GRACEFUL LABELING OF TREES

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M.Sc. in Mathematics

by

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APRIL 2024

Examined by: Seal of the School

DECLARATION BY STUDENT

I declare that this Dissertation report entitled, "GRACEFUL LABELING OF TREES" has been prepared by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Jessica Fernandes e Pereira and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University or its authorities will be not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Graceful Labeling Of Trees" is a bonafied work carried out by Miss Pallavi Uttam Naik under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences, Goa University.

Signature : ____________________

Supervisor : Dr. Jessica Fernandes e Pereira

Date: MAY 08, 2024

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT-651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is "GRACEFUL LABELING OF TREES." This survey is divided into 4 chapters. Each chapter has its own importance. The chapters are divided and defined in a logical and systematic manner to cover all the topics.

FIRST CHAPTER :

Introduction and history of this dissertation report in based on graph labeling in graph theory. We will also be discussing about problem of graph labeling and the origin of the problem.

SECOND CHAPTER:

In this chapter we have tried to give brief idea on recent results on graceful labeling. Some classes of trees are shown to be graceful like paths, caterpillar, etc.

THIRD CHAPTER:

In this chapter we have tried to give brief idea on new classes of trees like superstar extended superstar which are gracefully labeled.

FOURTH CHAPTER:

This chapter deal with transformed trees and transfer. The main aim is to determine the given tree to transformed trees and some classes of trees which are graceful.

ACKNOWLEDGEMENTS

First of all I would like to thank my supervisor, Dr. Jessica Fernandes e Pereira for introducing me to the interesting and diverse world of Graph Theory and its applications. Without her continuous supervision, guidance and advice it would not have been possible to complete this dissertation. I am especially grateful to her for giving her time whenever needed for her encouragement and help at times of disappointment and always providing continuous support in my effort.

I also want to thank the other members of department: Dr.M Kunhanandan, Dr.Manvendra Tamba, Dr. Mridini Gawas and Mr. Brandon Fernandes for helping me in understanding and learning to operate LaTeX software.

Last but not the least I am grateful to my parents, guardians, friends and family members for their support and encouragement during this period.

ABSTRACT

A tree is a connected acyclic graph on n vertices and $n-1$ edges. Graceful labeling of a tree is a labeling of its vertices with the numbers from 0 to $n-1$ so that no two vertices share a label, labels of edges being absolute difference of the labels of its end points are also distinct. There is a famous conjecture named Graceful tree conjecture or Ringel-Kotzig Conjecture that says "All trees are graceful" Almost 50 year old conjecture is yet to be proved. However. researchers have been able to prove that many classes of trees are graceful. In this dissertation we stydy that the classes of Superstar and Extended Superstar are graceful. A tree with one internal node and k leaves is said to be a star $S_{1,k}$ or a complete bipartite graph $K_{1,k}$. Superstar is a tree that consists of several stars all connected to a single star by sharing their leaves. If we remove all the leaves of a superstar then we will get a spider tree which has already been proved to be graceful. Extended superstar is a tree that consists of several superstars all connected to a single star by sharing their leaves. We also study that extended superstars are graceful.

KEYWORDS: Trees, Graceful labeling, Superstars, Spiders, Extended superstars.

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Chapter 1

INTRODUCTION

Graph theory deals with the study of graphs, which are mathematical structures representing a set of vertices or objects connected by any set of lines these lines are called edges. The study of graph is a very important tool for the applications of different subjects, such as chemistry, biochemistry, computer science, communication network, and coding theory.

Graph is collection of points called vertices and lines between those points called edges. Graph labeling is the assignment of labels where the vertices or edges or both are assigned real values subject to certain conditions have often been motivated by their use in various applied fields and their mathematical interest. Most graph labeling methods trace their origin to one introduced by Rosa in 1967 or given by Graham and Sloane in 1980. Graph labeling was first introduced in the mid 1960s.

Graph labeling where the vertices are assigned values subject to certain conditions have often been motivated by practical problems. Labeled graphs serve as useful mathematical models for a broad range of applications such as Coding theory which includes the design of good radar type codes, syncset codes, missile guidance codes and convolution codes with optimal auto correlation properties. They facilitate the optimal nonstandard encoding of integers.

An injection mapping f:$V \to \{0, 1, ..., m\}$ is said to be graceful if the induced edge function is defined by g_f (uv)=$|f(u) - f(v)|$ whenever $uv \in E$ and the resulting edge labels are all distinct and are from the set $\{1, 2, ..., m\}$. The graph which admits such a labeling is called a graceful graph. In other words, A graceful labeling of a graph with m edges is a labeling of its vertices with some subset of the integers from 0 to m inclusive, such that no two vertices share a label, and each edge is uniquely identified by the absolute difference between its endpoints, such that this magnitude lies between 1 and m inclusive. TREE: Connected acyclic graph is a tree. GRACEFUL LABELING OF A TREE is a labeling of its vertices with the numbers from 0 to $n-1$ so that no two vertices share a label.

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Many classes of trees have been shown to be graceful. However, it has not yet been possible to prove the conjecture for all trees. A lot of work have been done by many researchers towards proving this conjecture. So far some special classes of trees have been shown to be graceful. For example, paths, caterpillars, symmetrical trees, spider trees, star trees, banana trees etc.

There are many graph labeling techniques like Graceful Labeling, Harmonious Labeling, Magic-type Labeling, Antimagic-type Labeling, Prime and Vertex Prime Labelings, Edge-graceful Labelings, Radio Labelings, Line-graceful Labelings, k-sequential Labelings, Product and Divisor Cordial, Edge Product Cordial, Difference Cordial Labelings, Prime Cordial labelings, Geometric labelings, Mean Labelings, Irregular Total Labelings, Square Sum Labelings and Square Difference Labeling and so on. However we shall concentrate on graceful labeling that has received attention of a wider scientific community. The name "Graceful Labeling" has come up thanks to Solomon W. Golomb.

In graph theory, a major unproven conjecture is the Graceful Tree conjecture (GTC) or Ringel–Kotzig conjecture, named after Gerhard Ringel and Anton Kotzig which hypothesizes that "all trees are graceful". The Ringel-Kotzig conjecture is also known as the "Graceful Labeling Conjecture".

Chapter 2

CLASSES OF GRACEFUL TREES

2.1 Path

A path is a tree whose vertices can be ordered as $v_1, v_2, ..., v_n$ and the edges are v_i, v_{i+1} for $i = 1, 2, ..., n-1$. This sequence represents a connected series of vertices in the tree structure.

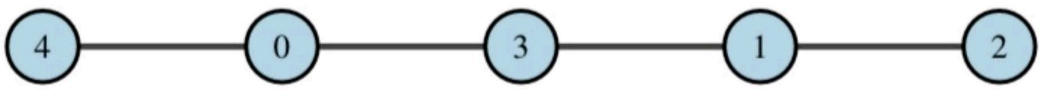

Figure 2.1: Path

Theorem 2.1.0.1. *All paths are graceful.*

Proof: Let P_n be the path with n edges and $n+1$ vertices. Label P_n by starting at one *end of the path and alternate between the smallest and largest remaining label along the path.*

Example: 2.1.0.2. *Graceful labeling of paths P_4 and P_9 is shown below*

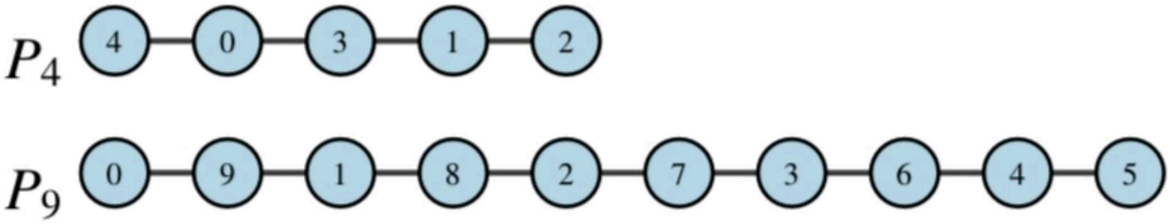

Figure 2.2: Gracefully Labeled Path

2.2 Caterpillars

A caterpillar is a tree where after removing all its leaves the remaining graph is a path.

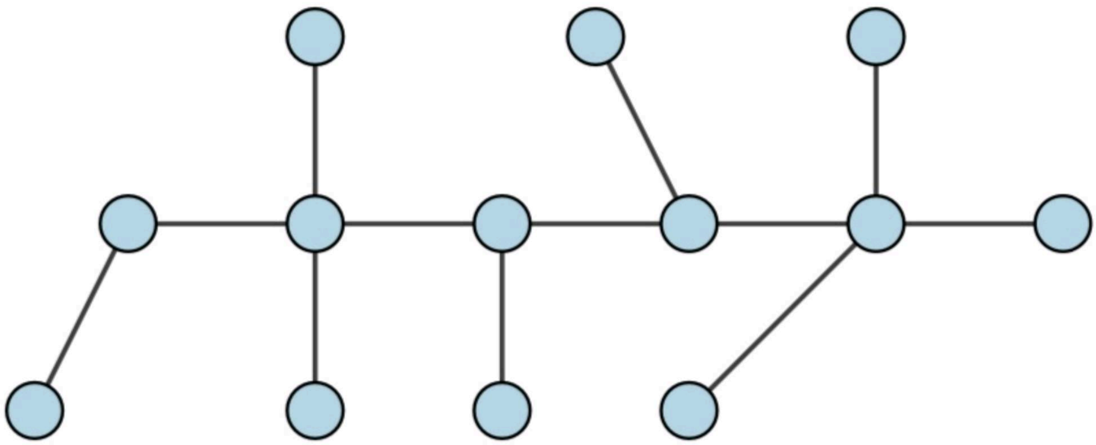

Figure 2.3: Caterpillar

Theorem 2.2.0.1. *All caterpillars are graceful.*

Proof. Let C be a caterpillar on *n* vertices. A caterpillar is labeled from one end with 0 and its adjacent vertices are labeled using unused largest label ending in labeling the next vertex on the path with the smallest of the largest labels used. Its adjacent vertices are labeled using the smallest so far unused labels alternately. While we are labeling the vertices largest unused edge labels are generated. An example of gracefully labeled caterpillar is shown in the figure 2.4. □

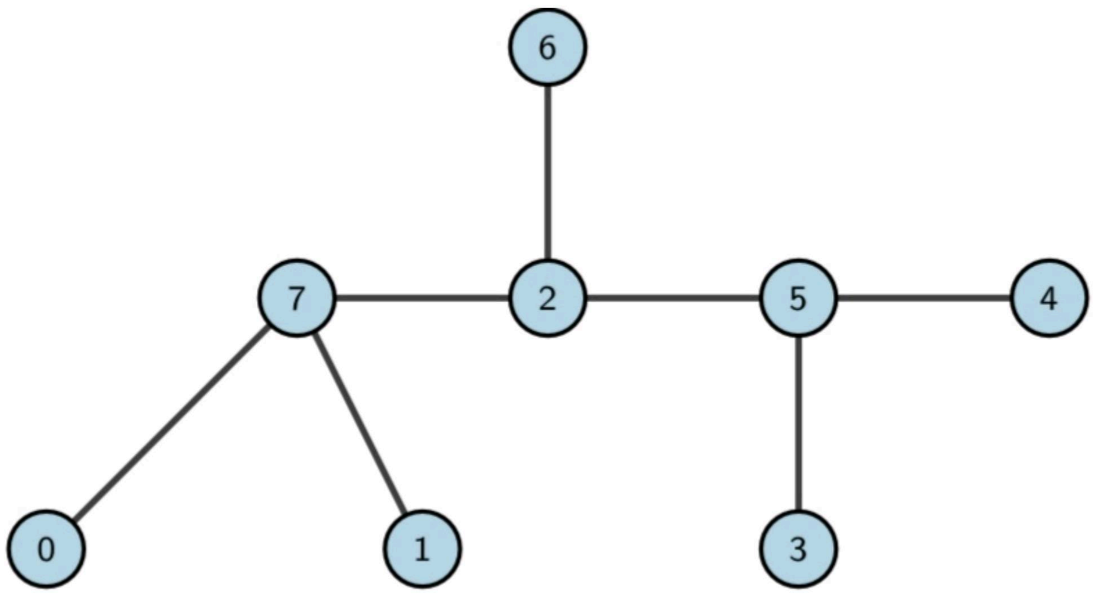

Figure 2.4: Gracefully Labeled Caterpillar

2.3 Super Caterpillars

Let T_0 be any arbitrary caterpillar and T_i $i = 1, ..., k$ be caterpillars with $|Ti| = m$ number of vertices and sum total of vertices is the same in odd levels of all pairs T_{2i+1} and T_{2i+2}. In case k being an odd number one caterpillar will be without a pair. Let one end of each caterpillar be joined to the vertex v by an edge. Then the resulting tree is called a super caterpillar. An example of supercaterpillar is shown in the figure 2.5 and 2.6.

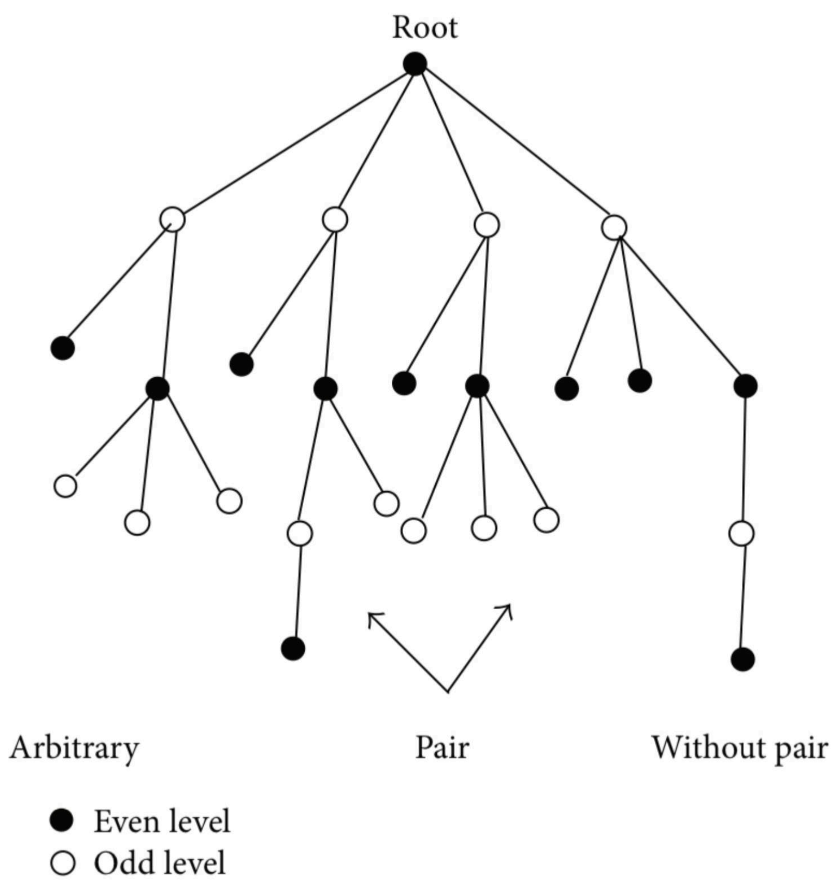

Figure 2.5: Supercaterpillar with Odd k

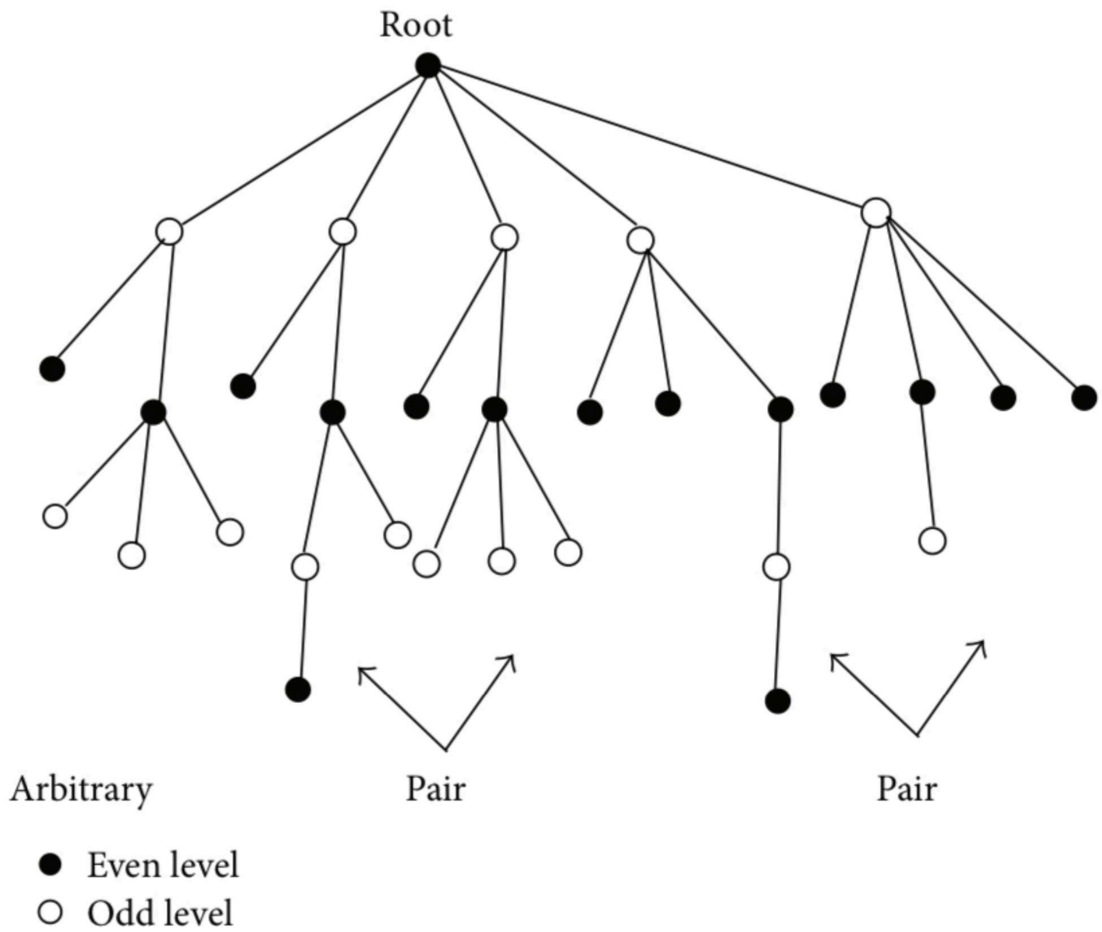

Figure 2.6: Supercaterpillar with Even k, with an Arbitrary Caterpillar Joined with a Root

Theorem 2.3.0.1. *All super caterpillars are graceful.*

Proof. Let us assume that we have k caterpillars joined to V and that each caterpillar has the same number of m vertices also a total of s_i vertices in odd levels s_i being equal for each pair of caterpillars $T_{2_{i+1}}$ and $T_{2_{i+2}}$. We also denote the vertex of T_j connected to V by V_j. Let us label V by 0. Now vertices of caterpillars will be in turn labeled using the smallest and largest labels following caterpillar labeling schemes so that edge numbers are generated in descending order. The endpoint of T_1 connected to V is labeled k_m, and we will use up m labels in T_1 of which s_1 large namely, $k_m, k_{m-1}, ..., k_m - s_1 + 1$ and m-s_1 small labels namely $1, ..., m$-s_1.

Thus the last edge label created is k_m - s_1 - (m-s_1) =$(k-1)m+1$. In the next caterpillar, both the smallest and largest labels will differ by 1 from the labels used in the previous tree. One vertex will get label k_m -s_1 + 1 and the other one m - s_1 resulting in edge label k_m-s_1 - (m-s_1 +1)=$(k-1)m-1$ missing the label$(k-1)m$. We are going to use up m-s_2 large and s_2 small labels in T_2 in bottom up way.

Note that s_2=s_1 T_2 will be labeled in such a way that V_2 ends getting the smallest unused label, that is, label m. In this way in tandem vertices of odd levels will be numbered by big and small numbers, respectively, for odd and even indexed caterpillars. This will result in label (v_i)= $(k-[(i-1)/2])m$ for odd i whereas label (v_i)=$im/2$ is for even ones. For each pair of caterpillars labels $(k+[(k-1)/2])m$ and $im/2$ will be missing which will be generated on edges incident to vertex v.

Now assume that we have one arbitrary caterpillar T_0 and any k caterpillars having the same number of vertices m and that the last caterpillar has m vertices but not necessarily having equal number of vertices in odd levels as all previous pairs have. One end of backbone of each caterpillar is connected to V. So total number of edges in the tree will be $k_m + m_0$. Now start labeling T_0 in such a way that we end up labeling V by s_0 shifting label of V by s_0. Large labels have also been shifted by m_0-$(m_0)-(s_0)$, thus producing $(k-[(i-1/2)])m$ labels on the edge incident to V, which we have missed in moving from one caterpillar to the next. Now for T_1 we are left with numbers from s_0+1 to k_m+m_0-m+s_0= $(k-1)m$+m_0+s . This way all V, adjacent to root V,will be labeled by multiples of m. After we have completed labeling all pairs of caterpillars, we will be left with consecutive integers to label the vertices of the unpaired caterpillar and generate the remaining smallest possible edge labels. An example of gracefully labeled supercaterpillar is shown in the figure 2.7. □

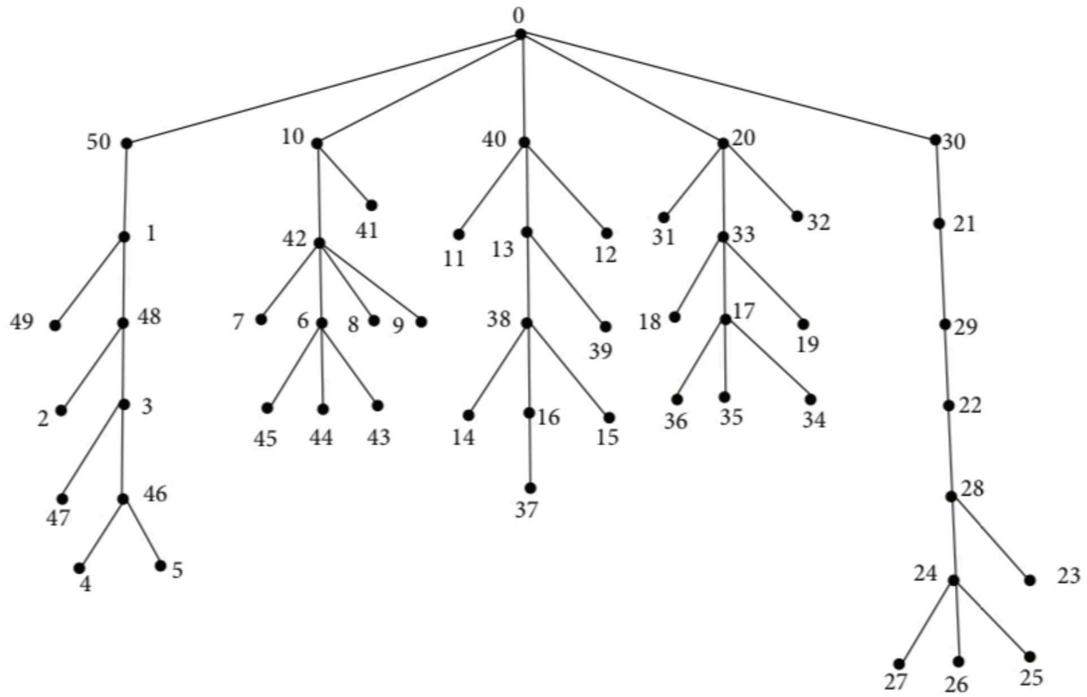

Figure 2.7: Gracefully Labeled Super Caterpillar

2.4 Extended Super Caterpillars

Let there be an even number k_p caterpillars, each having m vertices and sum total number of vertices in odd (or even) levels of those caterpillars are the same. These caterpillars are grouped in k groups each having p caterpillars. Let the group i of caterpillars be connected to a vertex v_i which is connected to vertex v. Then the resulting tree is called a extended super-caterpillar. An example of extended super caterpillar is shown in the figure 2.8.

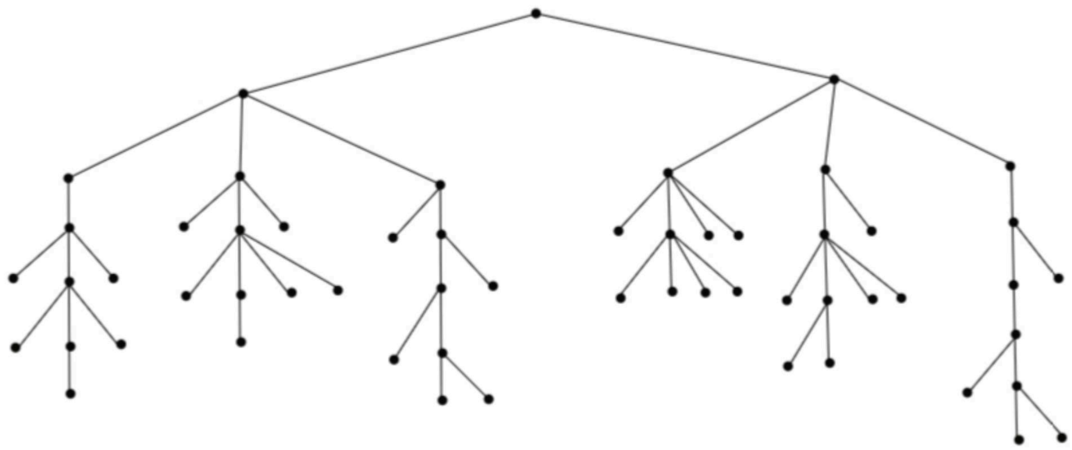

Figure 2.8: Super Caterpillar

Theorem 2.4.0.1. *All extended super caterpillar are graceful.*

Proof. Theorem 2.3.0.1 asserts that we can label each group k of caterpillars with root v_i for all $i = 1,2.....,k$ gracefully. Root v, connected to all v_i, is labeled 0 and labels of the remaining vertices in the tree are incremented by one. Let s be the total number of vertices in each group. For group i, we will be labeling all odd level vertices by adding an offset $(k-1)i$ and for even level vertices $(i-1)s$ where $i = 1,2,3,.....,k$. So v_i gets label $(k-1)i$, and therefore edge (v,v_i) gets label $(k-i)s$.

We will get the consecutive edge labels having the largest differences $(k-0)s-1$ to $(k-2)s+1$ to in groups T_i and T_{k-i+1} for $i = 1$ except $(k-1)s$ and k_s . Similarly, next consecutive differences $(k-2)s-1to(k-4)s+1$ are found in T_i and T_{k-i+1} for $i = 2$ again missing $(k-3)s$ and $(k-2)s$.

So in general we can say that differences $(k-2(d-1))s-1$ to $(k-2d)s+1$ are generated while missing $(k+2d+1)s$ and $(k+2(d-1))s$ where $i = 1,2,.....,[k/2]$. These numbers are multiples of s which have already been generated in edges (v,v_i) $i = 1,.....,k$.

An example of gracefully labeled extended super caterpillar is shown in the figure 2.9. □

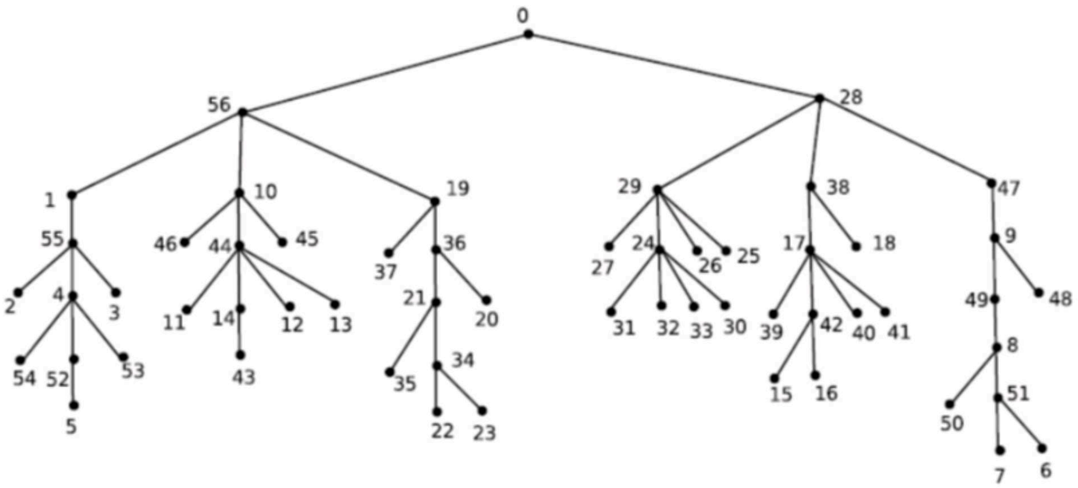

Figure 2.9: Gracefully Labeled Extended Super Caterpillar

2.5 Symmetrical Trees

A rooted tree in which every level contains vertices of same degree is called symmetrical tree.

Theorem 2.5.0.1. *All symmetrical trees are graceful.*

Proof. The proof has been shown by induction on the number of layers that all symmetrical trees are graceful and there exists a graceful labeling which assigns the number 1 to the root.

If T is a symmetrical tree with 0 layers then it consists of 0 edges and just one vertex, and clearly there is a graceful labeling which assigns 1 to that vertex. Suppose we have proved that for some $l > 0$ all symmetrical trees with $\leq l-1$ layers are graceful and each of them has a graceful labeling which assigns the number 1 to the root.

The idea of the induction step is to consider a rooted symmetrical tree for which we know that its k children $T_1, T_2, ..., T_k$ are graceful. We label the children with their graceful labeling and then add certain numbers to each of the vertices .

We order the children from left to right. Then if n is the number of vertices in each child, we start from the 0^{th} layer of the children and add $(k-1)n$ to the root of T_1, $(k-2)n$ to the root of T_2,....., and 0 to the root of the k^{th} one. Then for the first layer we start from right to left and add $(k-1)n$ to each of the vertices in the 1st layer of T_k then we add $(k-2)n$ to each of the vertices in the 1st layer of T_{k+1},....., and 0 to each of the vertices in the first layer of T_1.

So then we go on with the second layer and we start from left to right and so on until we finish with the last layer. Then we write $nk+1$ on the root of the new tree. Then we do the transformation $x \rightarrow nk+2+x$ to each of the vertices so that we can have 1 at the root and the resulting labeling as we show in the sequel is graceful. An example of gracefully labeled symmetrical tree is shown in the figure 2.10. □

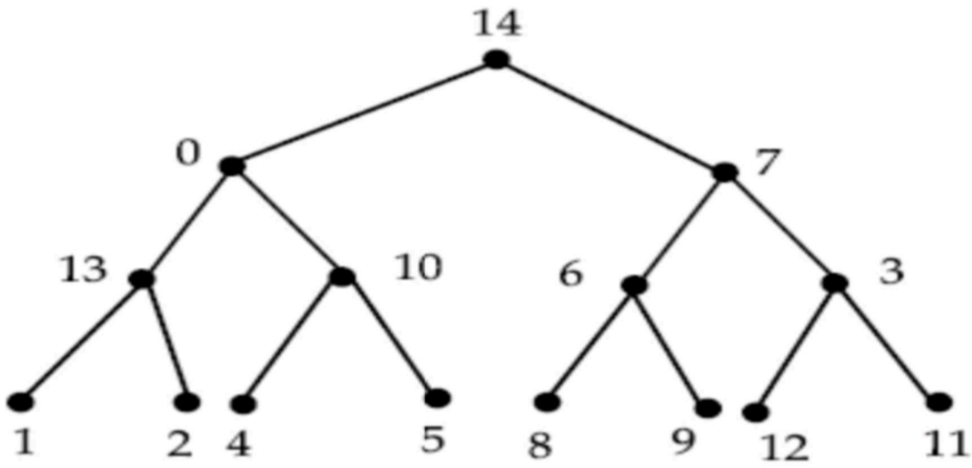

Figure 2.10: Gracefully Labeled Symmetrical Trees

2.6 Spider Trees

A spider tree is a tree with at most one vertex of degree greater than 2. If such a vertex exists, it is called the branch point of the tree. A leg of a spider tree is any one of the paths from the branch points to a leaf of the tree.

Step 1: Label the centre vertex with label 0.

Step 2: Follow any one path and label the first vertex of that path which is adjacent to the centre with maximum label.

Step 3: Then choose the next path and label the first vertex of that path which is adjacent to centre with least label among all.

Step 4: For the next path we label the first vertex which is adjacent to the centre with maximum label which is unused.

Step 5: For this path we label the first vertex which is adjacent to the centre with minimum label which is unused.

Step 6: Follow the above procedure and label the remaining vertices of the path.

Step 7: An example is shown in the figure 2.11.

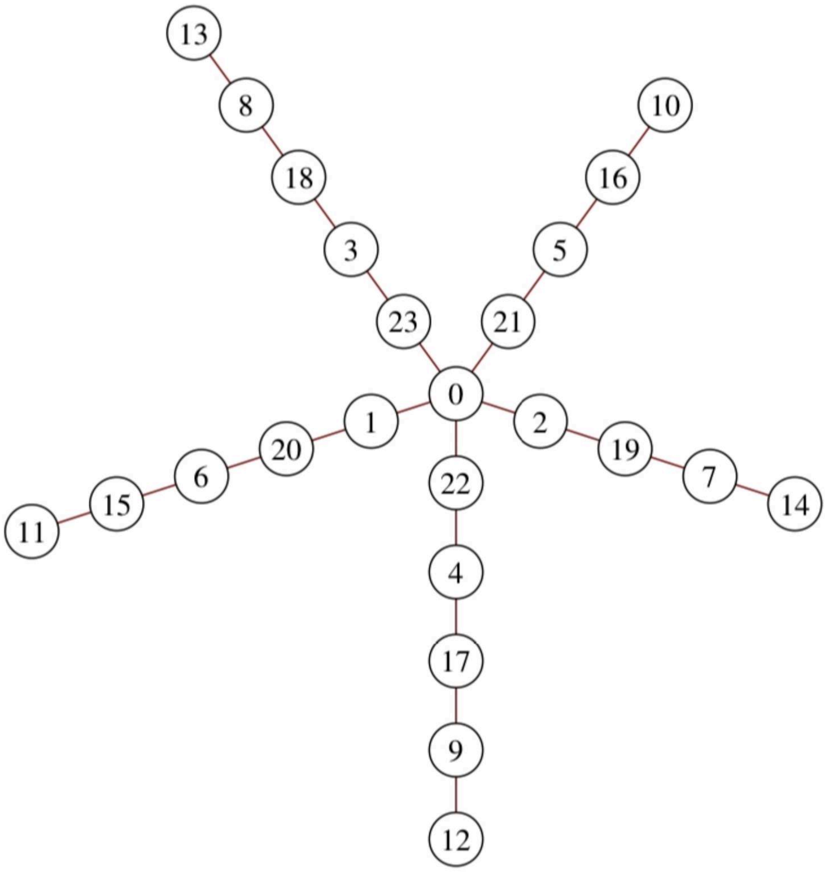

Figure 2.11: Gracefully Labeled Spider Tree

2.7 Banana Trees

A banana tree consists of a vertex v joined to one leaf of any number of stars. An example is shown in the figure 2.12.

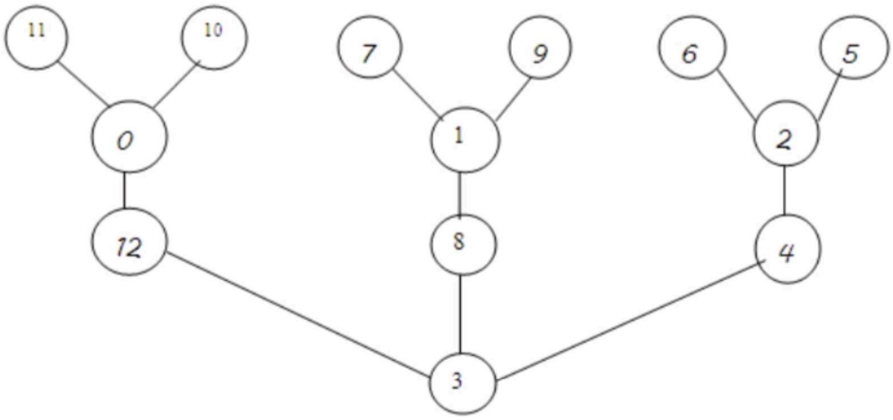

Figure 2.12: Gracefully Labeled Banana Tree

2.8 Coconut Trees

A coconut Tree $CT(m,n)$ is the graph obtained from the path P_n by appending m new pendent edges at an end vertex of P_n. An example is shown in the figure 2.13.

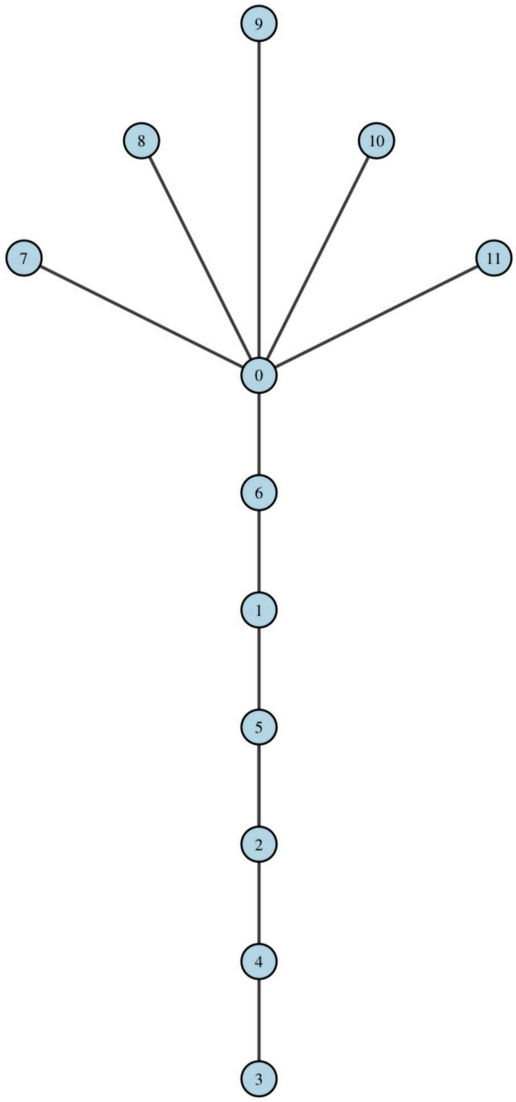

Figure 2.13: Gracefully Labeled Coconut Tree

2.9 Olive Trees

An olive tree T_k is a spider tree with k legs with lengths $1, 2, ..., k$ respectively. An example is shown in the figure 2.14 and 2.15.

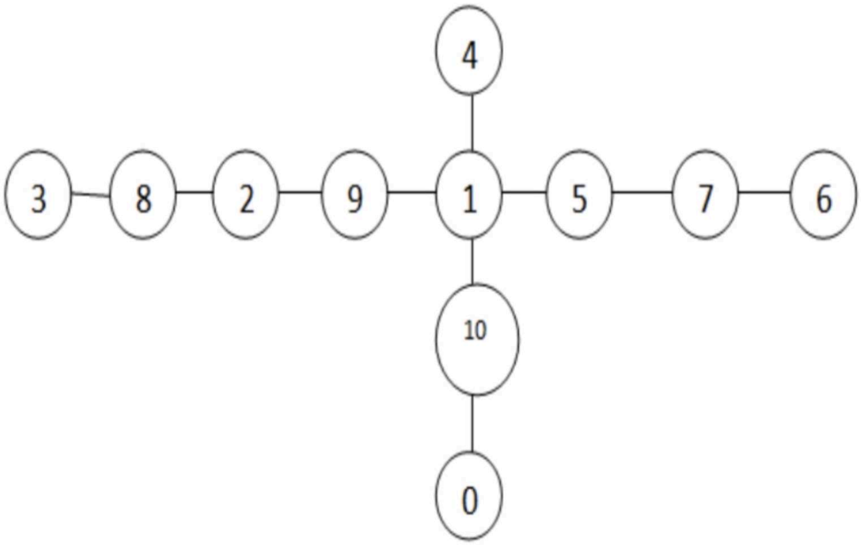

Figure 2.14: Gracefully Labeled Olive Tree

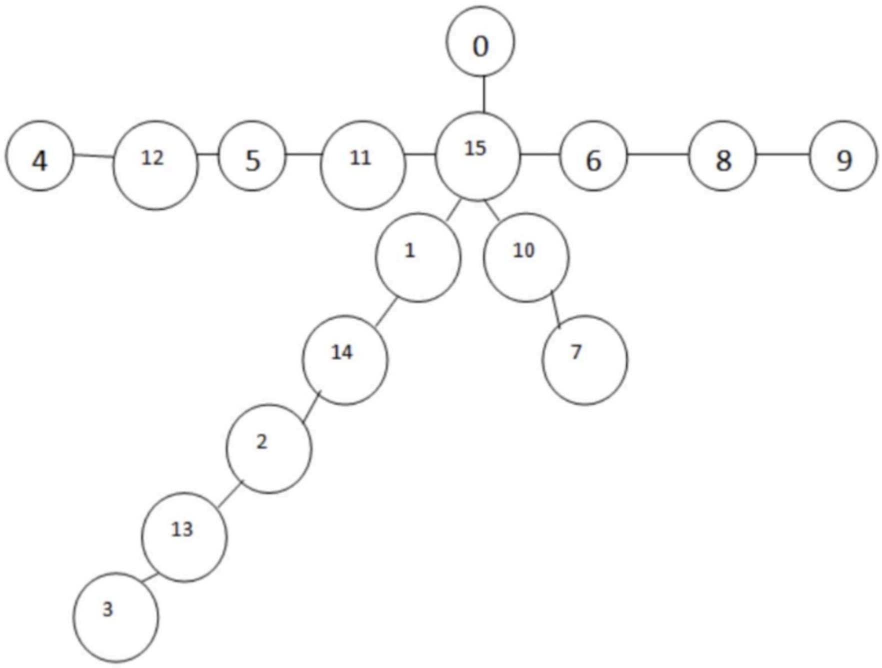

Figure 2.15: Gracefully Labeled Olive Tree

Chapter 3

NEW CLASSES OF GRACEFUL TREES

3.1 Star

A tree with one internal node and k leaves is termed as star, denoted as $S_{1,k}$ which is essentially a complete bipartite graph $K_{1,k}$. An example of gracefully labeled star is shown in the figure 3.1.

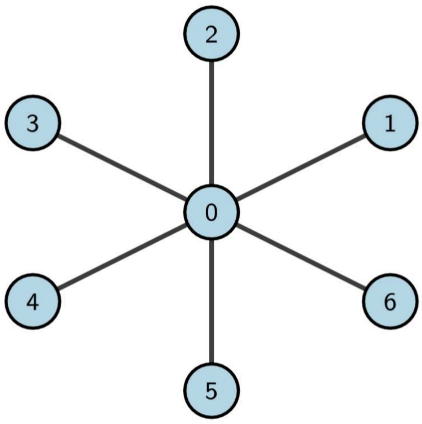

Figure 3.1: Gracefully Labeled Star

3.2 m-Star

A m-Star has a single root node with any number of paths of length m attached to it. An example of gracefully labeled m-star is shown in the figure 3.2

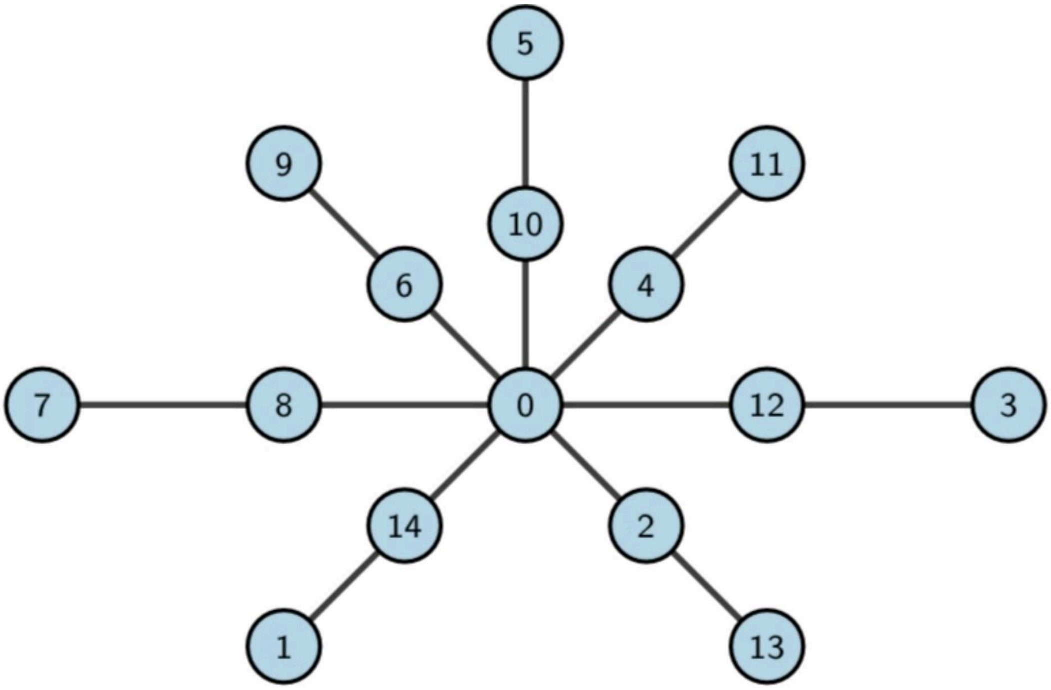

Figure 3.2: Gracefully Labeled m-Star

3.3 Superstar

Let a tree T consist of stars $S(i,k_i)$, $i = 0,1,...,I$ with k_i leaves. Each S_i,k_i $i = 1,...,I$ shares exactly one leaf with S_0,k_0. This S_0,k_0 is called the root star whereas S_i,k_i, $i = 1,...,I$ are called leaf stars. Then T is said to be a superstar denoted by SS. An example of superstar is shown in the figure 3.3.

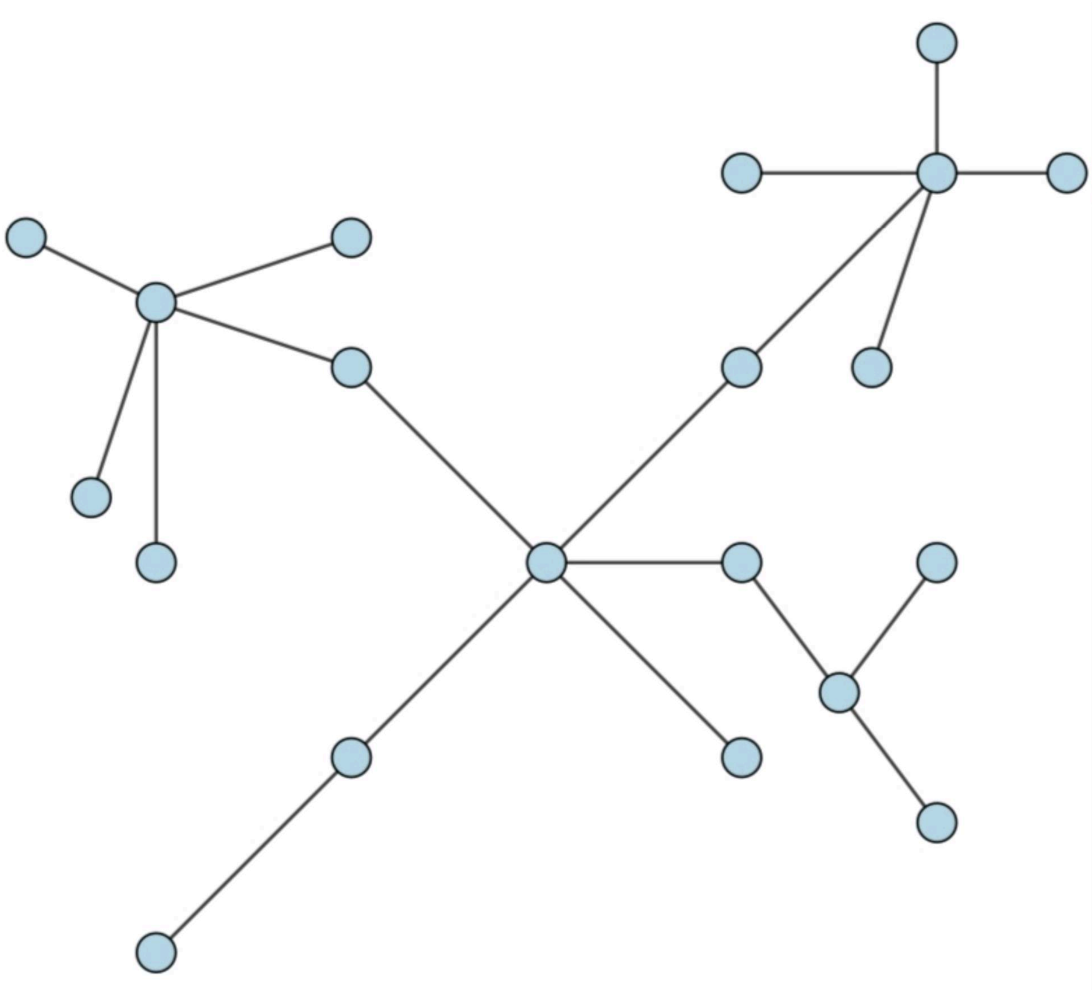

Figure 3.3: Superstar

3.3.1 Steps to Label the Superstar

Step 1

First of all we have to find out the value of $I$and Max_k and the center vertex of root star, l_0 will be labeled by $min\{\ I, Max_k\}$. As for the below example $I = 5$ and $Max_k = 6$, therefore, l_0=5 which is shown in the below figure 3.4.

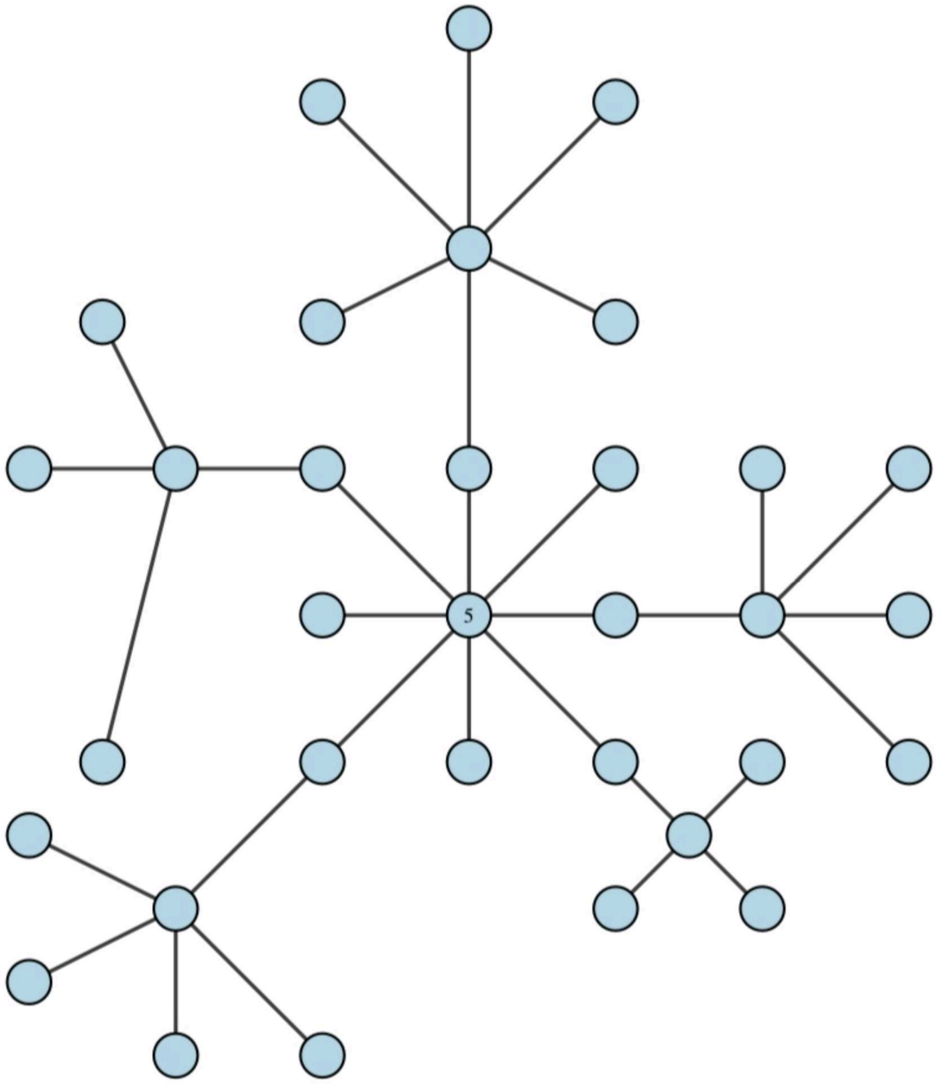

Figure 3.4: Step 1 *SS*

Step 2

Next we have to find an arbitrary star and we have to label the center vertex of all the leaf stars by l_i where l_i=$0, 1, ..., I-1$. Therefore, we labeled all the center vertex of leaf star by 0,1,2,3,4 which is shown in the below figure 3.5.

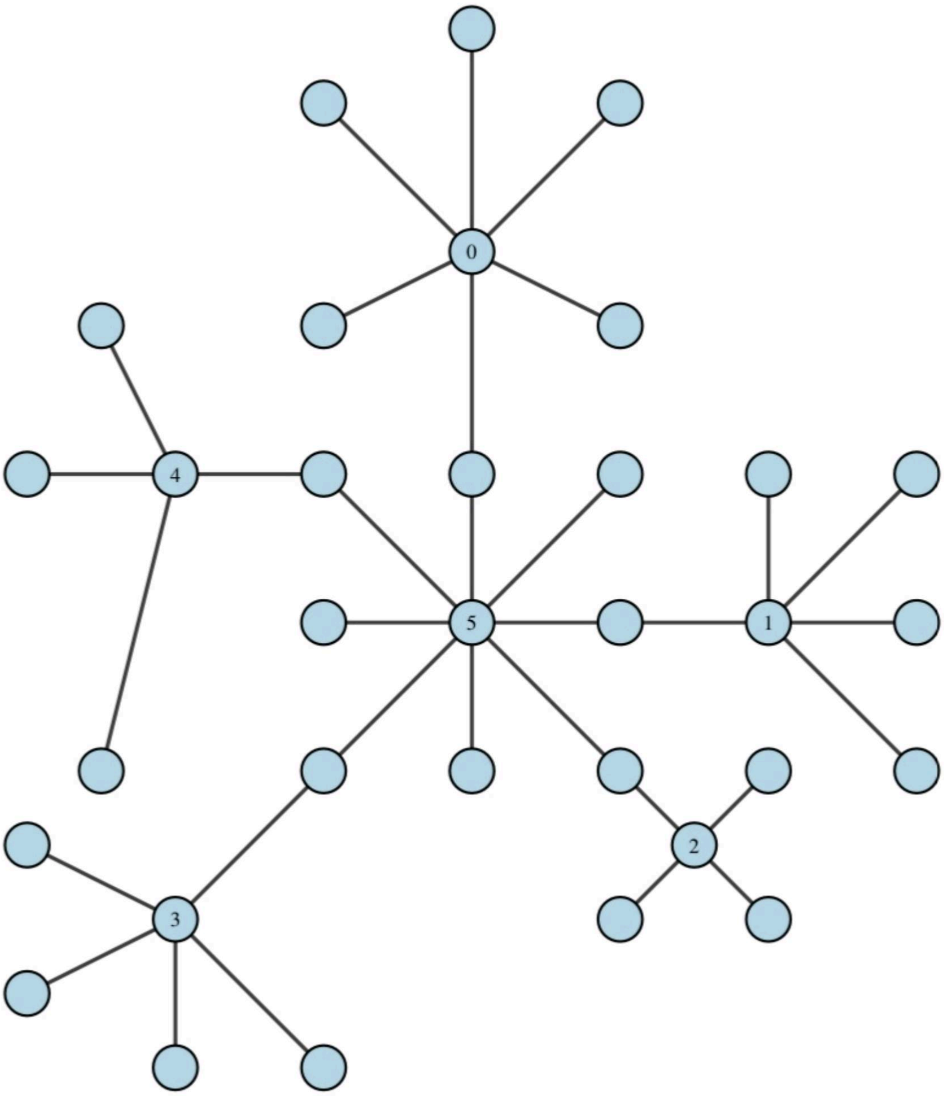

Figure 3.5: Step 2 *SS*

Step 3

Now we have to label the leafs of all star starting from the star whose center vertex is labeled by 0 and then 1,2,3,4,5. Therefore, we label them starting with maximum label and so on which is shown in the below figure 3.6.

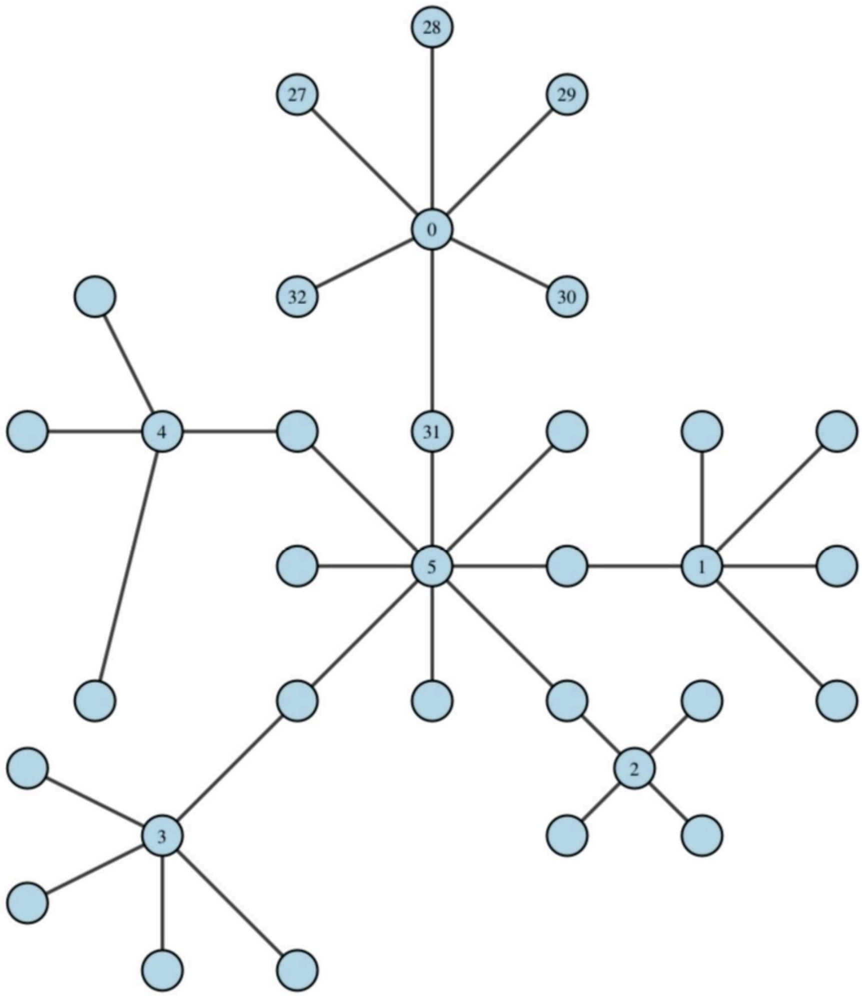

Figure 3.6: Step 3 *SS*

Step 4

Now we have labeled the leafs of star i for which l_i=1, 2, 3 which is shown in the below example. If we follow the procedure to label all the vertices then we get the edge label in descending order starting from $m, \ldots, 1$ consecutively. In this way we label super star gracefully. An example of gracefully labeled SS is shown in the figure 3.7.

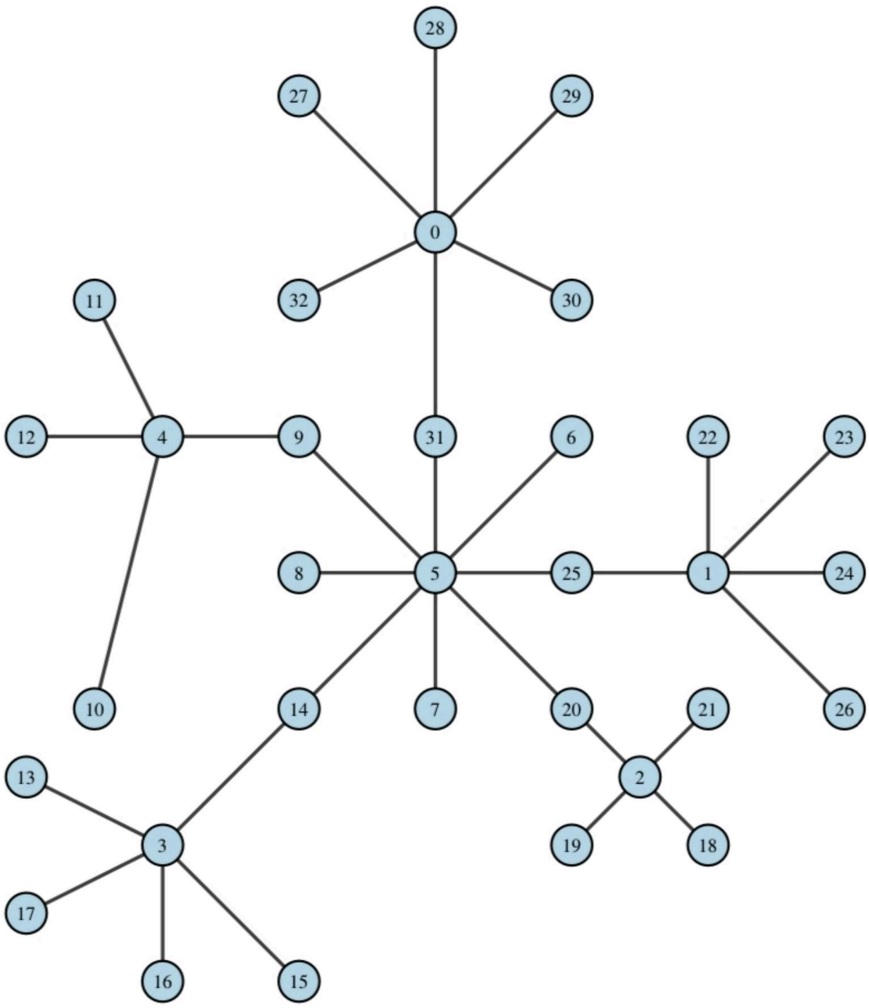

Figure 3.7: Step 4 SS

Algorithm 1

1. $I+1 \leftarrow$ total number of star of superstar, where $i = 0,1,.....,I$.

2. $m \leftarrow$ total number of edge of a superstar.

3. k $\leftarrow$ number of leaf of each star $S_{i,k}$.

4. $S_{i,k} \leftarrow$ leaf star of a Superstar

5. $SS_i \leftarrow$ root star

6. $SS_{i,k} \leftarrow$ number of leaf of root star.

7. $Max_k \leftarrow 0$

8. $l_0 \leftarrow$ root of SS_i

9. $C = 0$, where $C = 0,1,...,I - Max_k$

10. for $i = 0$ to I do

11. Count k_i for each $S_{i,k}$

12. if $Max_k < k_i$ then

13. $Max_k = k_i$

14. end if

15. end for

16. if $Max_k > I$ then

17. l_0=I

18. for $i = 0$ to $I - 2$ do
19. repeat
20. Find $S_{i,k}$
21. until $k_i \geq l_0$-i
22. Root of $S_{i,k}$=i
23. Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - i$-k_i+l_0
24. for $m+1$ down to m-k_i+1 do
25. Leaf of $S_{i,k}$= $m - 1$
26. $m = m - 1$
27. end for
28. from down to $m - I$ do
29. Leaf of SS_i=$m - 1$
30. $m = m - 1$
31. end for
32. else
33. $l_0 = Max_k$
34. for $i = 0$ to Max_k-1 do
35. repeat

36. Find S_i, k
37. until $k_i \geq l_0 - i$
38. Root of $S_i, k = i$
39. Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - i$ -k_i+ l_0
40. for $m + 1$ down to m-k_i +1 do
41. Leaf of $S_{i,k}$= $m - 1$
42. $m = m - 1$
43. end for
44. for m down to m-$SS_{i,k}$ do
45. Unshared Leaf of SS_i=$m - 1$
46. $m = m - 1$
47. end for
48. for $i = Max_k$+ 1 to I do
49. repeat
50. Find $S_{i,k}$
51. until $l_0 - i \geq ki$
52. Root of $S_i, k = i$
53. Connect l_0 to the leaf of $S_{i,k}$ labeled with $m - C$

54. $C = C + 1$

55. for $m + 1$ down to $m - k_i$ +1 do

56. Leaf of $S_{i,k}$= $m - 1$

57. $m = m - 1$

58. end for

59. end if

Lemma 3.3.1.1. *Algorithm 1 labels center vertices of stars by labels* $0, 1, ..., J$ *whereas leaves of stars with root labels* $0, 1, ..., i$ *labeled consecutively with labels from m to* $m - \sum_{j=1}^{i} k_i$*+1 and edges get labels m down to* $m - \sum_{j=1}^{i} k_i$*-i.*

Proof. Let us label leaves of the star centre of which has been labeled $i = 1$. Since l_0-$l_1$less thank_1, the leaf common to root star and the star being labeled can be labeled in a way that root star edge gets label mk_1, leaves get labels from m down to $m - k_1$, edges are labeled consecutively from m down to m-k_1-1+1.

Assume that we have labeled $i + 1$ stars with vertex labels of centres from 0 to i and leaf labels from m down to $m\sum_{j=1}^{i} k_i$+1 inducing edge labels from m down to $m - \sum_{j=1}^{i} k_i$-i. Now we are labeling leaves of star centre of which has been labeled $i + 1$.

We label the leaf common to root star and star centre of which has been labeled by $i + 1$ in such a way that it induces edge label $m - \sum_{j=1}^{i} k_i - i$, then the other edge labels up to $m - \sum_{j=1}^{i} k_i - i$ can be generated by using vertex labels from $j = 1$ $m - \sum_{j=1}^{i} k_i$ to $m - \sum_{j=1}^{i+1} k_i$. In case root star vertices are labeled not at the last then its yet $j = 1$ unlabeled vertices should be labeled. □

Theorem 3.3.1.2. *All superstars are graceful.*

Proof. By lemma 3.3.1.1 systematically labels leaves with labels from m down to $J+1$, centres of stars already labeled by 0 to J. This induces edge labels from m down to 1. Hence this is a graceful labeling of a superstar. □

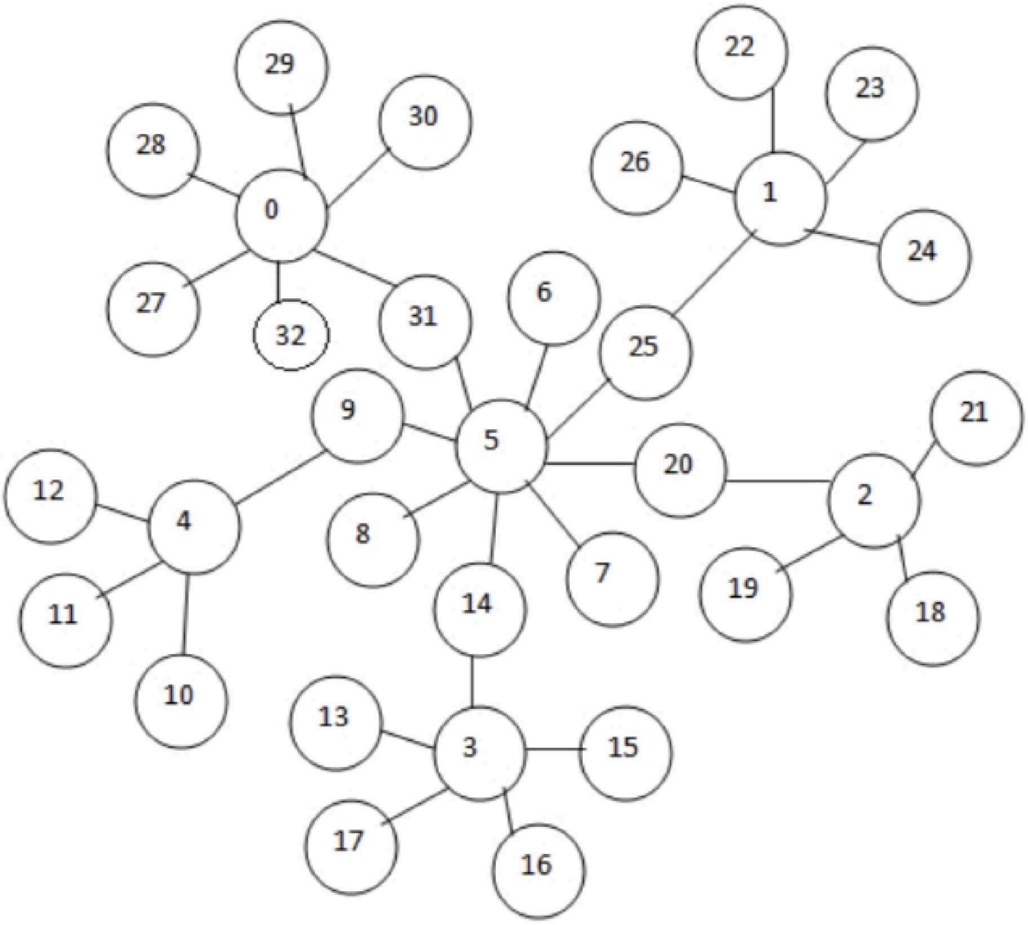

Figure 3.8: Gracefully Labeled Superstar

3.4 Extended Superstar

Let ESS be an extended superstar with m edges and stars $S_{i,ki}$ $i \in I_j$ contained in the superstar SS where $j = 1, 2, ..., J$. Among all the stars $S(i, k_i$ one star is a root star and rest of them are included in leaf superstars. Therefore total number of leaf superstars is J. If all the leaf of superstars SS_j share exactly one leaf with the leaf of the root then the resulting tree is called an Extended Superstar ESS. An example of ESS is shown in the figure 3.9

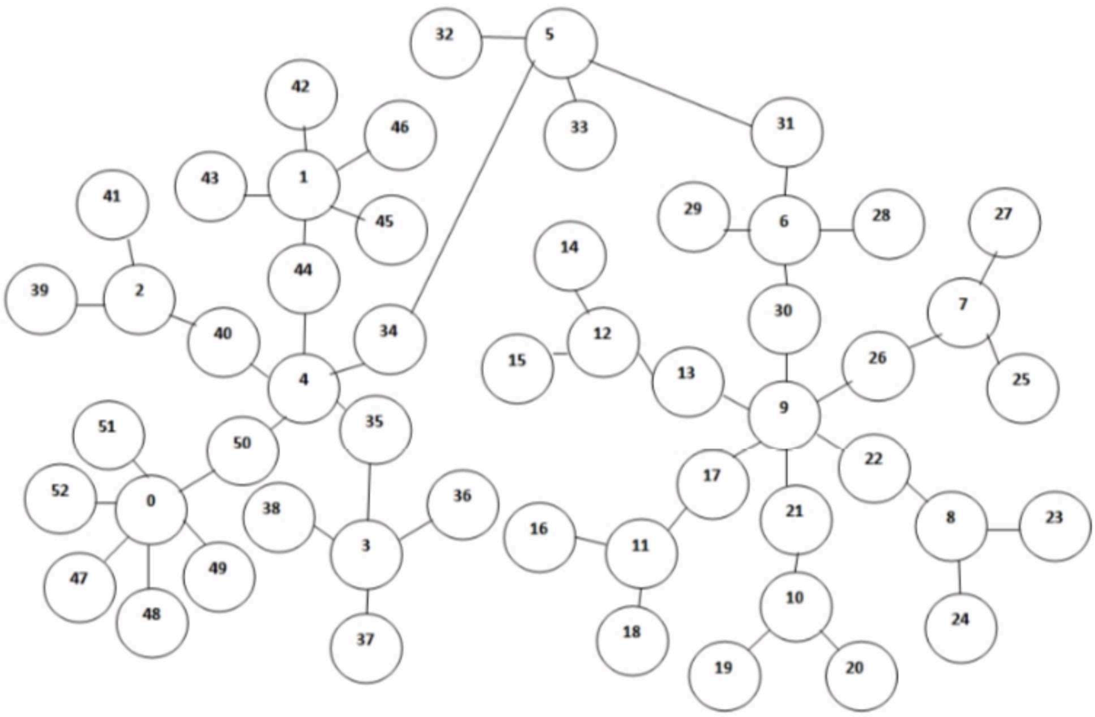

Figure 3.9: Extended Superstar

3.4.1 Steps to Label Extended Superstar

Step 1

First we have to identify total number superstar in the given extended superstar which is $j = 3$ for the below example and total number of edges $m = 69$. Therefore, We have to take an arbitrary superstar $j = 1$ and label the superstar gracefully using 1 where $p = m$. After labeling j^{th} superstar the value of $p = 51$ and $i = 4$. Then we have to label the center vertex of root star, $l_{0,0}$ by $i = 5$ and shared vertex of root star by p which is shown in the figure 3.10.

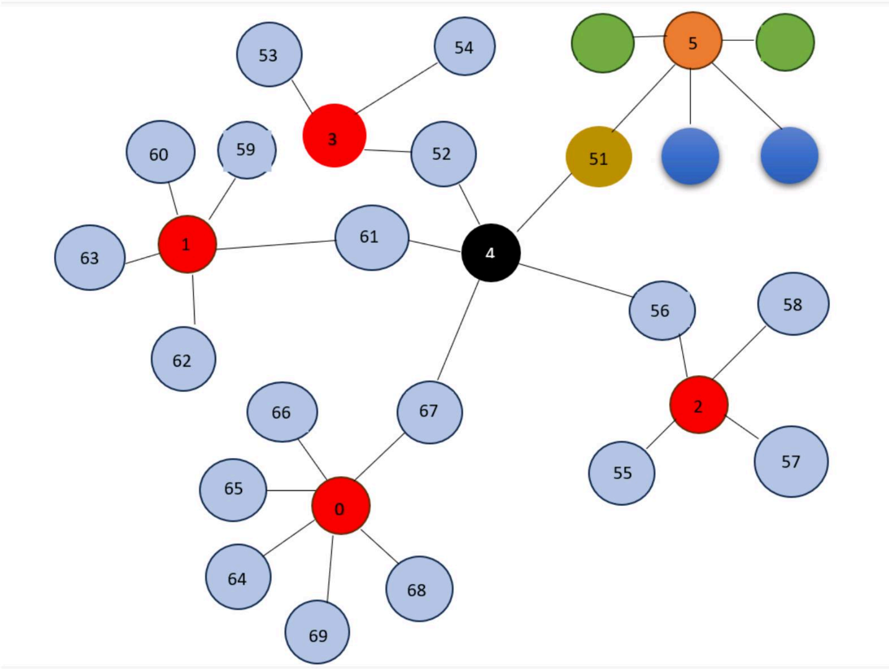

Figure 3.10: Step 1

Step 2

Now we have to label the unshared leaf of root star by p down to p- total number of unshared leaf of root star that is 2. Therefore the unshared leaf of root star will be labelled by 50 and 49 which is shown in the figure 3.11.

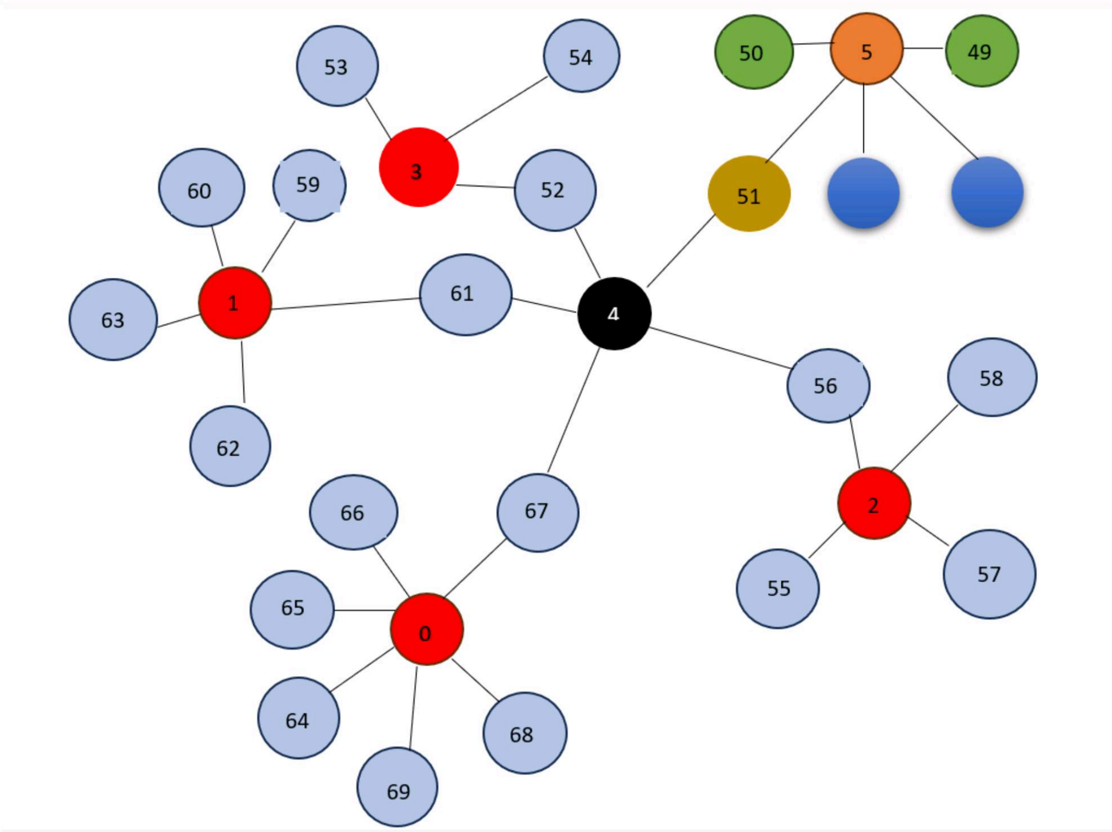

Figure 3.11: Step 2

Step 3

Now take another superstar and first label the shared leaf of root star by $p = 48$. Now $p = 47$ and $i = 6$. Then label the superstar in the same way which is shown in the figure 3.12.

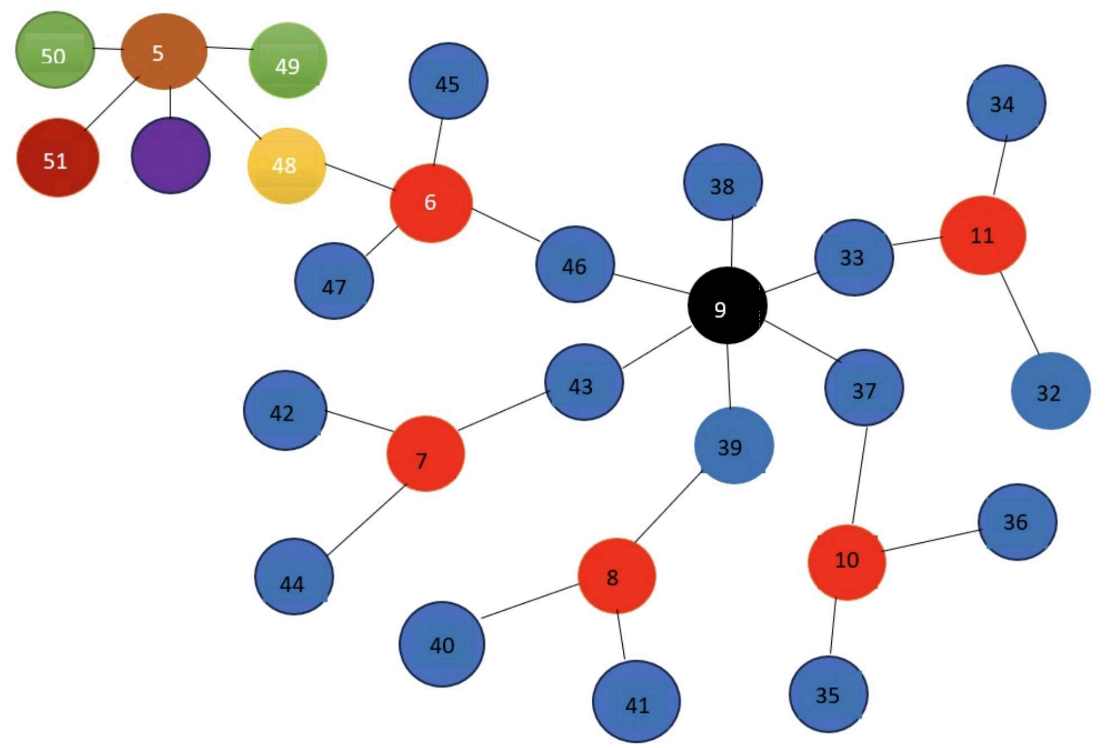

Figure 3.12: Step 3

Step 4

In this step we have to label the Superstar for $j = 3$ and we have $p = 31$ and $i = 12$. As $j > 2$, therefore, first we have to label the superstar gracefully in the same way then have to label the shared vertex of root star by $p-i+1+l_{0,0}$=25 which is shown in the figure 3.13.

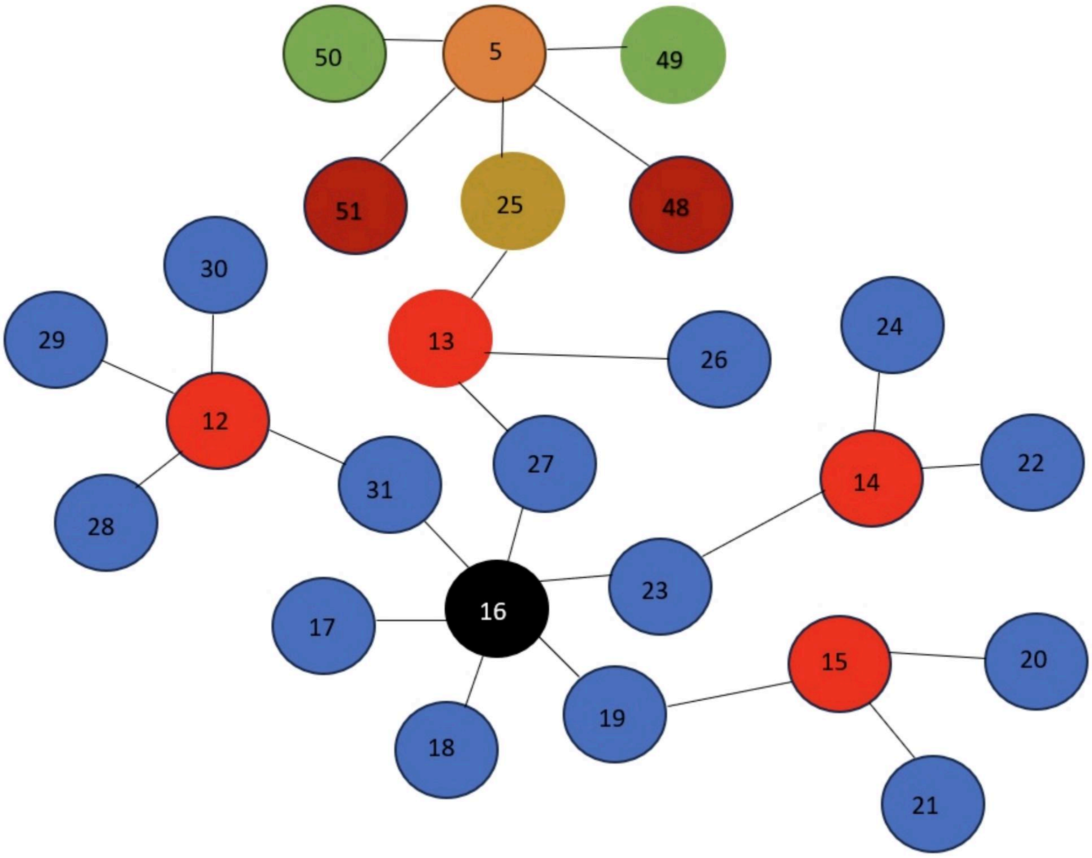

Figure 3.13: Step 4

Algorithm 2

1. i=← least possible label
2. $p = m \leftarrow$ largest possible label
3. call Superstar(i, p)

4. label center of the root star $\leftarrow i$

5. label shared vertex of root star and leaf Superstar $SS_1 \leftarrow p$

6. label shared vertex of root star and leaf superstar $SS_2 \leftarrow p-1$

7. call superstar(i, p)

8. for $j = 3$ to J do

9. call superstar (i, p)

10. label shared vertex of root star and leaf superstar $SS_j \leftarrow p-1-i+l_{0,0}$

11. end for

Theorem 3.4.1.1. *All extended superstars are graceful.*

Proof. We have already discussed how to label superstar gracefully therefore, we have omitted the labeling technique of superstar here. Let us assume for simplicity, SS_j is a superstar and ESS is an extended superstar. Now let $l_{0,0}$ be the label of the center of root star. First take an arbitrary superstar SS_j for $j = 1, 2, \ldots.., J$ and using previously described Algorithm 1 label the superstar gracefully. Let p be the largest possible label yet be used after labeling SS_1. Then label the center of root star, $l_{0,0}$=i where i is the immediate next least possible label and label the shared leaf of root star and superstar SS_1 by p. Now we have to label the unshared leaf of root star from p down to total number of unshared leaf of root star. After that, the shared leaf of root star and Superstar SS_2 will be labeled by $p-1$. Then for $j=2$ take another arbitrary superstar and label the superstar in the same way. But for $j=3,.....,J$ the shared leaf of root star and superstar SS_j will be labeled by $p-1-i+l_0,0$ and the remaining superstars will be labelled by using the label of superstar. Therefore all the label of vertices and edges of extended superstar will be distinct and from the set $1,2,...,m$ and $0,2,...,m$ respectively. This way we have labeled all the vertices and edges of the extended superstar gracefully. $\square$

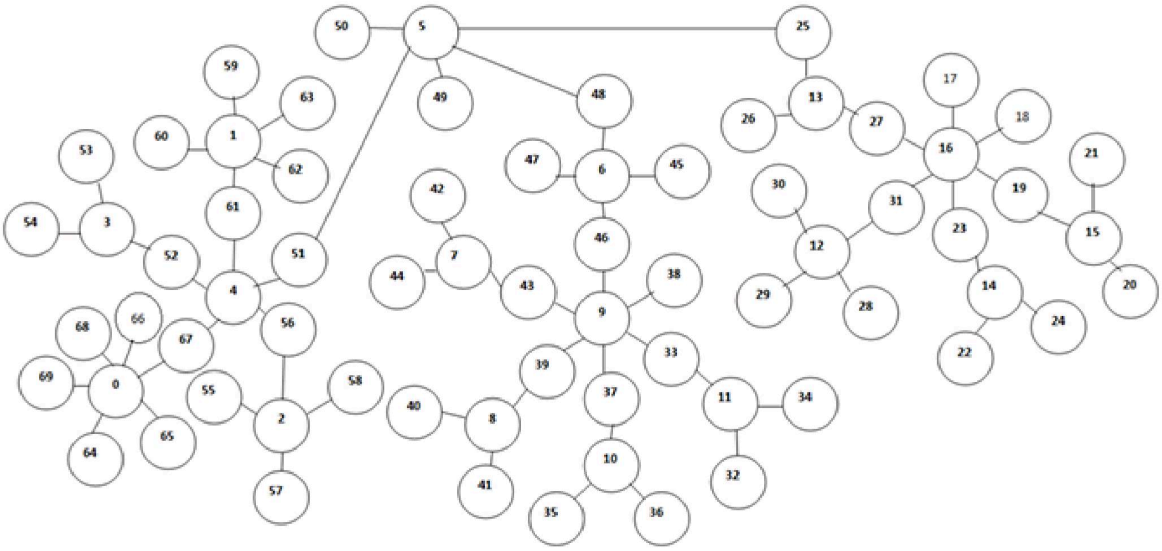

Figure 3.14: Gracefully Labeled Extended Superstar

Chapter 4

TRANSFORMED TREES

4.1 Transformed Trees

Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of u_0u path is equal to the length of v_0v path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv then the transformation of T is called an elementary parallel transformation (an ept) and the edge u_0v_0 is called a transformable edge. If by a sequence of ept's T can be reduced to a path then T is called a T_P-tree(transformed tree) and any such sequence of mappings (ept's) denoted by P, is called a parallel transformation of T. The path of T under P is denoted as $P(T)$.

A class of tree called T_P trees (transformed trees) are created by taking a gracefully labeled chain and shifting some of the edges.

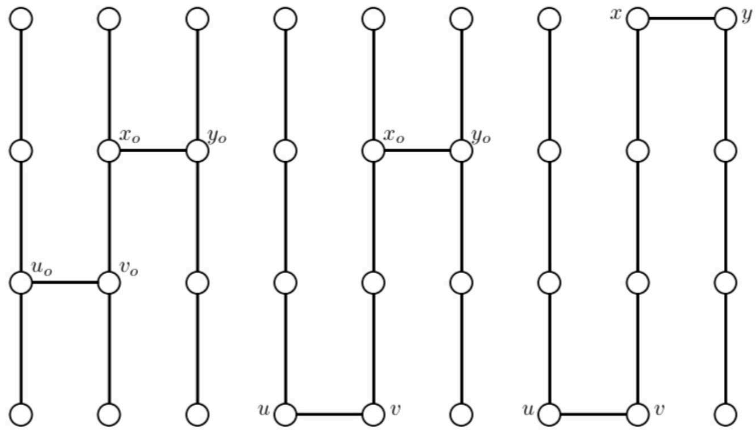

Figure 4.1: Transformed Tree

4.1.1 Diameter Four Trees with Central Vertex of Even Degree

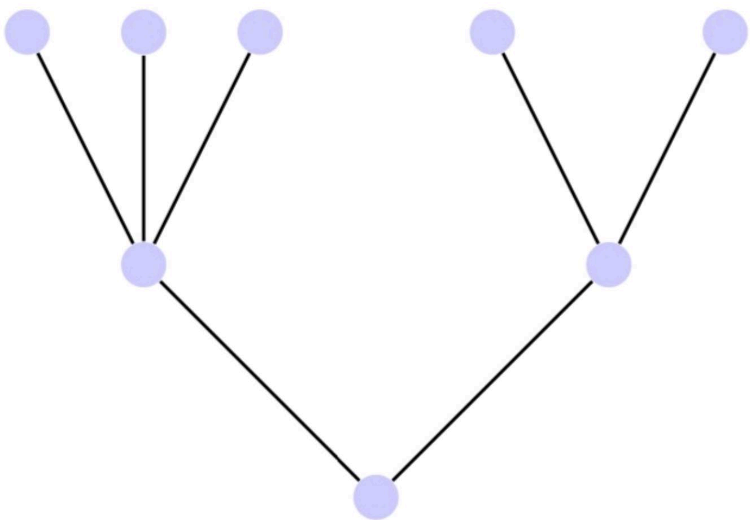

Figure 4.2: A Rooted Tree T with a Central Vertex of Degree 2

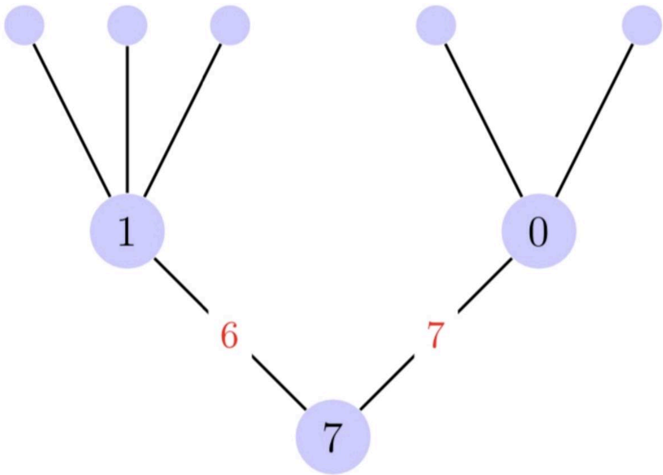

Figure 4.3: The First Partial Labeling of T

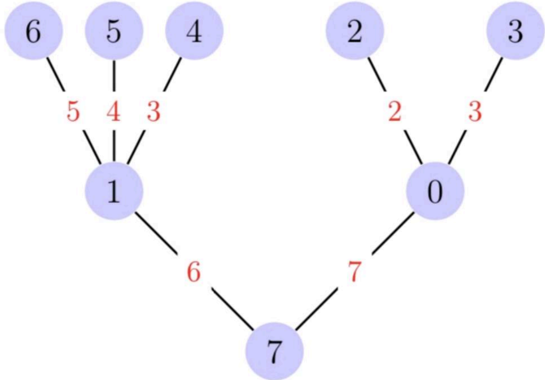

Figure 4.4: The First Attempt to Label T Gracefully Fails

Theorem 4.1.1.1. *Let T be a diameter four tree with a central vertex of degree 2 and 4k+1 vertices for some k∈N. Then T has a graceful labeling where the central vertex has the maximum label.*

Proof. Since T has $4k+1$ vertices, the maximum vertex label to be used in a graceful labeling of T is $4k$. We assign $4k$ to the central vertex and 0

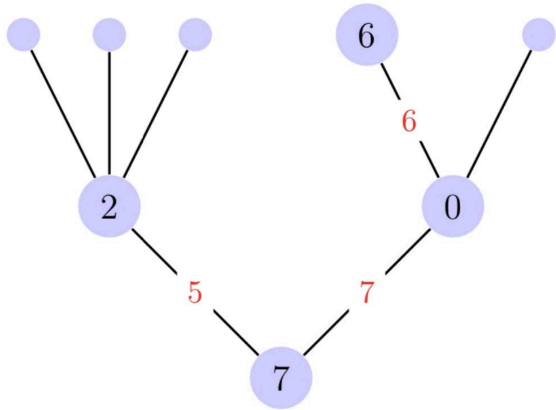

Figure 4.5: A Second Partial Labeling of *T*

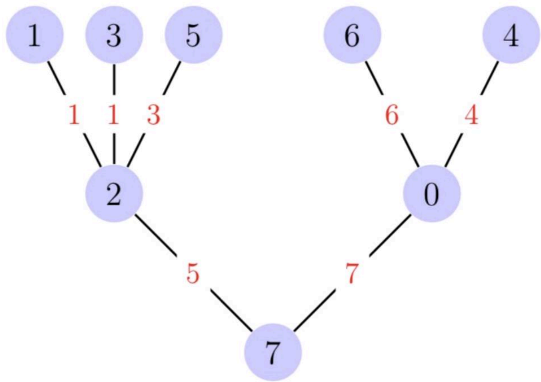

Figure 4.6: The Second Attempt to Label *T* Gracefully Fails

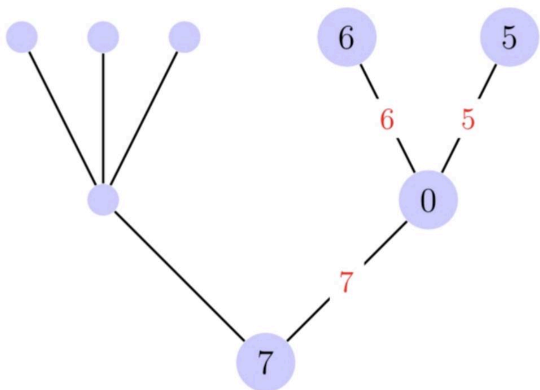

Figure 4.7: The Third Partial Labeling of *T*

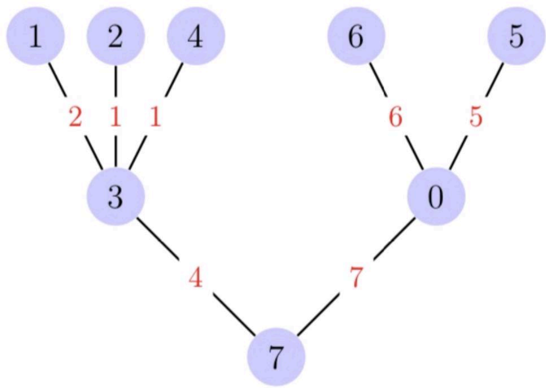

Figure 4.8: The Third Attempt to Label T Gracefully Fails

and $2k$ to its two neighbours.

If T is a symmetrical tree, we may label it as assigning the labels $2k+1, 2k+2, \ldots, 4k-1$ to the neighbours of 0 and the labels $1, 2, \ldots, 2k-1$ to the neighbours of $2k$. Otherwise, a graceful labeling of T can be obtained by beginning with this graceful labeling of the symmetrical tree and performing the particular $2k{\rightarrow}0$ transfer required. More specific if 0 and $2k$ each have an odd yet unequal number of leaves, a graceful labeling of T can be obtained by performing a $2k{\rightarrow}0$ transfer of the second type.

Otherwise if 0 and $2k$ each have an even number of leaves, then a graceful labeling of T can be obtained by performing a $2k \rightarrow 0$ transfer of the first type.

□

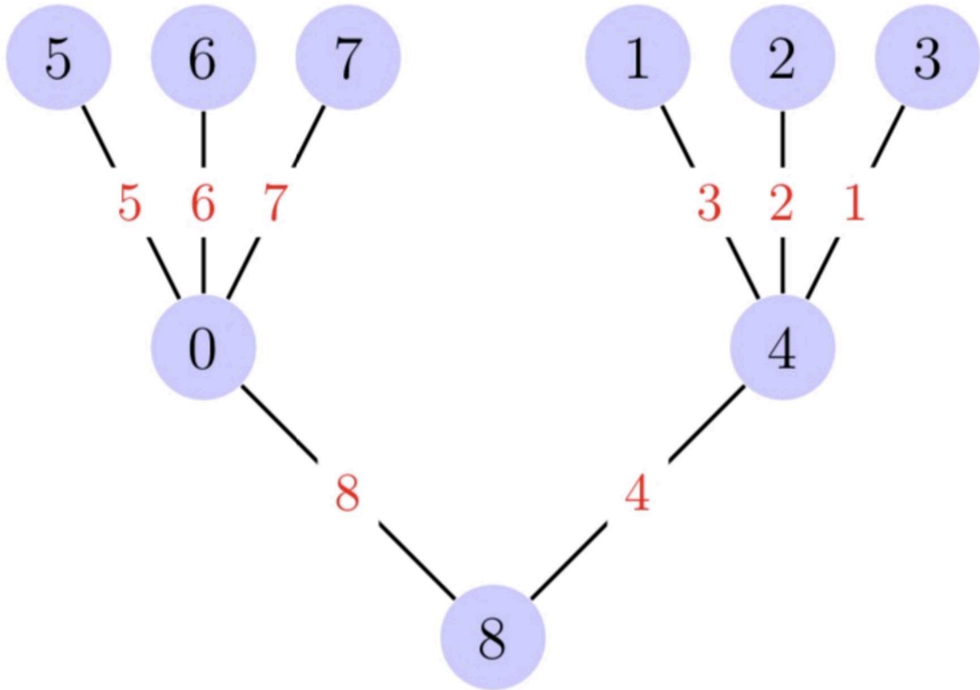

Figure 4.9: A Graceful Labeling of a Diameter Four Tree with 9 Vertices and Central Vertex of Degree 2

4.2 Transfer

A $u \to v$ transfer is a transfer of end edges from vertex u to vertex v. A $u \to v$ transfer followed by a $v \to w$ transfer may be denoted by $u \to v \to w$ etc.

Step of transfers is illustrated in figure 4.10, 4.11, 4.12 , 4.13 and 4.14.

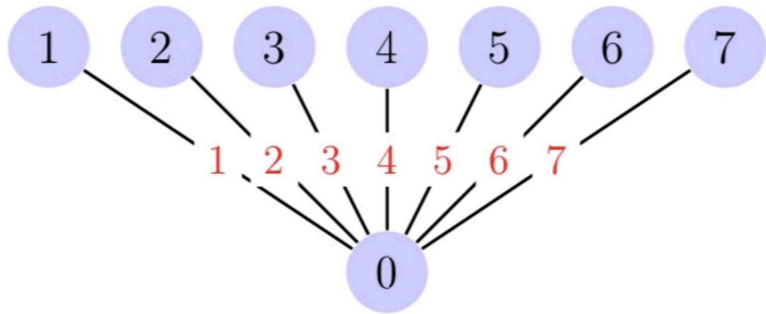

Figure 4.10: Step 1 Transfer

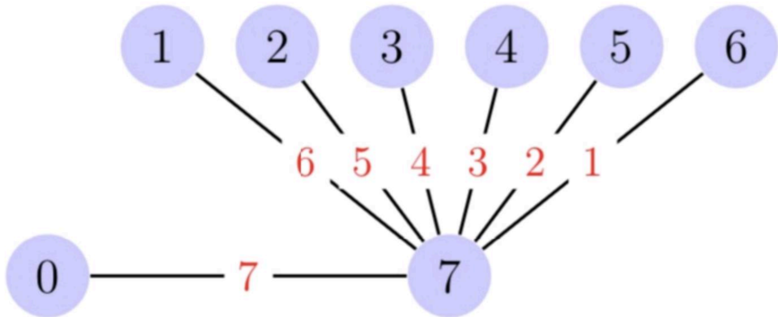

Figure 4.11: Step 2 T_1^1 0 to 7

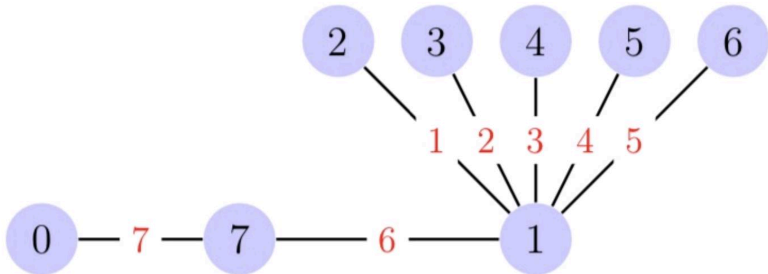

Figure 4.12: Step 3 T_1^2 7 to 1

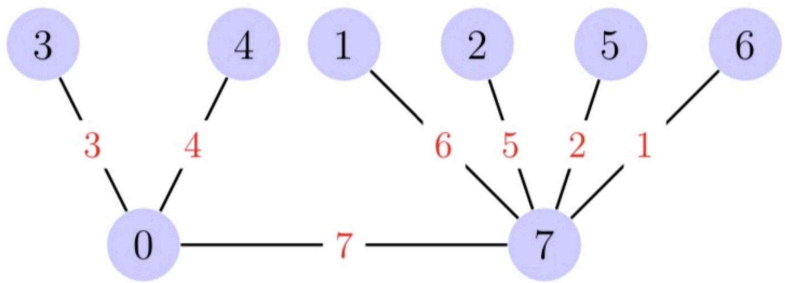

Figure 4.13: Step 4 T_2^1 0 to 7

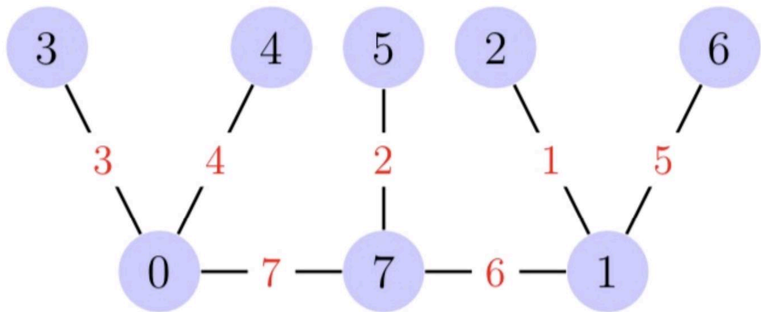

Figure 4.14: Step 5 T_2^2 7 to 1

4.3 Graceful Trees Built By Transfers

4.3.1 Crabs

Let G be a tree such that its central vertex v_0 is of degree 3, and two of its neighbours, call them v_1 and v_2 each have k neighbors that are not v_0 and $k-1$ of which are leaves, while the third neighbour of v_0 call it v_3 has degree 2, i.e it has a single neighbour that is not v_0 and this neighbour is a leaf. Let the non-leaf neighbours of v_1 and v_2 that are not v_0 be denoted u_1 and u_2 respectively. We further suppose that u_1 and u_2 are of degree

2, and that their second neighbors (i.e neighbors other than v_1 and v_2, respectively) are leaves. We say that such a tree G is a crab leg with index k. Consider a graph formed by joining two identical crab legs together by an edge connecting their central vertices to be a crab. Example of crab is shown in the figure 4.15. Steps to label crab is shown in the figure 4.16, 4.17, 4.18, 4.19.

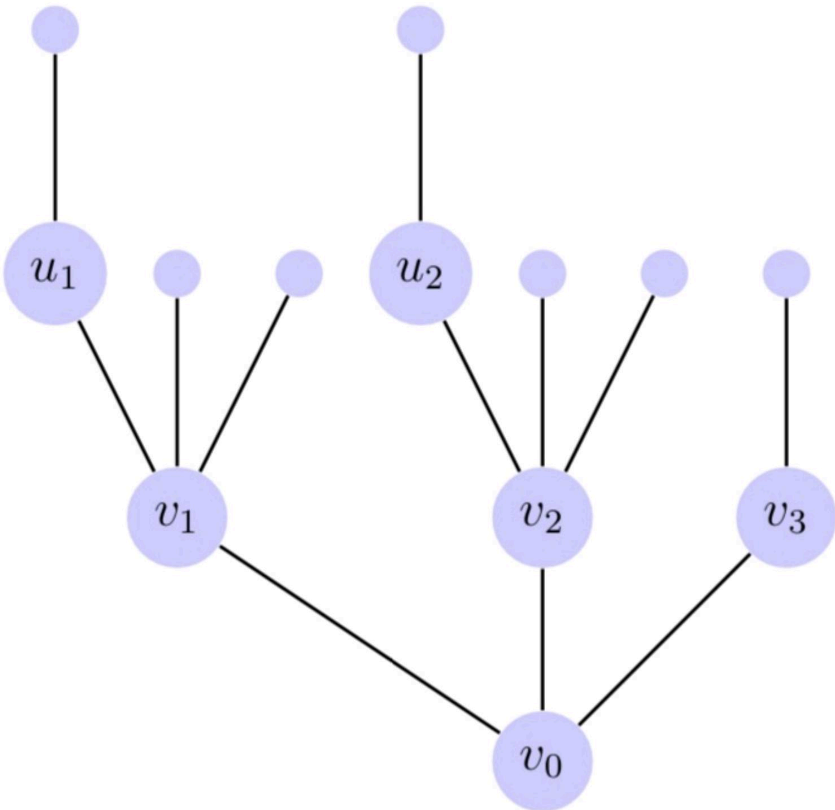

Figure 4.15: Crab

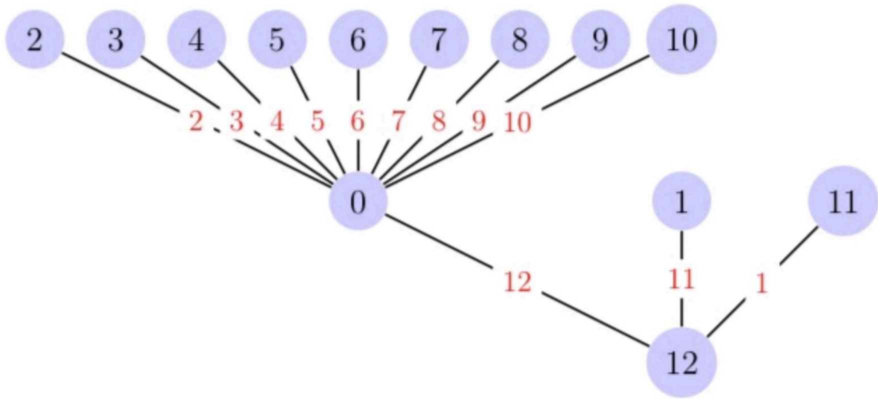

Figure 4.16: Starting Point to Label the Crab Leg Graph with Index $k = 3$

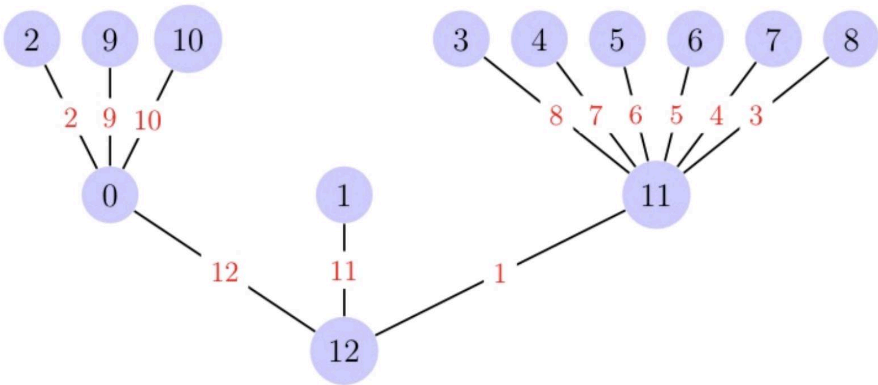

Figure 4.17: Step that Performs a 0→11 Transfer

4.3.2 Butterflies

Let G be a tree constructed by identifying three copies of P_2 and two copies of P_3 with the central vertex v_0 (i.e three 3 copies of P_2 and 2 copies of P_3 are extending from v_0). Furthermore, attach a leaf to each of the 3 vertices that are not v_0 in 1 of the copies of P_3. Finally let v_0 have an even number of leaves. We say that such a graph G is a butterfly wing.

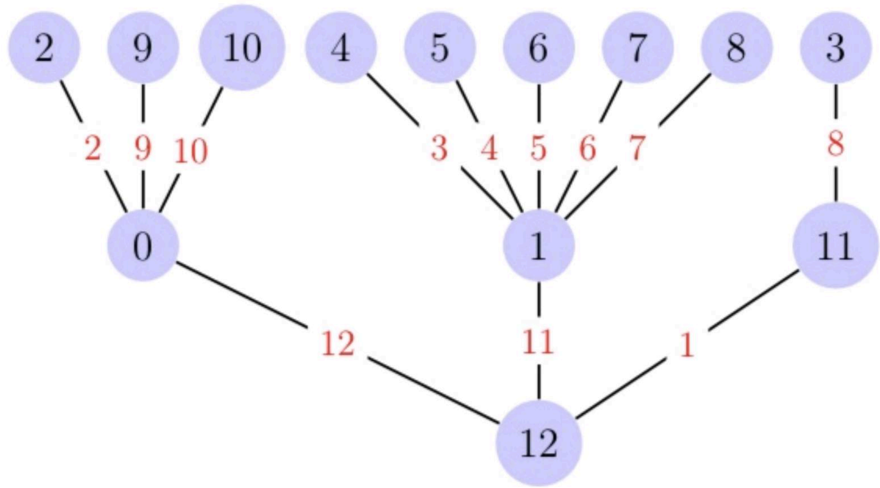

Figure 4.18: Step that Performs a 11→1 Transfer

Consider a graph formed by joining two identical butterfly wings together by an edge connecting their central vertices to be a butterfly. Example of butterfly and gracefully labeled butterfly is illustrated in figure 4.20 and 4.21.

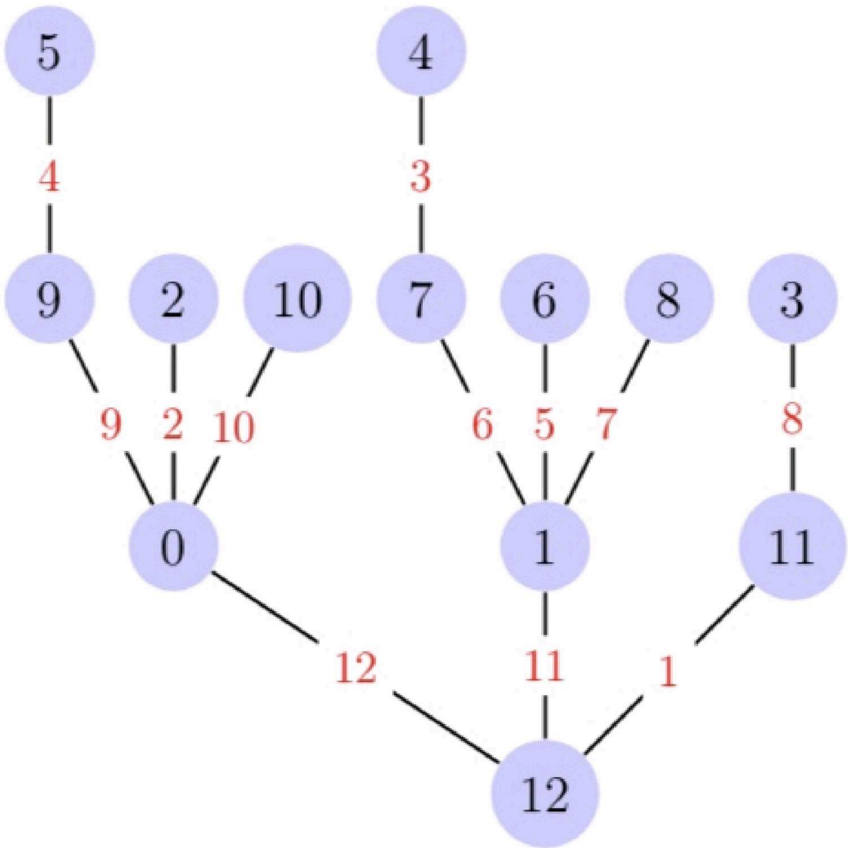

Figure 4.19: A Gracefully Label Crab Leg Graph with Index $k = 3$

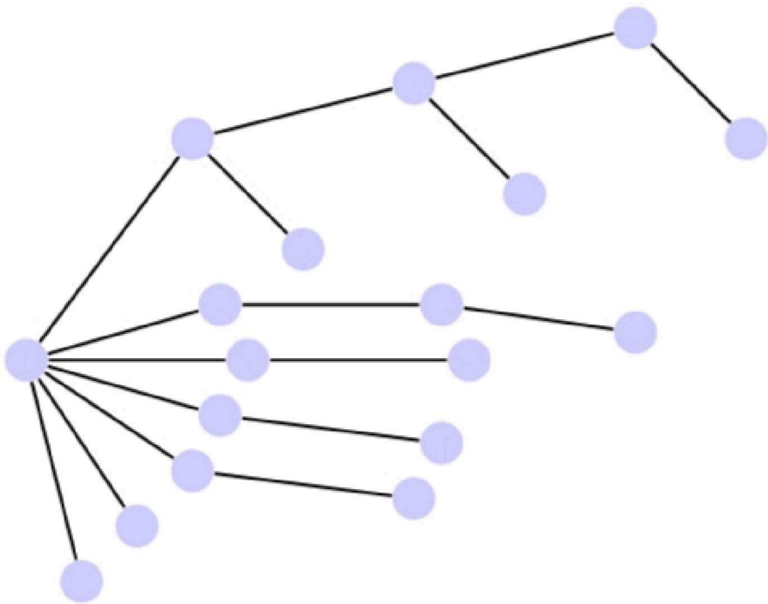

Figure 4.20: Butterfly

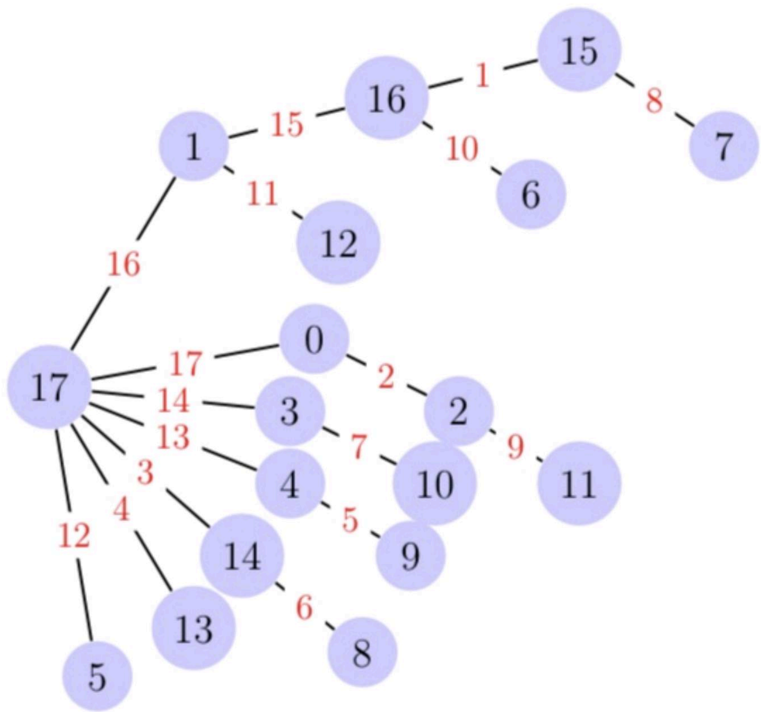

Figure 4.21: Gracefully Labeled Butterfly

Chapter 5

CONCLUSION

In **Chapter 1** we have introduced the problem of graceful labeling and the origin of this problem is discussed. Chapter also discussed history of Graph Labeling.

In **Chapter 2** we have discussed different classes of trees that have been proved graceful to strengthen the Graceful Tree Conjecture that is "All trees are Graceful". Classes of trees like paths, caterpillars, symmetrical trees, etc have been shown graceful.

In **Chapter 3** we have introduced new classes of trees like superstar extended superstar that are proved graceful which implies a partial effort to prove Graceful Tree Conjecture.

In **Chapter 4** we have talked about transformed trees and transfer. Some classes of trees are also discussed.

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