A Study of
Continued Fraction and Its Applications

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by

Miss. GAURI SURENDRA NARVEKAR

22P0410017

ABC ID : 602-984-449-595

201802583

Under the Supervision of

Dr. Manvendra Tamba

School of Physical & Applied Sciences

Mathematics Discipline

GOA UNIVERSITY

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, " A Study of Continued Fraction and its applications " is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Manvandna Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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Student Name: Gauri Surendra Narvekar

Seat no: 22P0410017

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "A Study of Continued Fraction and its applications" is a bonafide work carried out by Miss. Gauri Surendra Narvekar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

Signarure :

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Date:

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of M.Sc. in Mathematics in the academic year 2023-2024. The topic assigned for the research report is: " A Study of Continued Fraction and its applications." This survey is divided into three chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of the Continued fraction and the history of continued fraction.

SECOND CHAPTER:

This chapter deals with the Finite simple continued fraction. In this chapter we will dicuss some general definitions of continued fraction.Also some important properties ,theorems and examples of continued fraction

THIRD CHAPTER:

In this chapter we will studying convergents and will use convergents and some related theorems to solve linear Diophantine equations.

FOURTH CHAPTER:

In this chapter we will be solving Linear Diophantine equation with the help of continued fraction. Also discussing some cases of linear Diophantine Equation.

FIFTH CHAPTER:

In the last chapter we will discuss applications of Continued fraction.Here we apply continued fractions in calendar construction and in music .

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First of all, I would like to express my sincere gratitude to my supervisor Dr. Manvendra Tamba, for helping me to choose my dissertation topic and to find relevant reference material; also for the valuable guidance, continuous support and encouragement throughout my work. His helpful nature and motivation made me to complete this dissertation work within given period of time . His vast knowledge, extensive experience, and professional competence enabled me to successfully accomplish my work on this dissertation. This endeavour would not have been possible without his help and supervision.

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ABSTRACT

This is an expository study of continued fractions collecting ideas from several different sources including textbooks and journal articles. This study focuses on several applications of continued fractions from a variety of levels and fields of mathematics. Studies begin with looking at a number of properties that pertain to continued fractions and then move on to show how applications of continued fractions is relevant.In this dissertation, we study historical background of continued fractioin in the beginning of the project.Then we deals with the finite simple continued fraction.The definitions, notations, and basic results are shown.Then we study the convergents, their properties and some examples on them .Also, we use convergents and some related theorems to solve linear Diophantine equations.Finally, we apply continued fractions in calendar construction and in music.

Keywords: Continued fraction;finite simple continued fraction;convergents;linear Diophantine equation;calendar construction

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Notations and Abbreviations

Entity	**Abbreviation**
Continued Fraction	CF
Finite simple continued fraction	$[a_0, a_1, ...a_n]$
Infinite Simple continued faction	$[a_0, a_1, a_2, ...]$

Chapter 1

Introduction

1.1 Introduction

Continued fraction is a different way of looking at numbers. It is one of the most powerful and revealing representations of numbers that is ignored in mathematics that we have learnt during our study stages.

A continued fraction is a way of representing any real number by a finite (or infinite) sum of successive divisions of numbers. Continued fractions have been used in different areas of mathematics. Continued fractions are used in solving the Diophantine and Pell's equations. Moreover, there is a connection between continued fractions and chaos theory as Robert M. Corless wrote in his paper in 1992.

The use of continued fractions is also important in mathematical treatment to problems arising in certain applications, such as calendar construction, astronomy, music and others.

1.2 History

Mathematics is constantly built upon past discoveries. So, inorder to understand and to make contributions towards continued fractions, it is necessary to study its history. We can find examples of Continued Fractions throughout mathematics as far back as 2000 years ago. However, there is no systematic development of the subject, and because of this the origin of continued fractions is hard to pin down. Much of it's historical part is taken from the book C.D.Olds [Old63,Ch.1].

The origin of the continued fractions is traditionally placed at the time of the discovery of the Euclidean Algorithm [Old63,Ch.1]. The Eucldean Algorithm is used to find the greatest common divisor(gcd) of two integers say a and b.

For more than a thousand years, any work that used continued fractions was restricted to specific examples.The Indian mathematician Aryabhata (d.550AD) used a continued fraction to solve a linear indeterminate equation [Old3, Ch.1](equations that have more than one solution such as $ax+by=c$).Rather than generalizing this method,his use of continued fractions was used solely in specific examples .

Others that used continued fraction are Rafael Bombelli(b. c. 1530) and Pietro Cataldi(1548-1626)[Old63,Ch.1].Bombelli was the first mathematician to make use of the concept of continued fractions in his book **L'Algebra** that was published in 1572. His approximation method of the square root of 13 produced what we now interpret as a continued fraction. Cataldi did the same for the square root of 18.Besides these examples, however, both of them failed to examine closely the properties of continued fractions.

Daniel Schwenter (1585-1636) found approximations to $\frac{177}{233}$ by finding the gcd of 177 and 233, and from these Calculations he determined the convergents $\frac{79}{104}, \frac{19}{25}, \frac{3}{4}, \frac{1}{1}$ and $\frac{0}{1}$ [Old63,Ch.1].

In 1761, Lambert proved the irrationality of π using a continued fraction of tan x. He also generalized Euler work on e to show that both and tan x are irrationals if x is nonzero

rational.
Wallis represented the identity $\frac{4}{\pi}$. In his book Opera Mathematica (1695), Wallis explained how to compute the n^{th} convergent and discovered some of the properties of convergents. On the other hand, Brouncker found a method to solve the Diophantine Equation $x^2 - Ny^2 = 1$.
In the nineteenth century, the subject of continued fractions was known to every mathematician and the theory concerning convergents was developed. In 1813, Carl Friedrich Gauss derived a very general complex -valued continued fraction by a clever identity involving the hypergeometric function. Henri Pade defined Pade approximant in 1892. In fact, this century can probably be described as the golden age of continued fractions. Jacobi, Perron, Hermite, Cauchy, Stieljes and many other mathematicians made contributions to this field.
In 1776, Lagrange used continued fractions in integral calculus where he developed a general method for obtaining the continued fraction expansion of the solution of a differential equation in one variable. The field of continued fractions continues to grow, and is useful in a variety of fields. Rob Corless analyzed the connection between continued fractions and chaos theory in his paper “Continued Fractions and Chaos” [Cor92, Pg.213]. Additionally, continued fractions are used in computer algorithms for computing rational number approximations to real numbers.

Chapter 2

Finite Simple Continued Fractions

2.1 General Definitions

Definition 2.1.0.1. A **continued fraction (c.f.)** is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ddots}}}}$$

where $a_0, a_1, ..., b_0, b_1, ...$ can be either real or complex numbers.

Definition 2.1.0.2. A **simple(regular)** continued fraction is a continued fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

The numbers a_i , $i = 0, 1, 2, ...$ are called **partial quotient** of the **c.f.**.
A simple continued fraction can have either a **finite** or **infinite** represntation.

Definition 2.1.0.3. A **finite** simple continued fraction is a simple continued fraction with a finite number of terms.In symbols:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

It is called an n^{th} - order continued fraction and has $(n+1)$ elements (partial quotients).
It is also common to express the finite simple continued fraction as
$a_0 + \frac{1}{a_1+} \frac{1}{a_2+} .... \frac{1}{a_n}$ or simply as $[a_0, a_1, a_2, ..., a_n]$.

Example: 2.1.0.4.

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{6}}}}$$

Definition 2.1.0.5. An **infinite** simple continued fraction is a simple continued fraction with an infinite number of terms. In symbols:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\ddots}}}}$$

It can be also expressed as $a_0+\frac{1}{a_1+}\frac{1}{a_2+}$.... or simply as $[a_0,a_1,a_2,....]$.

Example: 2.1.0.6.

$$6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{6+\cdots}}}}$$

Definition 2.1.0.7. A **segment** of an n^{th}- order simple continued fraction is a continued fraction of the form $[a_0,a_1,a_2,....,a_k]$ where $0\leq k\leq n$ and arbitrary $k\geq 0$ if the continued fraction is infinite.

A **remainder** of an n^{th}- order finite simple continued fraction $[a_r, a_{r-1}, ..., a_n]$ where $0 \leq r \leq n$.
Similarly $[a_r, a_{r-1}, ...]$ is a remainder of an infinite simple continued fraction for arbitrary $r \geq 0$.

Examples:

1. $[0,1,2]$ is a segment of the finite simple continued fraction $[0,1,2,1,4]$ and $[2,1,4]$ is a remainder of it.

2. $[6,1,5,1]$ is a segment of the infinite simple continued fraction $[6,1,5,1,5,1,5,....]$ and $[5,1,5,1,5,....]$ is a remainder of it.

2.2 Properties and Theorems

Every rational number can be expressed as a finite simple continued fraction.Before we prove it and explain the way of expansion, we will introduced the continued fractions by studying the relationship between Euclidean Algorithm, the Jigsaw puzzle and continued fraction.Jigsaw puzzle uses picture analogy to clarify how to convert a rational number into a continued fraction. The explanation of the puzzles's steps is through the following example.

Example:

Find greatest common divisor of 64 and 17.

Solution:

Using Euclidean algorithm, we have:

$$64 = 3 \times 17 + 13 \tag{2.1}$$

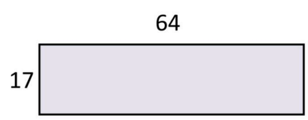
$$17 = 1 \times 13 + 4 \tag{2.2}$$

$$13 = 3 \times 4 + 1 \tag{2.3}$$

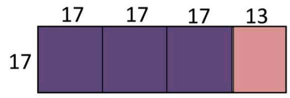
$$4 = 4 \times 1 + 0 \tag{2.4}$$

Then gcd $(64, 17) = 1$.

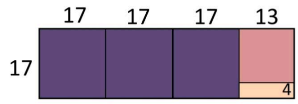
Now, consider a 64 by 17 rectangle.



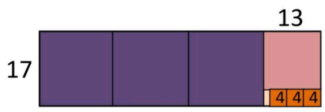
In terms of pictures, split the rectangle into 3 squares each of side length 17 and only one 17 by 13 rectangle.



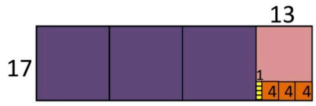
Next, it is clear that we can split the 17 by 13 rectangle into one square of one side length 13 and only one 13 by 4 rectangle.



Similarly , split the 13 by 4 rectangle into 3 squares each of side length 4 and a 4 by 1 rectangle.



Finally, we can place 4 squares , each of side length 1, inside the 4 by 1 rectangle with no remaining rectangles.



We can notice that each divisor q in the Euclidean algorithm represents the length of the side of a square. For instance, the divisor 17 in equation () represents the length of the sides of the squares that we obtain from the first splitting step. Moreover, gcd $(64, 17)$ is the length of the side of the smallest square which equals 1.

Now, divide equation (2.1) by 17 to get,

$$\frac{64}{17} = 3 + \frac{13}{17}$$

Also, divide equation (2.2) by 13 to obtain,

$$\frac{17}{13} = 1 + \frac{4}{13}$$

Repeat in the same manner for equation (2.3) and (2.4),

$$\frac{13}{4} = 3 + \frac{1}{4}$$

and

$$\frac{4}{1} = 4$$

Then write each proper fraction in the previous equations in terms of its reciprocal as follows:

$$\frac{64}{17} = 3 + \frac{1}{\frac{17}{13}} \tag{2.5}$$

$$\frac{17}{13} = 1 + \frac{1}{\frac{13}{4}} \tag{2.6}$$

$$\frac{13}{14} = 3 + \frac{1}{\frac{4}{1}} = 3 + \frac{1}{4} \tag{2.7}$$

Substitute equation (2.7) into equation (2.8) to obtain the following :

$$\frac{17}{13} = 1 + \cfrac{1}{3 + \cfrac{1}{4}} \tag{2.8}$$

Then, substitute equation (2.8) into equation (2.5) to get,

$$\frac{64}{17} = 3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

This is the continued fraction representation of the rational number $\frac{64}{17}$.
Note that by writing $\frac{64}{17} = [3,1,3,4]$, we do not mean an equality , but just a representation of the rational number $\frac{64}{17}$ by its continued fraction $[3,1,3,4]$.
This expression relates directly to the geometry of the rectangle as squares with the jigsaw pieces as follows:
3 squares each of side length 17, 1 square of side length 13, 3 squares each of side length 4 and 4 squares each of side length 1.
So, it's clear that the partial quotients of the continued fraction $[3,1,3,4]$ represent the number of squares that result from the splitting steps.
However, there is no need to use picture analogy each time we want to express a rational number as a continued fraction. The expansion of rational numbers into continued fractions is related to Euclidean algorithm as we have shown in the previous example.This relation will be studied closely in the proof of Theorem 2.0.0.4.
Now, to express any rational number $\frac{p}{q}$ as a continued fraction, we proceed in this manner.
We split the rational number into a quotient a_0 and a proper fraction, say $\frac{a}{b}$. If $a = 1$ or

$b = 1$, stop. Otherwise, repeat the process by considering the reciprocal $\frac{1}{\frac{b}{a}}$ of the proper fraction $\frac{a}{b}$ instead of $\frac{p}{q}$. Again, split $\frac{1}{(\frac{b}{a})}$ into a quotient "a_1" and a proper fraction, $\frac{a}{b}$ say again. Repeat this process until we get a proper fraction $\frac{1}{b}$, which is always the case for any rational number.

It is clear that if the rational number $\frac{p}{q}$ is positive and less than 1, then the continued fraction begins with zero, i.e.,$a_0 = 0$. Moreover, if the rational number is negative, then the continued fraction is $[a_1, a_2, a_3, ..., a_n]$ where $a_0 < 0$ and $a_1, a_2, ...a_n > 0$.

The Continued Fraction Algorithm:

This algorithm is a systematic approach that is used to find the continued fraction expansion of any rational number.Let y be any non-integer rational number .To find its continued fraction expansion, we follow the next steps:

Step 1: Set $y = y_0$ .The first partial quotient of the continued fraction is the greatest integer less than or equal to y_0(i.e. $a_0 = [[y_0]]$) where [[.]] is the greatest integer function.

Step 2: Define $y_1 = \dfrac{1}{y_0 - [[y_0]]}$ and set $a_1 = [[y_1]]$.

As long as y_j is non-integer, continue in this manner:

$y_2 = \dfrac{1}{y_1 - [[y_1]]}$, $a_2 = [[y_2]]$,

.

.

$y_k = \dfrac{1}{y_{k-1} - [[y_{k-1}]]}$, $a_k = [[y_k]]$, where $y_k - [[y_k]] = 0$.

Step 3: Stop when we find a value $y_k \in \mathbb{N}$

Note: This algorithm is also true for any real number. In this case, the process may continue indefinitely.

Example: 2.2.0.1. *Calculate the continued fraction expansion of* $\frac{102}{55}$ *using the continued fraction algorithm.*

Solution: Let $y_0 = \frac{102}{55} \approx 1.8545454545$. Then $a_0 = [[y_0]] = [[\frac{102}{55}]] = 1$.

$$y_1 = \frac{1}{y_0 - [[y_0]]} = \frac{1}{\frac{102}{55} - [[\frac{102}{55}]]} = \frac{1}{\frac{102}{55} - 1} = \frac{55}{47} \approx 1.170212766, a_1 = [[y_1]] = 1$$

$$y_2 = \frac{1}{y_1 - [[y_1]]} = \frac{1}{\frac{55}{47} - [[\frac{55}{47}]]} = \frac{1}{\frac{55}{47} - 1} = \frac{47}{8} \approx 5.875, a_2 = [[y_2]] = 5$$

$$y_3 = \frac{1}{y_2 - [[y_2]]} = \frac{1}{\frac{47}{8} - [[\frac{47}{8}]]} = \frac{1}{\frac{47}{8} - 5} = \frac{8}{7} \approx 1.142857143, a_3 = [[y_3]] = 1$$

$$y_4 = \frac{1}{y_3 - [[y_3]]} = \frac{1}{\frac{8}{7} - [[\frac{7}{8}]]} = \frac{1}{\frac{7}{8} - 1} = \frac{7}{1} = 7, a_4 = [[y_4]] = 7$$

We stop here since $y_4 = 7 \in \mathbb{N}$. Thus $[1,1,5,1,7]$ is the continued fraction representation of $\frac{102}{55}$

Conversely:

Given a continued fraction representation of a number y, we find y by using the following relationship repeatedly:

$$[a_0, a_1, \cdots, a_{n-1}, a_n] = [a_0, a_1, \cdots, a_{n-1} + \frac{1}{a_n}]$$

Example: 2.2.0.2. *Find the rational number who has the continued fraction representation* $[2,1,3,1]$.

Solution: $[2,1,3,1] = [2,1,3+\frac{1}{1}]$

$$= [2,1,4] = [2,1+\frac{1}{\frac{4}{1}}]$$

$= [2, 1 + \frac{1}{4}] = [2, \frac{5}{4}]$

$= [2 + \frac{1}{\frac{5}{4}}] = [2 + \frac{4}{5}] = [\frac{14}{5}]$

Theorem 2.2.0.3. *Every finite simple continued fraction represents a rational number.*

Proof. Let $[a_0, a_1, a_2, ..., a_n]$ be a given n^{th} - order finite simple continued fraction. We show that this continued fraction represents a rational number using induction on the number of partial quotients.

If $n = 1$, then $[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$

Since a_0 and a_1 are integers , then $\frac{a_0 a_1 + 1}{a_1}$ is a rational number.

Now assume any finite simple continued fraction with $k < n$ partial quotients represents a rational number.Then:

$$[a_0, a_1, ..., a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}] = a_0 + \frac{1}{Y}$$

where, $Y = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}} = [a_1, a_2, a_3, ..., a_k].$

Since $[a_0, a_1, ..., a_k]$ is a finite simple continued fraction with k partial quotients, it represents a rational number , say $\frac{d}{f}$. So, $Y = \frac{d}{f}, f \neq 0$.

Thus, $[a_0, a_1, a_2, ..., a_k] = a_0 + \frac{1}{Y} = a_0 + \frac{1}{\frac{d}{f}} = a_0 + \frac{f}{d} = \frac{a_0 d + f}{d}$ which is a rational number since a_0, d and f are integers.

So, any finite simple continued fraction $[a_0, a_1, ..., a_n]$represents a rational number for any $n \in \mathbb{N}$. □

Theorem 2.2.0.4. *Every rational number can be represented as a finite simple continued fraction in which the last term can be modified so as to make the number of terms in the expansion either even or odd.*

Proof. Let $\frac{p}{q}, q > 0$ be any rational number. By the Euclidean algorithm

$$p = q.a_1 + r_1, 0 < r_1 < q \tag{2.9}$$

$$q = r_1.a_2 + r_2, 0 < r_2 < r_1 \tag{2.10}$$

$$r_1 = r_2.a_3 + r_3, 0 < r_3 < r_2$$

$$r_2 = r_3.a_4 + r_4, 0 < r_4 < r_3$$

$$.$$

$$.$$

$$r_{n-3} = r_{n-2}.a_{n-1} + r_{n-1}, 0 < r_{n-1} < r_{n-2}$$

$$r_{n-2} = a_n.r_{n-1} + 0$$

The quotients $a_2, a_3, a_4, .., a_n$ and the remainders $r_1, r_2, r_3, .., r_{n-1}$ are positive integers, while a_1 can be positive integer, negative integer or zero.

Now, dividing equation (2.9) by q and then taking the reciprocal of the proper fraction we get :

$$\frac{p}{q} = a_1 + \frac{r_1}{q} = a_1 + \frac{1}{\frac{q}{r_1}}, 0 < r_1 < q$$

Also divide equation (2.10) by r_1 and take the reciprocal of the proper fraction to get:

$$\frac{q}{r_1} = a_2 + \frac{r_2}{r_1} = a_2 + \frac{1}{\frac{r_1}{r_2}}, 0 < r_2 < r_1 \tag{2.11}$$

Repeating the same process to each equation in the above Euclidean algorithm , we have :

$$\frac{r_1}{r_2} = a_3 + \frac{r_3}{r_2} = a_3 + \frac{1}{\frac{r_2}{r_3}}, 0 < r_3 < r_2 \tag{2.12}$$

$$\frac{r_2}{r_3} = a_4 + \frac{r_4}{r_3} = a_4 + \frac{1}{\frac{r_3}{r_4}}, 0 < r_4 < r_3 \tag{2.13}$$

.

.

$$\frac{r_{n-3}}{r_{n-2}} = a_{n-1} + \frac{r_{n-1}}{r_{n-2}} = a_{n-1} + \frac{1}{\frac{r_{n-2}}{r_{n-1}}}, 0 < r_{n-1} < r_{n-2} \tag{2.14}$$

$$\frac{r_{n-2}}{r_{n-1}} = a_n$$

Now, substituting $\frac{q}{r_1}$ and $\frac{r_{i-1}}{r_i}$ back into equations (2.11) through (2.14) yields:

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{r_1}} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{(\frac{r_2}{r_3})}}} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{(\frac{r_3}{r_4})}}}}$$

Continue in the same manner to get:

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{\left(\cfrac{r_{n-2}}{r_{n-1}}\right)}}}}}}$$

$$\Rightarrow \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

$= [a_1, a_2, a_3, ..., a_n]$.

Thus, every rational number can be represented as a finite simple continued fraction.

In fact, we can always modify the last partial quotient n of this representation so that the number of terms is either even or odd.

If $a_n = 1$, then $\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}} = \cfrac{1}{a_{n-1} + \cfrac{1}{1}} = \cfrac{1}{a_{n-1} + 1}$

and $\frac{p}{q} = [a_1, a_2, .., a_{n-1}, a_n] = [a_1, a_2, ..., a_{n-1} + 1]$.

Else, if $a_n > 1$, then $\cfrac{1}{a_{n-1} + \cfrac{1}{a_n}} = \cfrac{1}{a_{n-1} + \cfrac{1}{(a_n - 1) + 1}} = \cfrac{1}{a_{n-1} + \cfrac{1}{(a_n - 1) + \cfrac{1}{1}}}$

and $\frac{p}{q} = [a_1, a_2, .., a_{n-1}, a_n] = [a_1, a_2, ..., a_{n-1}, a_n - 1, 1]$. □

Theorem 2.2.0.5. *Let p and q be two integers such that* $p > q > 0$*. Then* $[a_0, a_1, ..., a_{n-1}, a_n]$ *is a continued fraction representation of* $\frac{p}{q}$ *if and only if* $\frac{q}{p}$ *has* $[0, a_0, a_1, ..., a_{n-1}, a_n]$ *as its continued fraction representation.*

Proof. Since $p > q > 0$, $\frac{p}{q} > 1$ and equals

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where a_0 is the greatest integer less than $\frac{p}{q} = [[\frac{p}{q}]] > 0$.

The reciprocal of $\frac{p}{q}$ is,

$$\frac{q}{p} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} = [0, a_1, a_2, .., a_{n-1}, a_n]$$

Conversely, since $p > q > 0$, $0 < \frac{q}{p} < 1$ and equals

$$\frac{q}{p} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

The reciprocal of $\frac{q}{p}$ is

$$\frac{p}{q} = \cfrac{1}{\cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_0, a_1, a_2, a_3 ..., a_n].$$

□

Chapter 3

Convergents

In order to have a thorough understanding of continued fractions, we must study some of their properties in details.

Consider the continued fraction representation $[2,2,7]$ of the rational number $\frac{37}{15}$. The segments of this continued fraction are:

$[2] = 2[2,2] = 2+\frac{1}{2}, [2,2,7] = 2+\cfrac{1}{2+\cfrac{1}{7}}$

Since each segment is a finite simple continued fraction, it represents a rational number. These segments are called convergents of the continued fraction $[2,2,7]$.

Definition 3.0.0.1. Let $[a_0, a_1, ..., a_n]$ be a finite simple continued fraction representation of a rational number $\frac{p}{q}$.Its segments:

$c_0 = [a_0] = a_0, c_1 = [a_0, a_1] = a_0 + \frac{1}{a_1}, c_2 = [a_0, a_1, a_2] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}, ...,$

$$c_n = [a_0, a_1, a_2, ...a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

are all called **convergents** of the continued fraction with c_k is the k^{th} **convergent**, $k = 0, 1, 2, ..., n$.

Note: Here we have $n+1$ convergents and each convergent c_k represents a rational number of the form $c_k = \frac{p_k}{q_k}$ where p_k and q_k are integers with $c_n = \frac{p}{q}$.
We shall use the representation of a convergent $c_k = [a_0, a_1, ..., a_k]$ and $\frac{p_k}{q_k}$ interchangeably to mean the same thing.

Example: 3.0.0.2. *Find all of the convergents for the continued fraction* $[3,5,1,7]$.

Solution:

$c_0 = [3] = 3$

$c_1 = [3,5] = 3 + \frac{1}{5} = \frac{16}{5}$

$c_2 = [3,5,1] = 3 + \cfrac{1}{5 + \cfrac{1}{1}} = 3 + \frac{1}{6}\frac{19}{6}$

$c_3 = [3,5,1,7] = 3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{7}}} = 3 + \cfrac{1}{5 + \cfrac{1}{\frac{8}{7}}} = 3 + \cfrac{1}{5 + \frac{7}{8}} = 3 + \frac{8}{47} = \frac{149}{47}$

Note that the 3^{rd} convergent $c_3 = \frac{149}{47}$ represents the fraction itself.

The following theorem gives a recursion formula to calculate the convergents of a continued fraction.

Theorem 3.0.0.3. ***(Continued Fraction Recursion Formula)***

Consider the continued fraction $[a_0, a_1, a_2, ...a_n]$ *of a given rational number. Define*

$$p_{-1} = 1, p_{-2} = 0$$

$$q_{-1} = 0, q_{-2} = 1$$

.Then

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

,for $k = 0, 1, 2, .., n$ where $p_0, p_1, ..., p_n$ are the numerators of the convergents of the given continued fraction and $q_0, q_1, ...q_n$ are their denominators.

Proof. We prove this theorem using induction on k.

For $k = 0$, we have :
$c_0 = \frac{p_0}{q_0} = a_0 = \frac{a_0}{1} = \frac{a_0.1+0}{a_0.0+1} = \frac{a_0.p_{-1}+p_{-2}}{a_0.q_{-1}+q_{-2}}$

Therefore, $p_0 = a_0 p_{-1} + p_{-2}$ and $q_0 = a_0 q_{-1} + q_{-2}$

For $k = 1$,we have :
$c_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0.a_1+1}{a_1} = \frac{a_1.a_0+1}{a_1.1+0} = \frac{a_1.p_0+p_{-1}}{a_1.q_0+q_{-1}}$
Therefore, $p_1 = a_1.p_0 + p_{-1}$ and $q_1 = a_1.q_0 + q_{-1}$

Thus , the formula

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

is true for $k = 0, 1$.

Assume the theorem is true for $k = 2, 3, ..., j$, where $j < n$.

i.e.

$$c_k = \frac{p_k}{q_k} = \frac{a_k.p_{k-1} + p_{k-2}}{a_k.q_{k-1} + q_{k-2}} \tag{3.1}$$

, for $k = 2,3,...,j$.

So, $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$

Now , we prove that the formula is true for the next integer $j+1$.

$$c_{j+1} = [a_0, a_1, ..., a_j, a_{j+1}] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_j + \cfrac{1}{a_{j+1}}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{(a_j + \cfrac{1}{a_{j+1}})}}}}$$

$$= [a_0, a_1, a_2, ...a_j + \frac{1}{a_{j+1}}].$$

This suggests that we can calculate c_{j+1} from the formula of c_j obtained from equation (3.1) after replacing k by j. Before we continue, we must make sure that the values of $p_{j-1}, p_{j-2}, q_{j-1}, q_{j-2}$ won't change if a_j in equation (3.1) is replaced by another number. To do this, first replace k in the equation by $j-1$, and then by $j-2$, $j-3$ to get:

$$c_{j-1} = \frac{p_{j-1}}{q_{j-1}} = \frac{a_{j-1}.p_{j-2} + p_{j-3}}{a_{j-1}.q_{j-2} + q_{j-3}}$$

$$c_{j-2} = \frac{p_{j-2}}{q_{j-2}} = \frac{a_{j-2}.p_{j-3} + p_{j-4}}{a_{j-2}.q_{j-3} + q_{j-4}}$$

$$c_{j-3} = \frac{p_{j-3}}{q_{j-3}} = \frac{a_{j-3}.p_{j-4} + p_{j-5}}{a_{j-3}.q_{j-4} + q_{j-5}}$$

We notice that p_{j-1} and q_{j-1} depend only on a_{j-1} while the numbers $p_{j-2}, p_{j-3}, q_{j-2}, q_{j-3}$ depend upon the preceding $a's$, $p's$ and $q's$. Thus, the numbers $p_{j-1}, p_{j-2}, q_{j-1}, q_{j-2}$ depend only on $a_0, a_1, .., a_{j-1}$ and not on a_j.This implies that they will remain the same when we replace a_j by $a_j + \frac{1}{a_{j+1}}$.

Back to equation (3.1), replace a_j by $a_j + \frac{1}{a_{j+1}}$ to get:

$$c_{j+1} = \frac{(a_j + \frac{1}{a_{j+1}}).p_{j-1} + p_{j-2}}{(a_j + \frac{1}{a_{j+1}}).q_{j-1} + q_{j-2}} = \frac{(\frac{a_j a_{j+1} + 1}{a_{j+1}}).p_{j-1} + p_{j-2}}{(\frac{a_j a_{j+1} + 1}{a_{j+1}}).q_{j-1} + q_{j-2}} \tag{3.2}$$

Multiply the numerator and denominator of equation (3.2) by a_{j+1} and rearrange the terms to obtain:

$$c_{j+1} = \frac{(a_j a_{j+1} + 1).p_{j-1} + a_{j+1} p_{j-2}}{(a_j a_{j+1} + 1).q_{j-1} + a_{j+1} q_{j-2}} = \frac{a_{j+1}(a_j p_{j-1} + p_{j-2}) + p_{j-1}}{a_{j+1}(a_j q_{j-1} + q_{j-2}) + q_{j-1}}.$$

But from our assumption, $a_j p_{j-1} + p_{j-2} = p_j$ and $a_j q_{j-1} + q_{j-2} = q_j$.

Then, $c_{j+1} = \frac{a_{j+1} p_j + p_{j-1}}{a_{j+1} q_j + q_{j-1}}$.

Thus, the formula is true for $k = j+1$. So, by induction, the theorem is true for $0 \leq k \leq n$. □

Note:

1. $\frac{p_{-1}}{q_{-1}}$ and $\frac{p_{-2}}{q_{-2}}$ are not convergents p_{-1}, p_{-2}, q_{-1} and q_{-2} are just initial values used to calculate c_0 and c_1.

2. $q_k > 0, k = 0, 1, 2, ..., n$.

3. Since $a_k > 0$ for $1 \leq k \leq n$ and $q_k > 0$ for $0 \leq k \leq n$, it follows that $q_k > q_{k-1}, k = 2, 3, .., n$.

Example: 3.0.0.4. *Find the convergents of the continued fraction representation of the rational number* $\frac{320}{171}$ *using Continued Fraction Recursion Formula.*

Solution:

First of all, the continued fraction representation of $\frac{320}{171}$ is $[1,1,6,1,3,2,2]$ and we have $a_0 = 1, a_1 = 1, a_2 = 6, a_3 = 1, a_4 = 3, a_5 = 2, a_6 = 2$.

With

$$p_{-1} = 1, p_{-2} = 0,$$

$$q_{-1} = 0, q_{-2} = 1$$

calculate p_k and q_k using the recursion formula.

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

Here, $k = 0, 1, ..., 6$.

For $k = 0$:

$$p_0 = a_0 p_{-1} + p_{-2} = 1 \times 1 + 0 = 1$$

$$q_0 = a_0 q_{-1} + q_{-2} = 1 \times 0 + 1 = 1$$

For $k = 1$:

$$p_1 = a_1 p_0 + p_{-1} = 1 \times 1 + 1 = 2$$

$$q_1 = a_1 q_0 + q_{-1} = 1 \times 1 + 0 = 1$$

For $k = 2$:

$$p_2 = a_2 p_1 + p_0 = 6 \times 2 + 1 = 13$$

$$q_2 = a_2 q_1 + q_0 = 6 \times 1 + 1 = 7$$

For $k = 3$:

$$p_3 = a_3 p_2 + p_1 = 1 \times 13 + 2 = 15$$

$$q_3 = a_3 q_2 + q_1 = 1 \times 7 + 1 = 8$$

For $k = 4$:

$$p_4 = a_4 p_3 + p_2 = 3 \times 15 + 13 = 58$$

$$q_4 = a_4 q_3 + q_2 = 3 \times 8 + 7 = 31$$

For $k = 5$:

$$p_5 = a_5 p_4 + p_3 = 2 \times 58 + 15 = 131$$

$$q_5 = a_5 q_4 + q_3 = 2 \times 31 + 8 = 70$$

For $k = 6$:

$$p_6 = a_6 p_5 + p_4 = 2 \times 131 + 58 = 320$$

$$q_6 = a_6 q_5 + q_4 = 2 \times 70 + 31 = 171$$

Thus, $c_0 = \frac{p_0}{q_0} = \frac{1}{1} = 1$, $c_1 = \frac{p_1}{q_1} = \frac{2}{1} = 2$, $c_2 = \frac{p_2}{q_2} = \frac{13}{7}$,

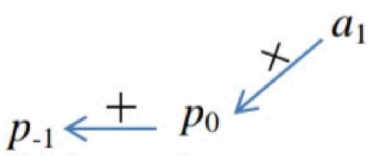
$c_3 = \frac{p_3}{q_3} = \frac{15}{8}$, $c_4 = \frac{p_4}{q_4} = \frac{58}{31}$, $c_5 = \frac{p_5}{q_5} = \frac{131}{70}$, $c_6 = \frac{p_6}{q_6} = \frac{320}{171}$.

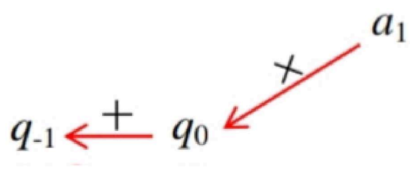
The last convergent, c_6 in this example, must be equal to the rational number the continued fraction represents.

However, a convergent table can be used to save time in calculating p_k and q_k. Following table explains the manner.

k	-2	-1	0	1	2	...	...	...	n
a_k			a_0	a_1	a_2	...	...	...	a_n
p_k	$p_{-2} = 0 \leftarrow$	$p_{-1} = 1 \leftarrow$	p_0	p_1	p_2	...	...	...	p_n
q_k	$q_{-2} = 1$	$q_{-1} = 0 \leftarrow$	q_0	q_1	q_2	...	...	...	q_n
c_k			c_0	c_1	c_2	...	...	...	c_n

The first row of the table is filled with the values of k that always range from -2 to n. In the second row, we write the partial quotients of the given continued fraction. Now, to fill the $3^r d$ and $4^t h$ rows, we write the values $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$ under $k = -2$, $k = -1$, respectively. Then we compute the values of $pk's$ and $qk's$ using the recursion formula. For example, to find p_1 and q_1 , we follow the arrows, (look at the table):





This manner gives us the following equations which we obtain when we set $k = 1$ in the recursion formula:

$$p_1 = a_1 p_0 + p_{-1}$$

$$q_1 = a_1 q_0 + q_{-1}$$

In the same process we find p_k and q_k for each value of k.
The last row contains the convergents $ck's$, where $c_k = \frac{p_k}{q_k}$, $0 \leq k \leq n$.
Back to our example, the table is filled in the same manner and the result is:

k	-2	-1	0	1	2	3	4	5	6
a_k			1	1	6	1	3	2	2
p_k	0	1	1	2	13	15	58	131	320
q_k	1	0	1	1	7	8	31	70	171
c_k			$\frac{1}{1} = 1$	$\frac{2}{1} = 2$	$\frac{13}{7}$	$\frac{15}{8}$	$\frac{58}{31}$	$\frac{131}{70}$	$\frac{320}{171}$

Theorem 3.0.0.5. ***(Difference of Successive Convergents Theorem)***

$$c_k - c_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}, 1 \leq k \leq n$$

To prove this theorem we need the following lemma.

Lemma 3.0.0.6. *Let $\frac{p_k}{q_k}$ be the k^{th} convergent of the continued fraction $[a_0, a_1, a_2, ...a_n]$, where p_k and q_k are defined as in Theorem 3.0.0.3. Then:*
$p_{k-1}q_k - p_k q_{k-1} = (-1)^k, -1 \leq k \leq n.$

Proof. This lemma will be proved by induction on k and using the formula that we have proved in the previous theorem. Direct calculations show the theorem is true for $k = -1$, 0 and 1.
For $k = -1 : p_{-2}q_{-1} - p_{-1}q_{-2} = 0.0 - 1.1 = -1 = (-1)^{-1}$
For $k = 0 : p_{-1}q_0 - p_0 q_{-1} = 1.1 - a_0.0 = 1 = (-1)^0$
For $k = 1 : p_0 q_1 - p_1 q_0 = a_0.a_1 - (a_0 a_1 + 1).1 = a_0 a_1 - a_0 a_1 - 1 = -1 = (-1)^1$
Assume the lemma is true for some integer $s < n$, i.e. $p_{s-1}q_s - p_s q_{s-1} = (-1)^s$.
Now, for $k = s + 1$, we have :
$p_s q_{s+1} - p_{s+1} q_s = p_s(a_{s+1}q_s + q_{s-1}) - (a_{s+1}p_s + p_{s-1})q_s.$
$= p_s a_{s+1} q_s + p_s q_{s-1} - a_{s+1} p_s q_s + p_{s-1} q_s = -1.(p_{s-1}q_s - p_s q_{s-1}) = -1.(-1)^s = (-1)^{s+1}.$
Therefore, the formula is true for $k = s + 1$ and so by induction the lemma is true for $-1 \leq k \leq n$.

□

Proof. of Theorem 3.0.0.5: For $1 \leq k \leq n$:

$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}} = -\frac{p_{k-1} q_k - p_k q_{k-1}}{q_k q_{k-1}}$$

Using Lemma 3.0.0.6,
$c_k - c_{k-1} = \dfrac{(-1)^k}{q_k q_{k-1}} = \dfrac{(-1)^{k+1}}{q_k q_{k-1}} = \dfrac{(-1)^{k-1}}{q_k q_{k-1}}$. □

Example: 3.0.0.7. *Verify Lemma 3.0.0.6 using the convergents of the continued fraction* $[1,1,6,1,3,2,2]$.

Solution: Using the values of $pk's =$ 1, 2, 13, 15, 58, 131, 320 and $qk's =$1, 1, 7, 8, 31, 70, 171 obtained in Example 3.0.0.4, we get:

For $k = -1$: $p_{-2}q_{-1} - p_{-1}q_{-2} = 0x0 - 1x1 = -1 = (-1)^{-1}$

For $k = 0$: $p_{-1}q_0 - p_0q_{-1} = 1x1 - 1x0 = 1 = (-1)^0$

For $k = 1$: $p_0q_1 - p_1q_0 = 1x1 - 2x1 = -1 = (-1)^1$

For $k = 2$: $p_1q_2 - p_2q_1 = 2x7 - 13x1 = 1 = (-1)^2$

For $k = 3$: $p_2q_3 - p_3q_2 = 13x8 - 15x7 = -1 = (-1)^3$

For $k = 4$: $p_3q_4 - p_4q_3 = 15x31 - 58x8 = 1 = (-1)^4$

For $k = 5$: $p_4q_5 - p_5q_4 = 58x70 - 131x31 = (-1)^5$

For $k = 6$: $p_5q_6 - p_6q_5 = 131x171 - 320x70 = (-1)^6$

Thus, $p_{k-1}q_k - p_kq_{k-1} = (-1)^k, -1 \leq k\ leq 6$.

Corollary 3.0.0.8. $c_k - c_{k-2} = \dfrac{(-1)^{k-1}a_k}{q_k q_{k-1}}, 2 \leq k \leq n.$

Proof. By Theorem (3.0.0.5), $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_k q_{k-1}}$ and $c_{k-1} - c_{k-2} = \dfrac{(-1)^{k-2}}{q_{k-1}q_{k-2}}$
Adding these two equations, we get:

$$\begin{aligned} c_k - c_{k-1} &= \frac{(-1)^{k-1}}{q_k q_{k-1}} + \frac{(-1)^{k-2}}{q_{k-1}q_{k-2}} \\ &= \frac{(-1)^{k-1}q_{k-2} + (-1)^{k-2}q_k}{q_k q_{k-1} q_{k-2}} \\ &= \frac{(-1)^{k-2}(q_k - q_{k-2})}{q_k q_{k-1} q_{k-2}} \end{aligned}$$

But from the continued fraction recursion formula, $q_k - q_{k-2} = a_k q_{k-1}$.
Thus,$c_k - c_{k-2} = \dfrac{(-1)^{k-2}(a_k q_{k-1})}{q_k q_{k-1} q_{k-2}} = \dfrac{(-1)^{k-2}(a_k)}{q_k q_{k-2}} = \dfrac{(-1)^k (a_k)}{q_k q_{k-2}}$ □

Corollary 3.0.0.9. *For* $1 \leq k \leq n, p_k$ *and* q_k *are relatively prime.*

Proof. Let $d = gcd(p_k, q_k)$.

Then d divides $p_{k-1}q_k - p_k q_{k-1} = (-1)^k, 1 \leq k \leq n$.

Hence,$d = 1 = gcd(p_k, q_k)$.So, p_k and q_k are relatively prime for all $1 \leq k \leq n$. To illustrate this property, consider the convergents of the continued fraction in Example 3.0.0.4 We find that

$gcd(p_1, q_1) = gcd(2, 1) = 1; gcd(p_2, q_2 = gcd(13, 7) = 1,$

$gcd(p_3, q_3) = gcd(15, 8) = 1; gcd(p_4, q_4 = gcd(58, 31) = 1,$

$gcd(p_5, q_5) = gcd(131, 70) = 1; gcd(p_6, q_6 = gcd(320, 171) = 1.$

Thus, p_k and q_k are relatively prime for each value of k, where $1 \leq k \leq 6$. □

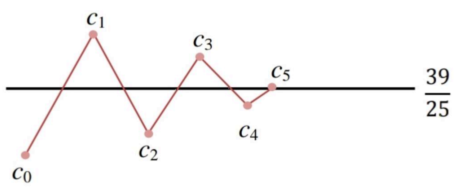
Example: 3.0.0.10. *Given* $[1,1,1,3,1,2]$ *is the continued fraction representation of the rational number* $\frac{39}{25}$*, find the convergents.*

Solution: Applying Theorem , we find the convergents of $[1,1,1,3,1,2]$:
$c_0 = 1, c_1 = 2, c_2 = \frac{3}{2}, c_3 = \frac{11}{7}, c_4 = \frac{14}{9}, c_5 = \frac{39}{25}$
Notice that:

1. The even convergents $1, \frac{3}{2}, \frac{14}{9}$ form an increasing sequence and approach the actual value $\frac{39}{25}$ from below, i.e. $c_0 < c_2 < c_4$.

2. The odd convergents $2, \frac{11}{7}, \frac{39}{25}$ form a decreasing sequence and approach the actual value $\frac{39}{25}$ from above, i.e. $c_1 > c_3 > c_5$.

3. The convergents c_k approach the actual value $\frac{39}{25}$ as k increases, where $0 \leq k \leq 5$. Moreover, they are alternatively less than and greater than $\frac{39}{25}$ except the last convergent c_5.
   Therefore, we conclude that $c_0 < c_2 < c_4 < \frac{39}{25} = c_5 < c_3 < c_1$.Following figure illustrates these notes.



These notes lead to the following theorem.

Theorem 3.0.0.11. *Let $c_0, c_1, \ldots, c_n$ be the convergents of the continued fraction.Then even–numbered convergents form an increasing sequence and odd -numbered convergents form a decreasing sequence. Moreover every odd-numbered convergent is greater in value than every even-numbered convergent. In other words:*
$c_{2m} < c_{2m+2}, c_{2m+3} < c_{2m+1}$ *and* $c_{2j} < c_{2r+1}, m, j, r \geq 0$.

Proof. By Corollary 3.0.0.8,

$$c_{2k} - c_{2k-2} = \frac{(-1)^{2k} a_{2k}}{q_{2k} q_{2k-2}}, k \geq 1. \tag{3.3}$$

Since a_k, q_k, $q_{k-2} > 0$, then $c_{2k} - c_{2k-2} \geq 0$. Hence,

$$c_{2k} \geq c_{2k-2}. \tag{3.4}$$

Thus, the even–numbered convergents form an increasing sequence $c_0 < c_2 < c_4, \ldots$. Similarly, by Corollary 3.0.0.8,

$$c_{2k+1} - c_{2k-1} = \frac{(-1)^{2k+1} a_{2k+1}}{q_{2k+1} q_{2k-1}}, k \geq 1. \tag{3.5}$$

and so

$$c_{2k-1} > c_{2k+1} \tag{3.6}$$

Thus, the odd–numbered convergents form an decreasing sequence $c_1 > c_3 > c_5, \ldots$. Finally, put $k = 2s+1, s \geq 0$ in Theorem 3.0.0.5 we obtain:

$$c_{2s+1} - c_{2s} = \frac{(-1)^{2s}}{q_{2s+1} q_{2s}} > 0,$$

With q_{2s+1},q_{2s},$(-1)^{2s} > 0$,we get

$$c_{2s} < c_{2s+1} \tag{3.7}$$

From (3.4),(3.6) and (3.7):

$$c_0 < c_2 < \ldots < c_{2k} < c_{2k+1} < c_{2k-1} < \ldots < c_3 < c_1, iFn = 2k+1$$

and

$$c_0 < c_2 < \ldots < c_{2k} < c_{2k+1} < c_{2k-1} < \ldots < c_3 < c_1, iFn = 2k$$

□

Chapter 4

Solving Linear Diophantine Equations

4.1 Linear Diophantine equation

Many puzzles, enigmas and trick questions lead to mathematical equations whose solutions are required to be integers.Such equations are called **Diophantine equations**, named after the Greek mathematician **Diophantus** who wrote a book about them.

Definition 4.1.0.1. Diophantine Equation is an algebraic equation in one or more unknowns with integral coefficients such that only integral solutions are sought. This type of equations may have no solution, a finite number or an infinite number of solutions.

Example: 4.1.0.2. *The following equations are Diophantine equations, where integral solutions are required for x, y and z.*
$3x+5y=7, x^2-y^2=z^2, -x+5y=9$

Definition 4.1.0.3. Linear Diophantine Equation "LDE" in two variables x and y is the simplest case of Diophantine equations and has the form $Ax+by=c$ where a, b and c are integers.

Example: 4.1.0.4. $3x+5y=1, 6x-4y=2, -5x+5y=8$ *are linear Diophantine equations in two variables.*

In this chapter we are interested in solving linear Diophantine equations in two variables. i.e., finding integral solutions of $Ax+by=c$. If a and b are both zeros, then the equation is either trivially true when $c=0$ or trivially false when $c=0$.
Moreover, if one of a or b equals zero, then the case is also trivial. So we omit these two cases and assume that both a and b are nonzero integers.
Geometrically, this equation represents a line in the Cartesian plane that is not parallel to either axis. Solutions of the equation $Ax+by=c$ are the points on the line with integral coordinates. Points with integral coordinates are called lattice points.
However, does every linear Diophantine equation Ax +by = c have an Integral solution? If not, what are the conditions necessary for a LDE to have a solution? The following theorem answers these questions.

Theorem 4.1.0.5. *Let a, b and c be integers with* $ab \neq 0$*. The linear Diophantine equation* $ax+by=c$ *is solvable if and only if* $gcd(a,b)$ *divides c. If* (x_0, y_0) *is a particular solution of the LDE, then all its solutions are given by:*

$$(x,y) = (x_0 + \frac{b}{gcd(a,b)}t, y_0 - \frac{a}{gcd(a,b)}t)$$

where t is an arbitrary integer.

Proof. First, we show that if the LDE $ax+by=c$ is solvable, then $gcd(a,b)$ divides c.
Suppose (x_1, y_1) is a solution of $ax+by=c$. Then, $ax_1+by_1=c$.
But $gcd(a,b)$ divides both a and b, then, $gcd(a,b)$ divides ax_1+by_1.
i.e. $gcd(a,b)$ divides c.
Next, we want to prove that if $gcd(a,b)$ divides c, then the LDE $ax+by=c$ is solvable.

Suppose that $gcd(a,b)$ divides c. Then $c = k.gcd(a,b)$ for some integer k.

Now, $\exists$ two integers m and n such that $ma+nb = gcd(a,b)$.

Multiply both sides of this equation by k to get: $kma+knb = kgcd(a,b) = c$.

Thus $x_0 = km$, $y_0 = kn$ is a solution of the LDE $ax+by = c$. Therefore, the LDE is solvable.

Now assume that (x_0, y_0) is a particular solution of $ax+by = c$, then
$x = x_0 + \frac{b}{gcd(a,b)}t$ and $y = y_0 - \frac{a}{gcd(a,b)}t, t \in \mathbb{Z}$ also satisfy the LDE:
$\Rightarrow ax+by = a(x_0 + \frac{b}{gcd(a,b)}t) + b(y_0 - \frac{a}{gcd(a,b)}t)$
$\Rightarrow ax+by = ax_0 + \frac{ab}{gcd(a,b)}t + by_0 - \frac{ab}{gcd(a,b)}t$
$\Rightarrow ax_0 + by_0 = c$
Thus, $(x_0 + \frac{b}{gcd(a,b)}t, y_0 - \frac{a}{gcd(a,b)}t)$ is a solution for any integer t.
Finally, we want to prove that any solution $(x?, y?)$ of the LDE $ax+by = c$ is of the form $(x_0 + \frac{b}{gcd(a,b)}t, y_0 - \frac{a}{gcd(a,b)}t)$for some integer t.
Since (x_0, y_0) and $(x?, y?)$ are solutions of $ax+by = c$, then:

$ax_0 + by_0 = c$ and $ax' + by' = c$. That is $ax_0 + by_0 = ax' + by'$

Hence,

$$a(x' - x_0) = b(y_0 - y') \tag{4.1}$$

Dividing both sides of this equation by $gcd(a,b)$, we have:
$\frac{a}{gcd(a,b)}(x' - x_0) = \frac{b}{gcd(a,b)}(y_0 - y')$
Note that $\frac{a}{gcd(a,b)} = a_1$ and $\frac{b}{gcd(a,b)} = b_1 \in \mathbb{Z}$ are relatively prime. So, we obtain $a_1(x' - x_0) = b_1(y_0 - y')$

This shows that b_1 divides $a_1(x' - x_0)$. But, since ,$gcd(a_1, b_1) = 1$, then b_1 divides $(x' - x_0)$.

Hence,

$$x' - x_0 = b_1 t = \frac{b}{gcd(a,b)}t, t \in \mathbb{Z} \tag{4.2}$$

That is $x' = x_0 + \frac{b}{gcd(a,b)}t$.
Similarly, $y' = y_0 - \frac{a}{gcd(a,b)}t$.
Thus, every solution $(x_0 + \frac{b}{gcd(a,b)}t, y_0 - \frac{a}{gcd(a,b)}t), t \in \mathbb{Z}$ of the linear Diophantine equation is of the desired form. □

Note: We conclude from this theorem that every solvable linear Diophantine equation $ax + by = c$ has infinitely many solutions. They are given by the general solution:

$$x = x_0 + \frac{b}{gcd(a,b)}t$$

and

$$y = y_0 - \frac{a}{gcd(a,b)}t$$

where t is an arbitrary integer.
By giving different values to t, we can find any number of particular solutions.

Corollary 4.1.0.6. *Suppose that $gcd(a,b) = 1$. Then the LDE $ax + by = c$ is solvable for all integers c. Moreover, if (x_0, y_0) is a particular solution, then the general solution is $x = x_0 + bt, y = y_0 - at, t \in \mathbb{Z}$.*

Example: 4.1.0.7. *Determine whether the following LDE?s are solvable.*

a)$6x + 18y = 30$

b)$2x + 3y = 7$

c)$6x + 8y = 15$

d)$59x - 29y = -5$

Solution: a)$gcd(6,18) = 6$ which divides 30, then the LDE $6x + 18y = 30$ is solvable.
b)$gcd(2,3) = 1$, so $2x + 3y = 7$ is solvable.

c)$gcd(6,8)=2$, but 2 does not divide 15, then $6x+8y=15$ is not solvable.

d)$gcd(59,29)=1$, so $59x-29y=-5$ is solvable.

4.2 How to find a particular solution to the LDE $ax+by=c$?

It is not difficult to find a particular solution. One of the methods that are used is the **Euclidean Algorithm method**.

To find a particular solution to a solvable LDE $ax+by=c$, we follow these steps.

Step 1: Write (a,b) as a linear combination of a and b. That is:

$$ar_0+bs_0=gcd(a,b)$$

where r_0 and s_0 are integers.

Step 2: multiply both sides of this equation by c and then divide it by $gcd(a,b)$:

$$a(\frac{r_0\times c}{gcd(a,b)})+b(\frac{s_0\times c}{gcd(a,b)})=c$$

Step 3: we obtain $(x_0=\frac{r_0\times c}{gcd(a,b)},y_0=\frac{s_0\times c}{gcd(a,b)})$ as a particular solution the linear Diophantine equation.

Note: LDE's were known in ancient China and India as applications to Astronomy and puzzles. The following puzzle is due to the Indian Mathematician Mahavira (ca. A.D. 850).

Example: 4.2.0.1. *Twenty-three weary travelers entered the outskirts of a lush and beautiful forest. They found* 63 *equal heaps of plantains and seven single fruits, and* *divided them equally.Find the number of fruits in each heap and the number of fruits received by each traveller.*

Solution:

Let x denote the number of fruits in a heap and y denote the number of fruits received from each traveller.

Then we get the linear Diophantine equation:

$$63x+7=23$$

i.e.

$$63x-23y=-7$$

Here x and y must be positive, so we are looking for positive integral solutions of the LDE.

Since $gcd(63,23)=1$, then, by Corollary 4.1.0.6, the LDE is solvable.

To find a particular solution, we apply the Euclidean Algorithm:

$$63=2\times 23+17 \tag{4.3}$$

$$23=1\times 17+6 \tag{4.4}$$

$$17=2\times 6+5 \tag{4.5}$$

$$6=1\times 5+1 \tag{4.6}$$

$$5=5\times 1$$

Now, use equations (4..3), (4.4), (4.5) and (4.6) in reverse order to get:

$$\begin{aligned}
1 &= 6-1\times 5\\
&= 6-1\times 5\\
&= 6-1(17-2\times 6)\\
&= 3\times 6-1\times 17\\
&= 3\times(23-1\times 17)-1\times 17\\
&= 3\times 23-4\times 17\\
&= 3\times 23-4\times(63-2\times 23)\\
&= 11\times 23-4\times 63
\end{aligned}$$

Thus, $63(-4)-23(-11)=1$. Multiplying both sides of Thus equation by -7, we have:

$63(-4\times -7)-23(-11\times -7)=-7$.

That is : $63(28)-(23)(77)=-7$.

Therefore, $(28,77)$ is a particular solution of $63x-23y=7$.

By Corollary 4.1.0.6,the general solution of the LDE is:

$(x,y)=(28-23t,77-63t)$, t is arbitrary integer.

Finally, since $x>0$ and $y>0$, then:

$28-23t>0$ and $77-63t>0$

$\Rightarrow t<\dfrac{28}{23}\approx 1.217$ and $t<\dfrac{77}{63}\approx 1.222$

So, $(x,y)=(28-23t,77-63t)$,where t is an integer less than or equal to 1, is a positive integral solution of the LDE $63x+7=23y$.

4.3 Continued Fractions and Linear Diophantine Equations

Another way to find a particular solution to a solvable LDE $ax+by=c$ is the continued fraction method. Our approach to explain this method will be a step-by-step process until we'll be able to find integral solutions to any solvable LDE of the form $Ax+by=c$. This method depends on the formula stated in Lemma 3.0.0.6

4.3.1. Solving the LDE $ax+by=1$; a and b are positive relatively prime integers

To solve this LDE, we express $\frac{a}{b}$ as a finite simple continued fraction.

$$\frac{a}{b}=[a_0,a_1,...,a_{n-1},a_n]$$

Then we calculate the convergents $c_0,c_1,c_2,\ldots,c_{n-1},c_n$. The last two convergents $c_{n-1}=\frac{p_{n-1}}{q_{n-1}}$ and $c_n=\frac{p_n}{q_n}$ with the relation stated in Lemma 2.1 are the key to the solution: $p_{n-1}q_n-p_nq_{n-1}=(-1)^n$

With $p_n=a$ and $q_n=b$ we have: $bp_{n-1}-aq_{n-1}=(-1)^n$

Or

$$a(-1)^{n-1}q_{n-1}+b(-1)^n p_{n-1}=1$$

Comparing this equation with the LDE $ax+by=1$, we conclude that:

$(x_0=(-1)^{n-1}q_{n-1},y_0=(-1)^n p_{n-1})$ is a particular solution of $ax+by=1$.

Therefore, if n is even, then $(x_0,y_0)=(-q_{n-1},p_{n-1})$ and if n is odd, then $(x_0,y_0)=(q_{n-1},-p_{n-1})$.

We have four cases $ax\pm by=c$ according to the sign of both a and b:

Case 1: $a > 0$ and $b > 0$

Equation : $ax + by = 1$

Solution : $(x_0, y_0) = ((-1)^{n-1}q_{n-1}, (-1)^n p_{n-1})$

Case 2: $a > 0$ and $b < 0$

Equation : $ax - by = 1$

Solution : $(x_0, y_0) = ((-1)^{n-1}q_{n-1}, (-1)^{n-1} p_{n-1})$

Case 3: $a < 0$ and $b > 0$

Equation : $-ax + by = 1$

Solution : $(x_0, y_0) = ((-1)^n q_{n-1}, (-1)^n p_{n-1})$

Case 4: $a < 0$ and $b < 0$

Equation : $-ax - by = 1$

Solution : $(x_0, y_0) = ((-1)^n q_{n-1}, (-1)^{n-1} p_{n-1})$

Example: 4.3.0.1. *Solve the LDE* $204x + 91y = 1$ *using continued fraction method.*

Solution:

First of all, $gcd(204, 91) = 1$, then the LDE is solvable.
To find a particular solution, we represent $\frac{204}{91}$ as a finite simple continued fraction.

$$\frac{204}{91} = [2, 4, 7, 3]$$

Then we construct the convergent table as shown in the following table

From this table $n = 3, p_{n-1} = p_2 = 65$ and $q_{n-1} = q_2 = 29$.

k	-2	-1	0	1	2	3
a_k			2	4	7	3
p_k	0	1	2	9	65	204
q_k	1	0	1	4	29	91
c_k			$\frac{2}{1}=2$	$\frac{9}{4}$	$\frac{65}{29}$	$\frac{204}{91}$

Thus, a particular solution to the LDE $204x - 91y = 1$ is:

$$x_0 = (-1)^2 \times 29 = 29$$

$$y_0 = (-1)^2 \times 65 = 65$$

Finally, by Corollary 2.3, the general solution is:

$$x = 29 + (-91)t = 29 - 91t$$

$$y = 65 - 204t$$

,t is an arbitrary integer.

Now, what if we replace the number 1 in any LDE in the cases above by another integer "c"? In other words, what is the particular solution of the LDE $ax+by=c$, $gcd(a,b)=1$?

4.3.2. Solving the LDE $ax+by=c$, where a, b and c are integers, $gcd(a,b)=1$.

The first step in solving this LDE is to find a particular solution (x_0, y_0) of the LDE $ax+by=1$ using the formulas we've studied and derived according to the case we have. From $ax_0+by_0=1$, we have: $a(cx_0)+b(cy_0)=c$.

Thus, (cx_0, cy_0) is a particular solution of the LDE $ax+by=c$.

4.3.3. Solving the LDE $Ax+By=C$, where A, B and C are integer $gcd(A,B) \neq 1$.

As we have proved in Theorem 2.8, the LDE $Ax+By=C$ is solvable if and only if $gcd(A,B)$ divides C. If so, divide both sides of the LDE by $gcd(A,B)$ to reduce it to the equation of the form:

$$ax+by=c \tag{4.7}$$

where a , b and c are integers, $gcd(a,b)=1$.

The solution of equation (3.5) has been discussed and is easy to solve.

Finally, any solution of this equation is automatically a solution of the original equation $Ax+By=C$.

Example: 4.3.0.2. *Solve the LDE* $65x-182y=299$ *using continued fraction method.*

Solution:

Here, $gcd(65,182)=13$, and 13 divides 299. So, the LDE $65x-182y=299$ is solvable.

Divide both sides of the equation $65x-182y=299$ by 13 to get the LDE $5x-14y=23$.

Now, we find a particular solution to the LDE $5x-14y=1$.

$\frac{5}{14}=[0,2,1,4]$.The following table is the convergent table.

From this table : $n=3, p_2=1, q_2=3$.

k	-2	-1	0	1	2	3
a_k			0	2	1	4
p_k	0	1	0	1	1	5
q_k	1	0	1	2	3	14
c_k			0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{14}$

Thus, a particular solution to the LDE $5x-14y=1$ is:

$$x_0 = (-1)^2 \times 3 = 3$$

$$y_0 = (-1)^2 \times 1 = 1$$

So, $(23x_0, 23y_0) = (69, 23)$ is a particular solution to the LDE $5x-14y=23$.
Finally, the general solution is:

$$x = 69 + (-14)t = 69 - 14t$$

$$y = 23 - 5t$$

,t is an arbitrary integer.

Note: The continued fraction method for finding a particular solution for a solvable LDE is equivalent to the Euclidean algorithm method. This is due to the fact that the continued fraction of $\frac{a}{b}$ is derived from the Euclidean. However, generating the convergents using the recurrence relations to solve a LDE is quicker than to find Euclidean algorithm equations and then use them in reverse order.

Chapter 5

Applications

5.1 Calendar Construction

The construction of a calendar that accurately determines the seasons by counting the days is an important issue for human beings since ancient time. Seasons depend on the revolution on the Earth around the Sun while days depend on the rotation of the Earth about its axis.

The Julian calendar used the approximation $365\frac{1}{4}$ days for 1 year. It was carried out by extending one extra day every four "common" years to form a "leap" year. After 1600 years, the error accumulated to 10 days. Pope Gregory XIII revised it by omitting one leap year every century except every fourth century.This is based on the approximation $365\frac{97}{400}$ days for a year. The Gregorian calendar is more accurate yet simple to use. By definition, a tropical year (the time it takes for the Earth to revolve around the Sun once) is

$$\frac{315569259747}{864000000} = [365, 4, 7, 1, 3, 5, 6, 1, 1, 3, 1, 7, 7, 1, 1, 1, 1, 2]$$

days long and the construction of a calendar reduces to selecting an approximation of the error

$$c = \frac{7750361}{32000000} = [0,4,7,1,3,5,6,1,1,3,1,7,7,1,1,1,1,2]$$

between the tropical year and the common year. The first few convergents of c are in the following table:

k	$\frac{p_k}{q_k}$
0	$\frac{0}{1}$
1	$\frac{1}{4}$
2	$\frac{7}{29}$
3	$\frac{8}{33}$
4	$\frac{31}{128}$
5	$\frac{163}{373}$

So the Julian calendar is just a realization of the first convergent. If our notation system was based on powers of 2 instead of powers of 10, it could be possible to design a calendar in which leap years occur every fourth year with every thirty-second leap year omitted. The annual error would be

$$c - \frac{31}{188} \approx 0.00001128$$

which amounts to be the loss of one day every hundred thousand years. However, this is not as easy to use as the Julian calendar. In fact, nobody used this calendar. Actually it is not proposed anywhere in the world besides in Russia by Russian astronomer Medler in 1864.

5.2 Music

In music, an octave is the interval between one musical note and another with double or half its frequency. The ancient Greeks realized that sounds which have frequencies in rational proportion are perceived as harmonious. The great scientist and philosopher Pythagoras noticed that subdividing a vibrating string into rational proportions produces consonant sounds. This is because the length of a string is inversely proportional to its fundamental frequency. If basic frequencies a and b have ratio a/b = m/n for some small integers m and n, the sound will be consonant as they will have overtones in common. The ancient Greeks found the consonance of "octaves" $\frac{a}{b} = \frac{2}{1}$) and "perfect fifths" ($a/b = 3/2$). They combined these to get a scale, but an approximation is needed to keep the scale finite. They had to find a power of 3/2 to approximate a power of 2.

$$\left(\frac{3}{2}\right)^m = 2^n \Rightarrow \frac{3}{2} = 2^{n/m} \Rightarrow \frac{b}{a} = \frac{log\left(\frac{3}{2}\right)}{log(2)}$$

Figure 5.1: Octave [36]

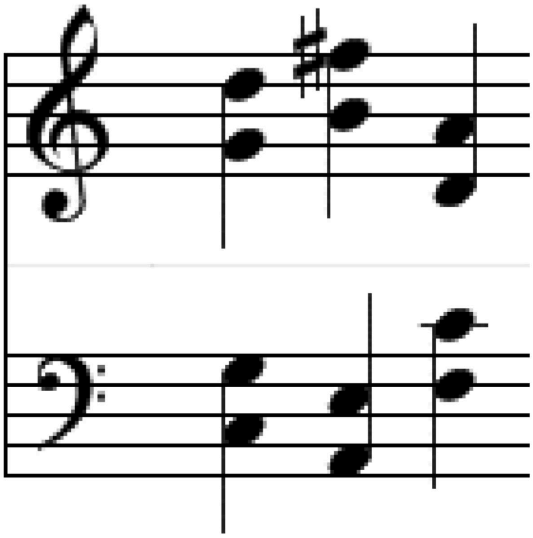


Figure 5.2: Perfect fifth [38]

It turns out that,

$$\frac{log\left(\frac{3}{2}\right)}{log(2)} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\ddots}}}}}}}$$

k	$\frac{p_k}{q_k}$
1	$\frac{1}{2}$
2	$\frac{3}{5}$
3	$\frac{7}{12}$
4	$\frac{24}{41}$

It is the approximation $\frac{7}{12} = 0.583$ which suggests an octave of 12 steps, with a perfect fifth equal to 7 semitones. If 5 is chosen then there will be too few notes, while choosing 41 gives too many notes. Actually the error of choosing $\frac{7}{12}$ amounts to

$$\frac{log_2\left(\frac{3}{2}\right) - \frac{7}{12}}{log_2\left(\frac{3}{2}\right)} \approx 0.002785$$

.

which is smaller than 0.3%.

Bibliography

[1] Rana Bassam Badawi. "On continued fractions and its applications". PhD thesis. An-Najah National University, 2016.

[2] Yu Tung Cheng. "Continued Fractions". PhD thesis. Cornell University, 2007.

[3] Thomas Koshy and Thomas Koshy. "Continued Fractions". In: *Pell and Pell–Lucas Numbers with Applications* (2014), pp. 57–78.

[4] Sagar Panda. "Diophantine equation". PhD thesis. 2011.

[5] R Walker. "Continued Fractions. By CD Olds. Pp. viii, 162. 1.95.1963.(*RandomHouseandtheLWSinge* In: *The Mathematical Gazette* 48.366 (1964), pp. 466–466.

[6] John J Watkins. *Number theory: a historical approach*. Princeton University Press, 2013.