x+1 \end{cases}$$

$$\text{and } f(b_i) = \begin{cases} n+2-i & \text{if } 1 \leq i \leq x \\ n+1-i & \text{if } i \geq x+1 \end{cases}$$

It can be easily verified that f is injective, the induced edge function g_f is also injective, the highest vertex label used is $n+1$ and the set of induced edge labels is $\{1, 2, ..., n-2, n-1, n+1\}$. Hence $m(C_n) = n-1$ and $M(C_n) = n+1$. □

Corollary 2.2.3. *$grac(C_n) = n+1$ for $n \equiv 1$ or $2(mod\ 4)$.*

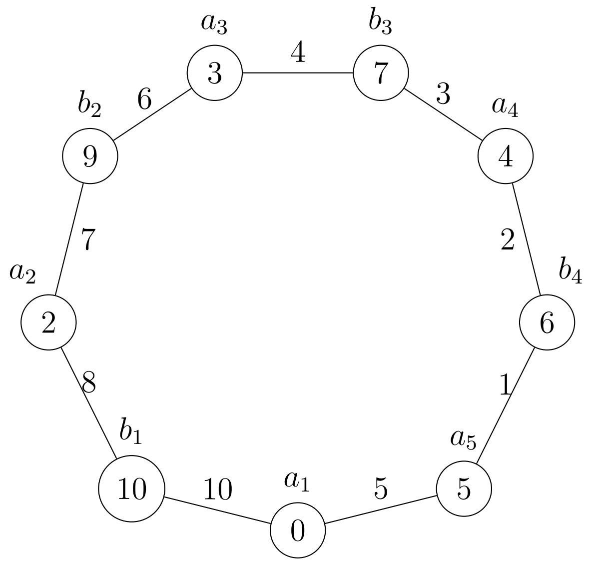


Figure 18: For C_9, $m(C_9) = 8$ and $M(C_9) = 10$.

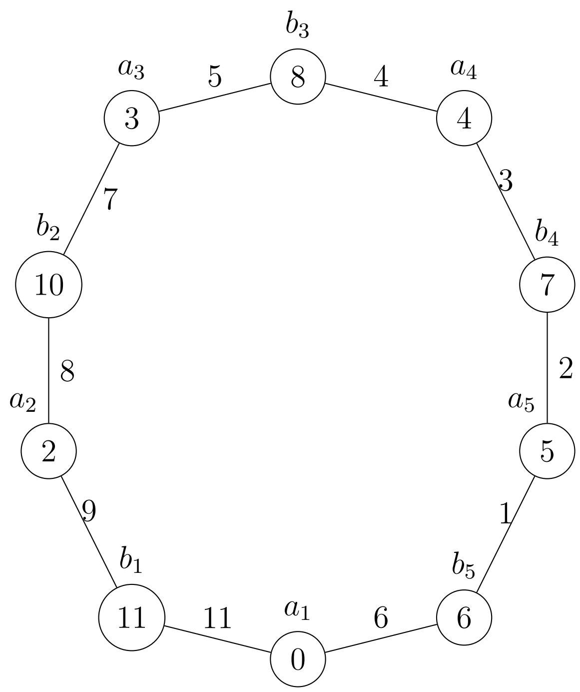


Figure 19: For C_{10}, $m(C_{10}) = 9$ and $M(C_{10}) = 11$.

Example 2.2.4. *Consider C_{21} and label its vertices as follows:*

$$f(a_1) = 0$$

$$f(a_i) = \begin{cases} i & \text{for } i = 2, 3, \ldots, 11 \end{cases}$$

and

$$f(b_i) = \begin{cases} 23 - i & \text{for } i = 1, 2, \ldots, 5 \\ 22 - i & \text{for } i = 6, 7, \ldots, 10 \end{cases}$$

It is easy to check that the set of induced edge labels is {1, 2, 3,..., 19, 20, 22}.

Hence $m(C_{21}) = 20$ and $M(C_{21}) = 22$.

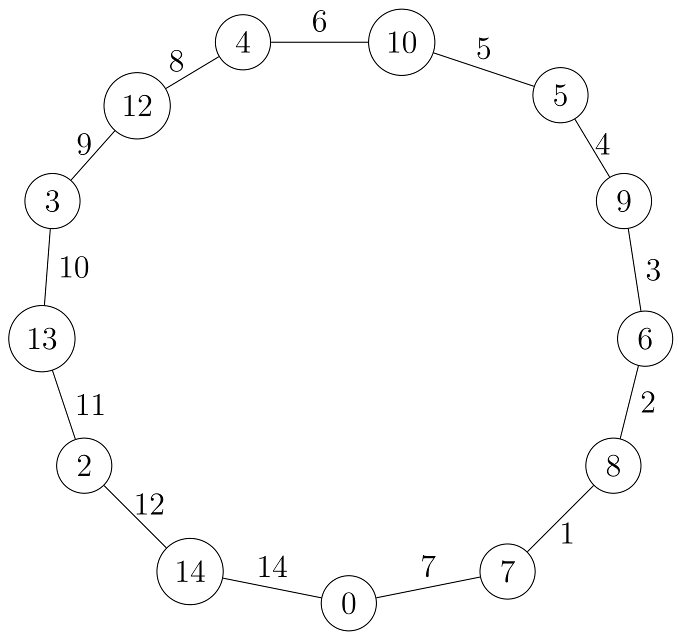


Figure 20: For C_{13}, $m(C_{13}) = 12$ and $M(C_{13}) = 14$.

Let F_k denote the friendship graph consisting of k triangles (a, u_i, v_i, a), $1 \leq i \leq k$. The graph F_k is nongraceful if $k \equiv 2$ or $3(mod\ 4)$ and in the following theorem we determine $m(F_k)$ and $M(F_k)$.

Theorem 2.2.5. *For the friendship graph F_k, we have $m(F_k) = 3k - 1$ and $M(F_k) = 3k + 1$ where $k \equiv 2$ or $3(mod\ 4)$.*

Proof. Let $f(a) = 0$, where a is the central vertex of F_k. We have the following two cases:

Case 1: $k \equiv 2(mod\ 4)$.

For F_2, F_6, F_{10} and F_{14} the labeling is given in the following Table 1.

Table 1: Labeling for Friendship Graphs F_2, F_6, F_{10}, F_{14}

Graph	u_i	v_i
F_2	1 2	4 7
F_6	1 2 3 4 5 6	15 19 11 13 12 16
F_{10}	1 2 3 4 5 6 7 8 9 10	13 31 14 25 22 26 23 27 24 28
F_{14}	1 2 3 4 5 6 7 8 9 10 11 12 13 14	19 43 35 26 36 23 37 24 38 25 39 33 40 34

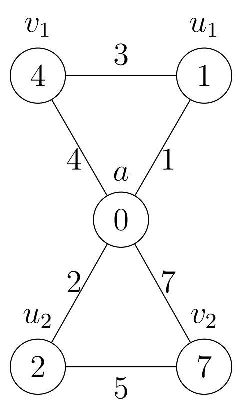


Figure 21: Labeling for friendship graph F_2

For $k \geq 18$, we define f as follows:

$$
\begin{aligned}
f(u_i) &= i && \text{for } i = 1, 2, \ldots, k; \\
f(v_1) &= \frac{5k+6}{4}; \\
f(v_2) &= 3k+1; \\
f(v_k) &= \frac{5k-2}{2}; \\
f(v_{k-2}) &= \frac{5k-4}{2}; \\
f(v_{k-1-2i}) &= 3k-2-i && \text{for } i = 0, 1, \ldots, \frac{k-4}{2}.
\end{aligned}
$$

We define $f(v_{2i})$ as follows:

$$
f(v_{2i}) = \begin{cases} \frac{3k}{2} + i & \text{for } i = 2, 3, \ldots, \frac{k-10}{4} \\ 2k-2 & \text{for } i = \frac{k-6}{4} \\ \frac{3k-2}{2} + i & \text{for } i = \frac{k-2}{4}, \frac{k+2}{4}, \ldots, \frac{k}{2} - 2 \end{cases}
$$

It can be easily verified that f is injective, the induced edge labeling g_f is also injective, the highest vertex label used is $3k+1$ and $m(F_k) = 3k-1$.

Case 2: $k \equiv 3(mod\ 4)$.

For F_3, F_7 and F_{11}, the labeling is given in the following Table 2.

Table 2: Labeling for Friendship Graphs F_3, F_7, F_{11}

Graph	u_i	v_i
F_3	1 2 3	6 10 7
F_7	1 2 3 4 5 6 7	9 22 17 15 18 16 19
F_{11}	1 2 3 4 5 6 7 8 9 10 11	14 34 27 19 28 18 29 25 30 26 31

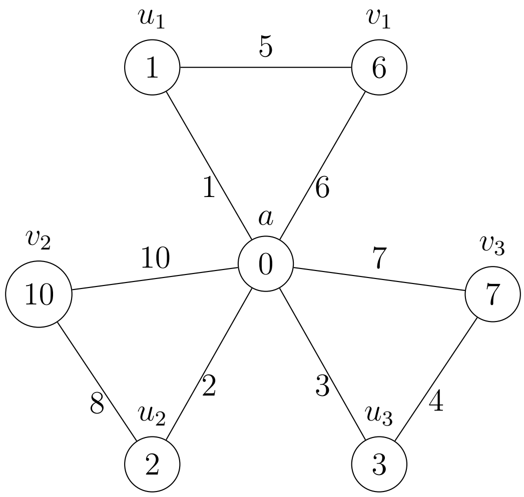


Figure 22: Labeling for friendship graph F_3

For $k \geq 15$, we label the vertices as follows: The function $f : V(G) \to \mathbb{N} \cup \{0\}$ is defined as follows:

$$
\begin{aligned}
f(u_i) &= i && \text{for } i = 1, 2, ..., k \\
f(v_1) &= \frac{5k+1}{4} \\
f(v_2) &= 3k+1 \\
f(v_{k-1}) &= \frac{5k-3}{2} \\
f(v_{k-3}) &= \frac{5k-5}{2} \\
f(v_{k-2i}) &= 3k-2-i && \text{for } i = 0, 1, ..., \frac{k-3}{2}
\end{aligned}
$$

$$f(v_{2i}) = \begin{cases} \frac{3k-1}{2} + i & \text{for } i = 2, 3, ..., \frac{k-7}{4} \\ 2k-3 & \text{for } i = \frac{k-3}{4} \\ \frac{3k-3}{2} + i & \text{for } i = \frac{k+1}{4}, \frac{k+5}{4}, ..., \frac{k-5}{4}; \end{cases}$$

It can be easily verified that f is injective, the induced edge function g_f is also injective, the highest vertex label used is $3k + 1$ and $m(F_k) = 3k - 1$. □

Corollary 2.2.6. *$grac(F_k) = 3k + 1 = q + 1$ for $k \equiv 2$ or $3(mod\ 4)$.*

If the vertices of the complete graph K_5 are labeled from the set $\{0, 2, 7, 8, 11\}$, then the set of edge labels is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$. Hence $M(K_5) = 11$, $grac(K_5) = 11$ and $m(K_5) = 9$.

Similarly if we label the vertices of K_6 either from the set $\{0, 1, 4, 10, 12, 17\}$ or $\{0, 4, 6, 9, 16, 17\}$, then the set of edge labels is $\{1, 2, \ldots, 13, 16, 17\}$. Hence we have $M(K_6) = 17$ and $m(K_6) = 13 = q - 2$.

For K_n, $6 \leq n \leq 8$ we have observed that $m(K_n) < q - 1$ and determining the exact value of $m(K_n)$ for $n \geq 7$ is an open problem.

2.3 On k-Hypergraceful Labelings of Complete Graphs

In this section, the characterization of k-hypergraceful complete graphs K_p when $p-4 \leq k \leq p-1$ is done. The results are present through a series of lemmas. The following notation is used.

Let $\pi_p = (a_1, a_2, \ldots, a_t)$ be a sequence of positive integers with $a_1 \leq a_2 \leq \ldots \leq a_t$ and $t = \begin{pmatrix} p \\ 2 \end{pmatrix}$. If a_i occurs r_i times in the sequence, then we write the sequence as $\pi_p = (a_1^{r_1}, a_2^{r_2}, \ldots, a_s^{r_s})$.

Lemma 2.3.1. *The complete graph K_p is $(p-4)$-hypergraceful if $p \geq 8$ and p is even.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 3, 4, 6\}$ and $T = \{8, 9, \ldots, p+3\}$. Note that the elements of the sets S and T are from the set $\{0, 1, \ldots, q\}$ which are the labels of the vertices of K_p. It can be easily verified that $\pi_8 = (1^4, 2^4, 3^4, 4^3, 5^3, 6^3, 7^2, 8^2, 9^1, 10^1, 11^1)$ and $\pi_{10} = (1^6, 2^6, 3^6, 4^5, 5^4, 6^4, 7^3, 8^3, 9^3, 10^2, 11^1, 12^1, 13^1)$. Now let $p \geq 12$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$, and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then

$L_1 = \{1, 2, 3^2, 4, 6\}$,

$L_2 = \left\{1^{p-5}, 2^{p-6}, 3^{p-7}, \ldots, (p-6)^2, (p-5)^1\right\}$,

$L_3 = \{2, 3, 4^2, 5^3, 6^3, 7^3, 8^4, 9^4, \ldots, (p-3)^4, (p-2)^3, (p-1)^3, p^2, p+1, p+2, p+3\}$.

Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+3)^{r_{p+3}})$ where

$$r_i = \begin{cases} p-4 & \text{if } 1 \le i \le 3; \\ p-5 & \text{if } i = 4; \\ p-6 & \text{if } i = 5, 6; \\ p-8 & \text{if } i = 7; \\ p-i & \text{if } 8 \le i \le p-4; \\ 4 & \text{if } i = p-3; \\ 3 & \text{if } i = p-2, p-1; \\ 2 & \text{if } i = p; \\ 1 & \text{if } i = p+1, p+2, p+3. \end{cases}$$

Clearly $r_i \le p-4$ for all i and $r_i \le r_j$ if $i \le j$. Hence π_p gives a $(p-4)$-hypergraceful decomposition of K_p. □

Lemma 2.3.2. *The complete graph K_p is $(p-4)$-hypergraceful if $p = 4t+1$, where $t \ge 2$.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 2, 4, 5\}$ and $T = \{8, 9, \ldots, p+3\}$. It can be easily verified that $\pi_9 = (1^5, 2^5, 3^4, 4^4, 5^3, 6^3, 7^3, 8^3, 9^2, 10^2, 11^1, 12^1)$. Now let $p \ge 13$. Let $L_1 = \{g_f(uv) :$

$u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$, and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then

$$L_1 = \{1, 2^2, 3, 4, 5\},$$

$$L_2 = \left\{1^{p-5}, 2^{p-6}, 3^{p-7}, \ldots, (p-6)^2, (p-5)^1\right\},$$

$$L_3 = \{3, 4^2, 5^2, 6^3, 7^3, 8^4, 9^4, \ldots, (p-2)^4, (p-1)^3, p^2, (p+1)^2, (p+2)^1, (p+3)^1\}.$$

Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+3)^{r_{p+3}})$, where

$$r_i = \begin{cases} p-4 & \text{if } i = 1, 2; \\ p-5 & \text{if } i = 3, 4; \\ p-6 & \text{if } i = 5; \\ p-7 & \text{if } i = 6; \\ p-8 & \text{if } i = 7; \\ p-i & \text{if } 8 \le i \le p-4; \\ 4 & \text{if } i = p-3, p-2; \\ 3 & \text{if } i = p-1; \\ 2 & \text{if } i = p, p+1; \\ 1 & \text{if } i = p+2, p+3. \end{cases}$$

Clearly $r_i \leq p-4$ for all i and $r_i \leq r_j$ if $i \leq j$. Hence π_p gives a $(p-4)$-hypergraceful decomposition of K_p. $\square$

Lemma 2.3.3. *The complete graph K_p is $(p-4)$-hypergraceful if $p = 4t+3$, where $t \geq 3$.*

Proof. The labeling given in Lemma 2.3.1 is also a $(p-4)$-hypergraceful labeling of K_p where $p = 4t+3$ and $t \geq 3$. $\square$

Lemma 2.3.4. *The complete graph K_{11} is 7-hypergraceful.*

Proof. Let f be the labeling of K_{11} with the elements of the set $S \cup T$ where $S = \{0, 4, 6, 7\}$ and $T = \{9, 10, 11, 12, 13, 14, 15\}$. It can be easily verified that $\pi_{11} = (1^7, 2^7, 3^7, 4^6, 5^5, 6^5, 7^4, 8^3, 9^3, 10^2, 11^2, 12^1, 13^1, 14^1, 15^1)$. Hence f gives a 7-hypergraceful labeling of K_{11}. $\square$

Lemma 2.3.5. *The complete graph K_7 is not 3-hypergraceful.*

Proof. Suppose there exists a 3-hypergraceful labeling of K_7 with label set S with decomposition G_1, G_2, G_3 of sizes (m_1, m_2, m_3). Since $7 - 2j$ is a perfect square when $j = 3$, it follows from Remark 1.0.4 that each m_i is odd. Hence the possible cases for (m_1, m_2, m_3) are:

- $(1, i, 20 - i)$ where $i = 1, 3, 5, 7$ or 9,
- $(3, i, 18 - i)$ where $i = 3, 5, 7$ or 9,

- $(5, i, 16 - i)$ where $i = 5$ or 7,
- $(7, 7, 7)$.

We claim that f does not induce any of the above twelve decompositions.

Case 1. $(m_1, m_2, m_3) = (1, 1, 19)$.

The sequence of edge labels is $(1^3, 2^1, 3^1, \ldots, 19^1)$. Without loss of generality, we may assume that $0, 19, 1 \in S$. Now to get the label 1 for the second edge, two consecutive integers $i, i+1$ must be in S for some $i \geq 2$. However, in this case, the edge label i occurs twice which is a contradiction. Therefore, f does not induce the decomposition $(1, 1, 19)$.

Case 2. $(1, 3, 17)$.

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^1, 5^1, \ldots, 17^1)$. Without loss of generality, we may assume that $0, 17, 1 \in S$. Now to get the label 1 for the second edge, i and $i + 1$ must be in S and $i \leq 3$. Hence $2, 3 \in S$. Now $4, 5$ cannot belong to S. So to get label 3 for the second edge, 6 must be in S. Now 7 cannot be an edge label. Therefore, f does not induce the decomposition $(1, 3, 17)$.

Case 3. $(1, 5, 15)$.

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^1, 7^1, \ldots, 15^1)$. Without loss of generality, we may assume that $0, 15, 1 \in S$. To get two more edges with label 1, we have the following possibilities:

(a) $2, 3 \in S$

(b) $3, 4, 5 \in S$

(c) $2, 4, 5 \in S$

In case (a), for edges to have edge labels 3 and 11, we must have $6, 11 \in S$ and the label 9 is repeated twice, which is a contradiction. In cases (b) and (c), label 5 for the second edge cannot appear. Therefore, f does not induce the decomposition $(1, 5, 15)$.

Case 4. $(1, 7, 13)$.

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1, \ldots, 13^1)$. Without loss of generality, we may assume that $0, 13, 1 \in S$. Now the set $\{6, 7\}$ cannot be a subset of S. To get label 1 for three edges we have the following possibilities:

(a) $2, 3 \in S$

(b) $3, 4, 7, 8 \in S$ or $4, 5, 7, 8 \in S$

(c) $2, i, i+1 \in S$ for $i = 4, 5$ or $j, j+1, j+2 \in S$ for $j = 3, 4$

In case (a), label 7 for two edges cannot be obtained. In case (b), the edge label 3 repeats more than twice which gives a contradiction. In case (c), if $2, i, i+1 \in S$ then label 10 for an edge cannot appear and if $j, j+1, j+2 \in S$ then label 11 for an edge cannot appear. Therefore, f does not induce the decomposition $(1, 7, 13)$.

Case 5. $(1, 9, 11)$.

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^1, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get label 9 for two edges, 2 and 9 must be in S. So to get label 8 for the second edge, 8 must be in S. Now the label 3 for the second edge cannot be obtained. Therefore, f does not induce the decomposition $(1, 9, 11)$.

Case 6. $(3, 3, 15)$.

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^1, 5^1, \ldots, 15^1)$. Without loss of generality, we may assume that $0, 15, 1 \in S$. To get label 1 for 3 edges, we have to include 2 and 3 in S and to get label 11 for an edge, we must have 11 in S. Now label 7 for an edge cannot be obtained. Therefore, f does not induce the decomposition $(3, 3, 15)$.

Case 7. $(3, 5, 13)$.

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^1, 7^1, \ldots, 13^1)$. Without loss of generality, we may assume that $0, 13, 1 \in S$. To get label 1 for two more edges, we have the following possibilities:

(a) $2, 3 \in S$

(b) $3, 4, 5 \in S$

(c) $2, 4, 5 \in S$

In case (a), to get label 9 for an edge, we must have $9 \in S$. Now the label 3 for an edge cannot be obtained. In case (b), label 11 for an edge cannot appear and in case (c), label 10 for an edge cannot appear. Therefore, f does not induce the decomposition $(3, 5, 13)$.

Case 8. $(3, 7, 11)$.

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1, \ldots, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get the edge label 9 we have the following two cases:

(a) $9 \in S$

(b) $2 \in S$

In case (a), to get label 7 for two edges, 4 and 7 must be in S. Now the label 1 for the second edge cannot appear. In case (b), to get label 8 for an edge, either 8 or 3 must be in S. If $3 \in S$, then label 7 for two edges cannot appear. If $8 \in S$, then to get label 7 for a second edge, either 4 must be in S or 7 must be in S. In either case, edge label 5 cannot appear. Therefore, f does not induce the decomposition $(3, 7, 11)$.

Case 9. $(3, 9, 9)$.

In order to get the sequence of edge labels as $(1^3, 2^3, 3^3, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2)$, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 9\}$, which is not possible, as we cannot get label 9 for two edges.

Case 10. $(5, 5, 11)$.

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^3, 5^3, 6^1, 7^1, \ldots, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get label 1 for two more edges, we have the following possibilities:

(a) $2, 3 \in S$

(b) $3, 4, 5 \in S$

(c) $2, 4, 5 \in S$

In case (a), label 5 for three edges cannot appear. In cases (b) and (c), label 5 for two edges cannot appear. Therefore, f does not induce the decomposition $(5, 5, 11)$.

Case 11. $(5, 7, 9)$.

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^3, 5^3, 6^2, 7^2, 8^1, 9^1)$. Without loss of generality, we may assume that $0, 9, 1 \in S$. To obtain label 7 for two edges, the integers 2 and 7 must be in S. Now to obtain label 6 for a second edge, either $6 \in S$ or $3 \in S$. In both cases, label 5 for another edge cannot be obtained. Therefore, f does not induce the decomposition $(5, 7, 9)$.

Case 12. $(7, 7, 7)$.

In this case, to get the sequence of edge labels $(1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3)$, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \dots, 7\}$. One can easily see that no labeling from this set can give label 7 for three edges. Thus we see that none of the above decompositions have a 3-hypergraceful labeling of K_7. Hence K_7 is not 3-hypergraceful. □

Theorem 2.3.6. *The complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$.*

Proof. Suppose $p \geq 8$. If $p = 2t$, $t = 4, 5, \ldots$, then by Lemma 2.3.1, K_p is $(p-4)$-hypergraceful. If $p = 4t+1$, $t \geq 2$, then by Lemma 2.3.2, K_p is $(p-4)$-hypergraceful. If $p = 4t+3$, $t \geq 3$, then by Lemma 2.3.3, K_p is $(p-4)$-hypergraceful. Finally, by Lemma 2.3.4, K_{11} is 7-hypergraceful. Therefore if $p \geq 8$ then K_p is $(p-4)$-hypergraceful. Conversely, Suppose that K_p is $(p-4)$-hypergraceful. We need to prove that $p \geq 8$. Instead, we prove the contrapositive statement. Suppose that $p < 8$. By Theorem 1.0.1, K_5 is nongraceful; by Theorem 1.0.6, K_6 is not 2-hypergraceful and by Lemma 2.3.5, K_7 is not 3-hypergraceful. Therefore, if $p < 8$ then K_p is not $(p-4)$-hypergraceful. □

Lemma 2.3.7. *The complete graph K_p is $(p-3)$-hypergraceful if $p \geq 7$.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 2\}$ and $T = \{5, 6, \ldots, p+2\}$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \{2\}$, $L_2 = \{1^{p-3}, 2^{p-4}, 3^{p-5}, \ldots, (p-4)^2, (p-3)^1\}$ and $L_3 = \{3, 4, 5^2, 6^2, 7^2, \ldots, p^2, (p+1)^1, (p+2)^1\}$. Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+2)^{r_{p+2}})$, where

$$r_i = \begin{cases} p-3, & \text{if } i = 1, 2; \\ p-4, & \text{if } i = 3; \\ p-5, & \text{if } i = 4; \\ p-i, & \text{if } 5 \leq i \leq p-2; \\ 2, & \text{if } i = p-1, p; \\ 1, & \text{if } i = p+1, p+2. \end{cases}$$

Clearly $r_i \leq p-3$ for all i and $r_i \leq r_j$ if $i \leq j$. Hence π_p gives a $(p-3)$-hypergraceful decomposition of K_p. □

Lemma 2.3.8. *The complete graph K_6 is 3-hypergraceful.*

Proof. Consider the labeling of K_6 with the elements of $\{0, 1, 3, 4, 5, 7\}$. One can easily verify that the corresponding sequence of induced edge labels is $(1^3, 2^3, 3^3, 4^3, 5^1, 6^1, 7^1)$. The decomposition G_1, G_2 and G_3 of K_6 have sizes $(4, 4, 7)$. □

Theorem 2.3.9. *The complete graph K_p is $(p-3)$-hypergraceful for all $p \geq 4$.*

Proof. The result follows from Theorem 1.0.1, Theorem 1.0.6, Lemma 2.3.7 and Lemma 2.3.8. □

Theorem 2.3.10. *The complete graph K_p is $(p-2)$-hypergraceful for all $p \geq 3$.*

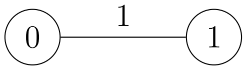
Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0\}$ and $T = \{2, 3, \ldots, p\}$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \emptyset$, $L_2 = \{1^{p-2}, 2^{p-3}, 3^{p-4}, \ldots, (p-4)^3, (p-3)^2, (p-2)^1\}$ and $L_3 = \{2, 3, \ldots, p-1, p\}$. Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, p^{r_p})$, where

$$r_i = \begin{cases} p-2, & \text{if } i = 1; \\ p-i, & \text{if } 2 \leq i \leq p-1; \\ 1, & \text{if } i = p. \end{cases}$$

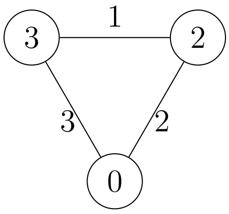
Clearly $r_i \leq p-2$ for all i and $r_i \leq r_j$ if $i \leq j$. Hence π_p gives a $(p-2)$-hypergraceful decomposition of K_p. □

Theorem 2.3.11. *The complete graph K_p is $(p-1)$-hypergraceful for all $p \geq 2$.*

Proof. We label the vertices of K_p from the set $\{0, 1, \ldots, p-1\}$. It can be easily verified that the sequence of edge labels $\pi_p = (1^{p-1}, 2^{p-2}, 3^{p-3}, \ldots, (p-2)^2, (p-1)^1)$. Hence K_p is $(p-1)$-hypergraceful for all $p \geq 2$. □

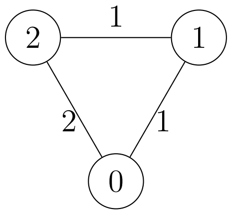


K_2 is 1-hypergraceful



K_3 is 1-hypergraceful

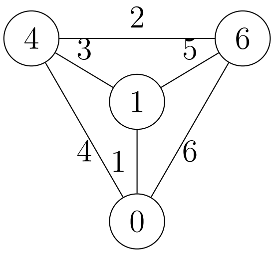
$\pi_3 = (1, 2, 3)$



K_3 is 2-hypergraceful

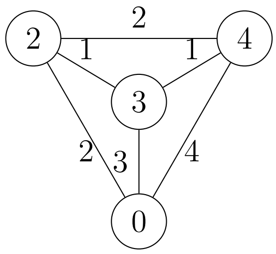
$\pi_3 = (1^2, 2)$

Figure 23: k-hypergraceful labelings of K_2 and K_3



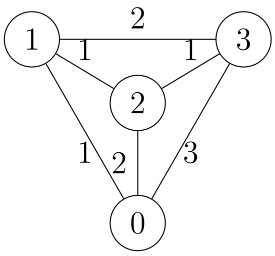
K_4 is 1-hypergraceful

$\pi_4 = (1, 2, 3, 4, 5, 6)$



K_4 is 2-hypergraceful

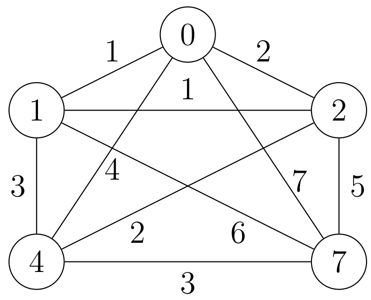
$\pi_4 = (1^2, 2^2, 3, 4)$



K_4 is 3-hypergraceful

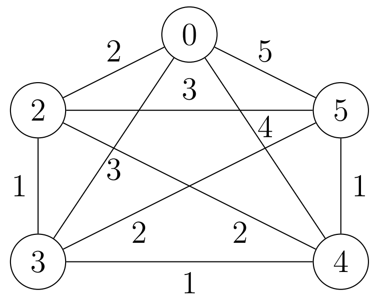
$\pi_4 = (1^3, 2^2, 3)$

Figure 24: k-hypergraceful labelings of K_4



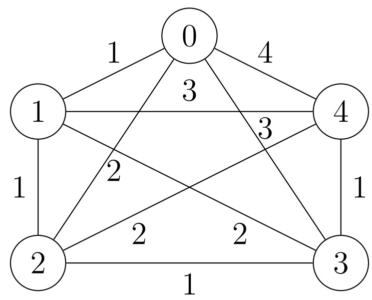
K_5 is 2-hypergraceful

$\pi_5 = (1^2, 2^2, 3^2, 4, 5, 6, 7)$



K_5 is 3-hypergraceful

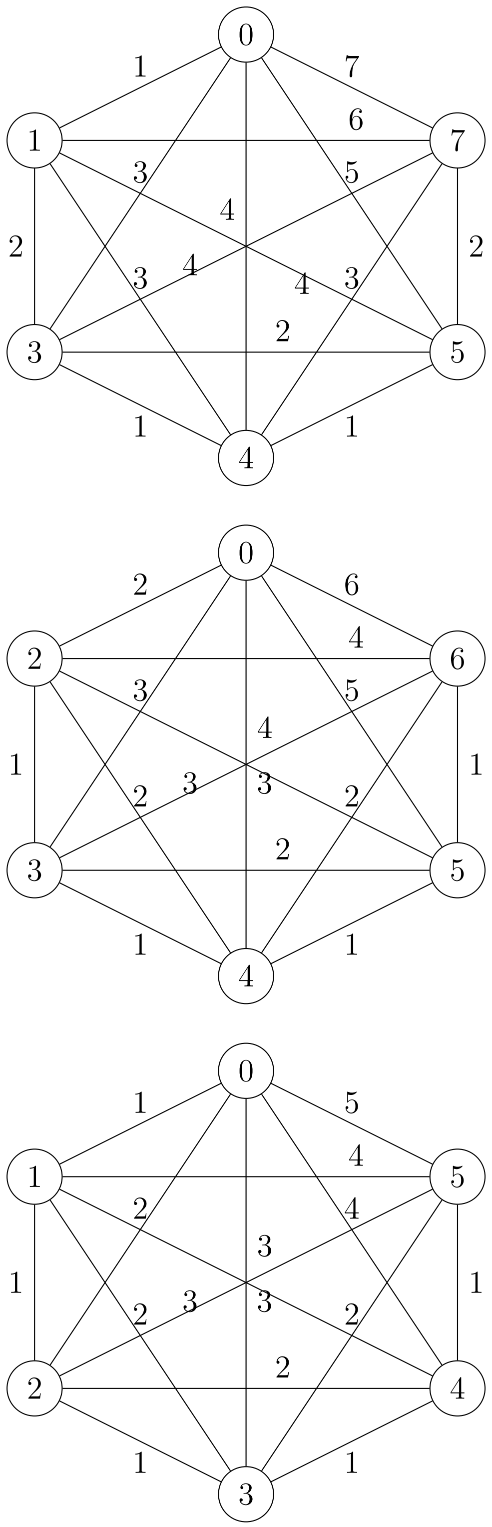
$\pi_5 = (1^3, 2^3, 3^2, 4, 5)$



K_5 is 4-hypergraceful

$\pi_5 = (1^4, 2^3, 3^2, 4)$

Figure 25: k-hypergraceful labelings of K_5



K_6 is 3-hypergraceful

$\pi_6 = (1^3, 2^3, 3^3, 4^3, 5, 6, 7)$

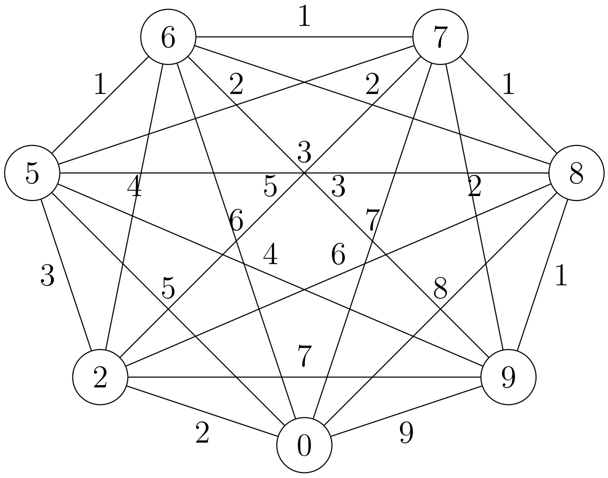
K_6 is 4-hypergraceful

$\pi_6 = (1^4, 2^4, 3^3, 4^2, 5, 6)$

K_6 is 5-hypergraceful

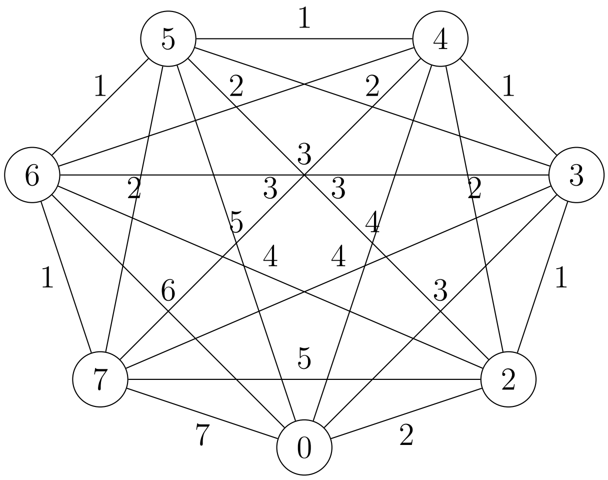
$\pi_6 = (1^5, 2^4, 3^3, 4^2, 5)$

Figure 26: k-hypergraceful labelings of K_6



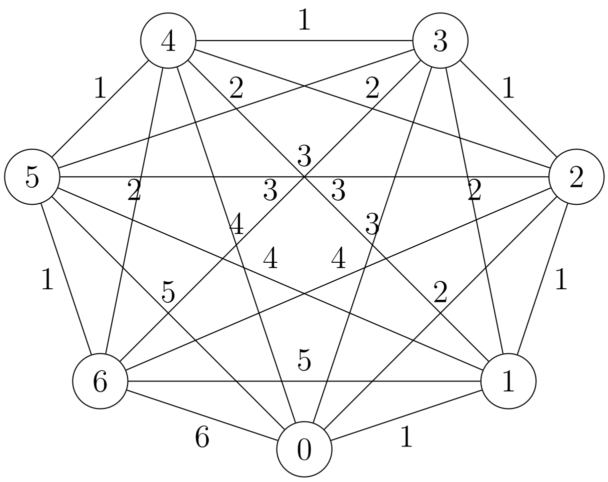
K_7 is 4-hypergraceful

$\pi_7 = (1^4, 2^4, 3^3, 4^2, 5^2, 6^2, 7^2, 8, 9)$



K_7 is 5-hypergraceful

$\pi_7 = (1^5, 2^5, 3^4, 4^3, 5^2, 6, 7)$



K_7 is 6-hypergraceful

$\pi_7 = (1^6, 2^5, 3^4, 4^3, 5^2, 6)$

Figure 27: k-hypergraceful labeling of K_7

The existence of k-hypergraceful labelings of cycles is investigated. It is known that if $n \equiv 1$ or $2(mod\ 4)$, then the cycle C_n is nongraceful [8] and if $n \equiv 1(mod\ 4)$, then C_n is also not 2-hypergraceful [2]. The following theorems determines the least k for which C_n, $n \equiv 1$ or $2(mod\ 4)$ is k-hypergraceful.

Theorem 2.3.12. *If $n \equiv 1(mod\ 4)$, then C_n is 3-hypergraceful.*

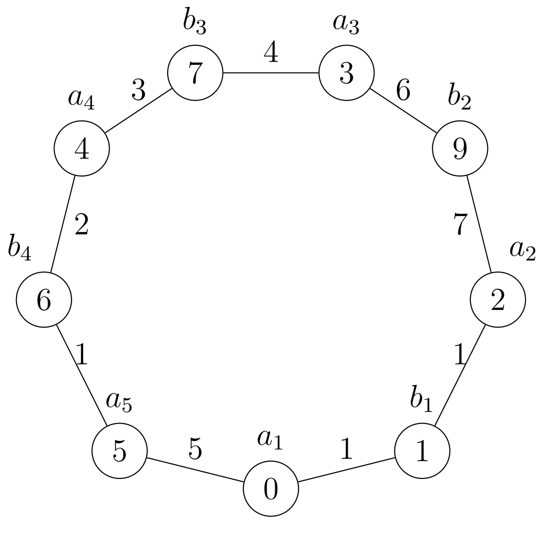
Proof. Let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}, a_1)$. Let $f : V(C_n) \to \{0, 1, \ldots, n\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0, & \text{for } i = 1; \\ i, & \text{for } 2 \leq i \leq \frac{n+1}{2}; \end{cases}$$

and

$$f(b_i) = \begin{cases} 1, & \text{for } i = 1; \\ n+2-i, & \text{for } 2 \leq i \leq \frac{n-1}{4}; \\ n+1-i, & \text{for } \frac{n-1}{4}+1 \leq i \leq \frac{n-1}{2}. \end{cases}$$

It can be easily verified that f is injective and the sequence of corresponding edge labels is $(1^3, 2^1, 3^1, \ldots, (n-2)^1)$. Hence C_n is 3-hypergraceful. □



$n \equiv 1(mod\ 4)$

C_n is 3-hypergraceful

$(1^3, 2, 3, 4, 5, 6, 7)$

Figure 28: 3-hypergraceful labeling of C_9

Theorem 2.3.13. *If $n \equiv 2(mod\ 4)$, then C_n is 2-hypergraceful.*

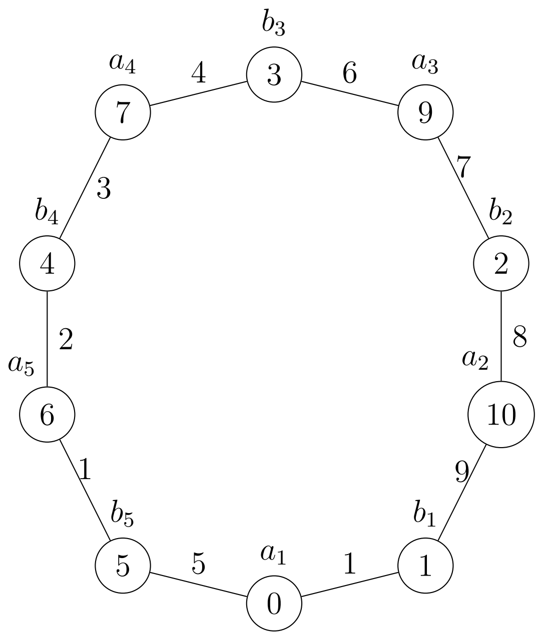
Proof. Let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{\frac{n}{2}}, b_{\frac{n}{2}}, a_1)$. Let $f : V(C_n) \to \{0, 1, \ldots, n\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0, & \text{for } i = 1; \\ n + 2 - i, & \text{for } 2 \leq i \leq \frac{n+2}{4}; \\ n + 1 - i, & \text{for } \frac{n+2}{4} + 1 \leq i \leq \frac{n}{2}. \end{cases}$$

and

$$f(b_i) = i, \text{ for } 1 \leq i \leq \frac{n}{2}.$$

It can be easily verified that f is injective and the sequence of corresponding edge labels is $(1^2, 2^1, \ldots, (n-1)^1)$. Hence C_n is 2-hypergraceful. □



$n \equiv 2(mod\ 4)$

C_n is 2-hypergraceful

$(1^2, 2, 3, 4, 5, 6, 7, 8, 9)$

Figure 29: 2-hypergraceful labeling of C_{10}

3 CONCLUSION AND SCOPE

The crossing numbers of the graphs $G_k \times P_n$, $k \in \{1, 2, 3\}$, for $n \geq 1$ are obtained i.e.

1. $cr(G_1 \times P_n) = 2(n - 1)$
2. $cr(G_2 \times P_n) = 3n - 1$
3. $cr(G_3 \times P_n) = 3n - 1$.

A new measure of gracefulness $m(G)$ of a graph G which is less than q, whereas $grac(G)$ is greater than q is introduced. The problem of determining $m(G)$ for several classes of nongraceful graphs remains open.

The existence of k-hypergraceful labeling of complete graphs and cycles is investigated. In this connection, the following conjecture.

Conjecture 3.0.1. *For any connected graph G, there exists a positive integer k such that G is k-hypergraceful.*

The hypergraceful index $h_i(G)$ is then defined to be the least positive integer k such that G is k-hypergraceful. Since $h_i(G) = 1$ if and only if G is graceful, this parameter gives another measure of gracefulness of graphs.

It follows from Theorem 2.3.6 that $h_i(K_p) \leq p - 4$ for all $p \geq 8$. Also, it follows from Theorem 2.3.12 and Theorem 2.3.13 that $h_i(C_n) = 2$ if $n \equiv 2(mod\ 4)$ and 3 if $n \equiv 1(mod\ 4)$.

The most well-known conjecture on graceful labeling is Kotzig's conjecture which states that every nontrivial tree is graceful; which still remains open. The following weaker conjecture.

Conjecture 3.0.2. *Every nontrivial tree is* 2*-hypergraceful.*

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