An Extension of Egorov's Theorem and The Lebesgue's Differentiation Theorem

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "An Extension of Egorov's Theorem & The Lebesgue's Differentiation Theorem" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. M. Kunhanandan and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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This is to certify that the dissertation report "An Extension of Egorov's Theorem & The Lebesgue's Differentiation Theorem" is a bonafide work carried out by Ms. Akansha Dhananjay Pednekar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " An Extension of Egorov's Theorem and The Lebesgue's Differentiation Theorem." This survey is divided into four chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical and systematic manner to cover the topic.

In this dissertation report we have made a humble approach in understanding two of the important concepts of Measure theory namely Egorov's theorem and it's extension and the Lebesgue's Differentiation theorem. While doing so we have stated some important and basic results of the topic I have studied from the references mentioned in the bibliography.

Before moving forward, we first assume some basic results in Analysis such as the $\varepsilon - \delta$ presentation of convergence, continuity on the real line and a rigorous definition of the Riemann integral. The first chapter is a brief introduction to measure theory and integration and some of it's basic properties. It includes concepts like Lebesgue measure, measurable sets and functions and also functions of bounded variation.

l In the subsequent chapter, we shall be coming across an extension version of the classical Egorov's theorem . We will be seeing what restrictions and conditions are required to prove the statement. We shall also be constructing a measurable sets and functions in this process.

In the next chapter, we will see a new mode of convergence different from the earlier modes of convergence we have come across. Some basic properties of convergence in measure would also be stated along with some useful results.

Finally, we see an important theorem in Real Analysis which is the "Lebesgue's Differentiation theorem". It can be said that Lebesgue differentiation theorem is an analogue, and a generalization, of the fundamental theorem of calculus in higher dimensions. However, we don't go into it's further details. With the help of [3] we prove the lemmas required to prove the theorem and also see how it's helpful to answer some important questions in Analysis.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my Mentor, Dr. M. Kunhanandan, for his guidance and support throughout the process of completing this dissertation. His insightful feedback, encouragement, and patience have been helpful in providing me the direction to complete my work and also improve the quality of it. I am very grateful for his mentorship, which has not only helped me grow academically but also in my professional growth. His vast knowledge in Real Analysis and Measure theory enabled me to successfully accomplish this work. This endeavour would not have been possible without his help and supervision.

ABSTRACT

The general theory of measure and integration plays an important role in diverse areas of mathematics, including probability theory, partial differential equations, functional analysis. This report consists of some major theorems and lemmas that we have proven using [1], [2], [3], [6], [4], [5], and made an approach to understand the same.

The main aim of this article is to note the differences in the classical statement of Egorov's Theorem and the extension of Egorov's theorem. In earlier courses of analysis and a basic course in measure theory one may have come across several modes of convergence for sequences of real valued function. In this study we shall be dealing with another mode of convergence and how it is established. Also, we will be studying a major theorem in analysis , namely the "Lebesgue's Differentiation Theorem".

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Notations and Abbreviations

$\mathbb{R}$	Set of Real numbers
$\mathscr{P}(\mathbb{R})$	Power set of Real numbers
$\mathbb{N}$	Set of Natural numbers
X	Abstract measure space
$\mathscr{M}$	Set of Measurable sets
μ	Lebesgue measure
μ^*	Lebesgue outer measure
L_1	Set of Lebesgue integrable functions
$V(f,P)$	Variation of a function f w.r.t partition P
$BV[a,b]$	Space of functions of bounded variation
ϕ	Empty set
Φ	Simple function
G_δ	Countable intersection of open sets
A^c	Complement of a set A
sup	Supremum (least upper bound)
inf	Infiimu(greatest lower bound)
$\|f\|$	Modulus of a function f
$\|\|f\|\|$	Norm of a function f
$\exists$	There exists
$\cap, \cup$	Intersection , Union
$\forall$	For all

Chapter 1

INTRODUCTION

1.1 Literature Review

Measure theory is a branch of mathematics that provides a systematic framework for understanding and quantifying the concept of "measure." It deals with the study of measurable sets, which are subsets of a given space that can be assigned a numerical value representing their size or extent. Central to measure theory is the notion of a measure, which is a function that assigns non-negative real numbers to sets in a consistent and intuitive manner. Measures generalize familiar notions of length, area, and volume to more abstract spaces, enabling the rigorous treatment of various mathematical concepts. Key topics in measure theory and integration include sigma algebras, Lebesgue measure, measurable functions, Lebesgue integration, convergence theorems, among others. These concepts play a fundamental role in various areas of mathematics, including analysis, probability theory, and functional analysis.

All the sets that will be considered in this and the following chapters are contained in $\mathbb{R}$, the real line unless stated otherwise. Note that the length of an interval I is defined to be the difference of the endpoints of I if I is bounded i.e. For any bounded interval I with left end point a and right end point b, we define the length of I to be $l(I) = b - a$. and if I is an unbounded interval then we define $l(I) = \infty$. In this chapter we will be constructing a collection of sets called the Lebesgue measurable sets and a set function of this collection called Lebesgue measure denoted by μ. Before doing so, let us see how we define a set function called the outer-measure denoted by μ^*

Definition 1.1.1. Let E be an arbitrary subset of $\mathbb{R}$. We define the Lebesgue outer measure (or simply the outer measure) of E to be the quantity

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} l(I) : (I_n) \text{ is any sequence of intervals finite or infinite s.t. } E \subset \bigcup_{n=1}^{\infty} I_n\right\}$$

Basic properties of outer measure:

1. $0 \le \mu^*(E) < \infty \quad \forall E \subset \mathbb{R}$
   Thus μ^* is a non-negative, extended, real valued function on the power set $\mathscr{P}(\mathbb{R})$ of $\mathbb{R}$.

2. If $E \subset F$ then $\mu^*(E) \le \mu^*(E)$.

3. $\mu^*(E+x) = \mu^*(E) \quad \forall E \subset \mathbb{R} \;\&\; x \in \mathbb{R}$

4. The outer measure of any countable set is zero..

5. $$\begin{aligned}\mu^*(E) &= \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\right\} \\ &= \inf\left\{\sum_{n=1}^{\infty} l(I) : (I_n) \text{ is any sequence of open intervals s.t. } E \subset \bigcup_{n=1}^{\infty} I_n\right\}\end{aligned}$$

Theorem 1.1.2. *The outer measure of any interval bounded or unbounded is its length.*

Note that the outer measure is <u>not</u> "**Countably additive**". But we have the following;

Theorem 1.1.3. *The outer measure is countably sub-additive. i.e. For any sequence $(E_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$, we have*

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

Proposition 1.1.4. *Let E be a subset of $\mathbb{R}$. Then*

(i) given $\varepsilon > 0, \exists$ an open set U containing E s.t. $\mu^*(U) \leq \mu^*(E) + \varepsilon$

(ii) $\exists$ a $G_\delta-$set G containing E s.t. $\mu^*(G) = \mu^*(E)$ [$G_\delta -$ set means a set that is a countable intersection of open sets.]

Definition 1.1.5. (**Measurable Set**):
A subset E of $\mathbb{R}$ is called a measurable set if given $\varepsilon > 0$, $\exists$ a closed set F and an open set U such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.

This definition is also equivalent to the Caratheodary's definiton of measurable sets which is given by,

Definition 1.1.6. A subset E of $\mathbb{R}$ is measurablre $\iff$ For every subset A of $\mathbb{R}$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Countable addivity

Theorem 1.1.7. *Let $(E_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint measurable sets. Then $\bigcup_{n=1}^{\infty} E_n$ is also measurable and we have*

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu^*(E_n)$$

Definition 1.1.8. (Sigma Algebra of sets)
Let X be a set and $\mathscr{A}$ be a subset of the power set $\mathscr{P}(X)$ of X. $\mathscr{A}$ is said to be a $\sigma-$algebra on X if

(i) $\mathscr{A}$ is closed under complementation i.e. $A \in \mathscr{A} \implies A^c = X \setminus A \in \mathscr{A}$

(ii) $\mathscr{A}$ is closed under countable union
i.e. if $(A_n)_{n=1}^{\infty}$ is a sequence of sets in $\mathscr{A}$ then $\bigcup\limits_{n=1}^{\infty} A_n \in$ A

Now we determine what it means for a set to be Lebesgue measurable.

Definition 1.1.9. (Lebesgue measure)
Let E be a measurable set in $\mathbb{R}$. We define the Lebesgue measure of E to be
$\mu(E) = \mu^*(E)$ =outer measure of E.

<u>Note:</u> We can understand the Lebesgue measure μ as a function defined on the $\sigma-$algebra $\mathscr{M}$ with values in the set of non-negative real numbers which is countably additive i.e. we have $\mu : \mathscr{M} \to \mathbb{R}$ satisfying the properties:

(i) $\mu(E) \geq 0 \quad \forall E \in \mathscr{M}$

(ii) If $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathscr{M}$ then
$\mu\left(\bigcup\limits_{n=1}^{\infty}\right) = \sum\limits_{n=1}^{\infty} \mu(E_n)$

Definition 1.1.10. Let $(E_n)_{n=1}^{\infty}$ be a sequence of sets. We define

(a) $\limsup\limits_{n\to\infty} E_n = \bigcap\limits_{n=1}^{\infty}\left(\bigcup\limits_{k=n}^{\infty} E_k\right)$

(b) $\liminf\limits_{n\to\infty} E_n = \bigcup\limits_{n=1}^{\infty}\left(\bigcap\limits_{k=n}^{\infty} E_k\right)$

Definition 1.1.11. (**Measurable function**):

Let E be a measurable subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be measurable if $f^{-1}((\alpha,\infty)) = \{x \in E : f(x) > \alpha\}$ is measurable $\quad ,\forall\alpha \in \mathbb{R}$.

This Definition is also equivalent to the following:

Proposition 1.1.12. *Let E be a measurable set. For a function $f : E \to \mathbb{R}$, the following are equivalent*

(1) f is measurable

(2) $f^{-1}([\alpha,\infty)) = \{x \in E : f(x) \geq \alpha\}$ is measurable $\quad ,\forall\alpha \in \mathbb{R}$.

(3) $f^{-1}((-\infty,\alpha)) = \{x \in E : f(x) < \alpha\}$ is measurable $\quad ,\forall\alpha \in \mathbb{R}$.

(4) $f^{-1}((-\infty,\alpha]) = \{x \in E : f(x) \leq \alpha\}$ is measurable $\quad ,\forall\alpha \in \mathbb{R}$.

Proposition 1.1.13. *Every Riemann integrable function is measurable*

Corollary 1.1.14. *Every function of bounded variation is measurable.*

<u>The concept of "almost everywhere true " property</u>

Definition 1.1.15. Let E be a measurable set. We say that a property P is true almost everywhere (a.e. in short) on E if $\{x \in E : Pisnottrue\}$ is a set of measure zero.

Proposition 1.1.16. *Let E be a measurable set and $f, g : E \to \mathbb{R}$ s.t $f = g$ a.e. on E. If f is measurable then g is also measurable.*

Measurability of extended real valued functions

Let E be a measurable set, consider $f : E \to [-\infty, +\infty] = \tilde{\mathbb{R}}$. Then we say that f is measurable if $f^{-1}((\alpha, \infty]) = \{x \in E : f(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

Proposition 1.1.17. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions. Then*

(i) $\sup\limits_{n} f_n$

(ii) $\inf\limits_{n} f_n$

(iii) $\liminf\limits_{n\to\infty} f_n$

(iv) $\limsup\limits_{n\to\infty} f_n$

are all measurable (extended real valued function)

Definition 1.1.18. (**Almost Uniform Convergence**):
A sequence of measurable functions $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f on the measurable set E if for each $\varepsilon > 0$,there is a measurable subset $A \subset E$, $\mu(A) < \varepsilon$ such that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $E \setminus A$.

Definition 1.1.19. (**Signed Measure**):
If $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets and if $E = \bigcup_{n=1}^{\infty} E_n$, then $\sum_{n=1}^{\infty} \mu^(E_n) = \sum_{n=1}^{\infty} \int_{E_n} f$ is absolutely convergent with sum $\mu(E) = \int_E f$. In this case , we say that μ is a signed measure.

Integration in the context of measure theory extends the classical concept of integration from calculus. Instead of integrating functions with respect to variables, measure-theoretic integration allows for integrating functions with respect to measures. This generalization provides a powerful tool for analyzing complex systems and functions defined on abstract spaces.

Definition 1.1.20. (**Simple functions**):

Let $E_1, E_2, \cdots, E_n$ be subsets of $\mathbb{R}$. Then a function of the form

$$\Phi(x) = a_1 \chi_{E_1}(x) + a_2 \chi_{E_2}(x) + \cdots + a_n \chi_{E_n}(x)$$

where $a_1, a_2, \cdots, a_n$ are real numbers, is called a simple function.

Definition 1.1.21. Let $f : \mathbb{R} \to [0, \infty)$ be a non-negative , measurable extended real valued function. Then we define the Lebesgue integral of f over $\mathbb{R}$ to be:

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} \Phi : 0 \leq \Phi \leq f, \Phi \text{ simple integrable } \right\}$$

We say that f is Lebesgue integrable over $\mathbb{R}$ if $\int_{\mathbb{R}} f < \infty$

Functions of Bounded Variation

Definition 1.1.22. (**Total Variation**):

Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \to \mathbb{R}$ be a function. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of [a,b] the sum $V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ is called the variation f in [a,b] over the partition P.

$V_a^b(f) = \sup\limits_P V(f, P)$ is called the total variation of f over [a,b].

Also, if $\sup\limits_P V(f, P) < \infty$, where the supremum is taken over all partitions P of [a,b] then we say that f is a function of bounded variation on [a,b] and denoted by BV[a,b]

Definition 1.1.23. Let $f \in BV[a,b]$. We define $v : [a,b] \to \mathbb{R}$ by

$$v(x) = \begin{cases} 0 & \text{for } x = a \\ V_a^x(f) & \text{for } a < x \leq b \end{cases}$$

v is called the total variation function for f over [a,b].

1.2 Basic Results

Proposition 1.2.1. *If f is a function of bounded variation over [a,b], then f is bounded on [a,b] and we have* $||f||_\infty \leq |f(0)| + V_a^b(f)$

Proposition 1.2.2. *Any monotonic function on [a,b] is a function of bounded variation .*

Theorem 1.2.3. ***(Jordan's Theorem)**:*
$f \in BV[a,b] \iff$ *f can be written as a difference of two monotonic increasing functions.*

Theorem 1.2.4. ***(Continuity Properties of Measure)**:*
(i) Let $(E_n)_{n=1}^\infty$ *be a sequence of measurable sets s.t.* $E_n \subset E_{n+1}\ \forall n$.
Let $E = \bigcup\limits_{n=1}^{\infty} E_n$. *Then* $\mu(E) = \lim\limits_{n\to\infty} \mu(E_n)$.
(ii) Let $(E_n)_{n=1}^\infty$ *be a sequence of measurable sets s.t.* $E_n \supset E_{n+1}\ \forall n$.
Assume that $\mu(E_n) < \infty$ *for some n. Then* $\mu(\bigcap\limits_{n=1}^{\infty} E_n) = \lim\limits_{n\to\infty} \mu(E_n)$.

Chapter 2

EGOROVS'S THEOREM AND IT'S EXTENSION

John Edensor Littlewood was a British mathematician. He worked on topics relating to analysis, number theory, and differential equations.Most of Littlewood's work was in the field of mathematical analysis.
Most of the results of the theory are fairly intuitive applications of the following ideas. The Littlewood's three principles can be expressed in the following terms:

- Every measurable set is almost a finite union of open intervals.
- Every measurable function is almost a continuous function.
- Every pointwise convergent sequence of measurable functions is almost uniformly convergent.

A precise realization of the Littlewood's last principle is the Egorov's Theorem.

The importance of the classical theorem of Egorov in measure theory is well appreciated:

It establishes a condition for the almost uniform convergence of a pointwise convergent sequence of measurable functions. Although we will be concerned only with real valued function our study also will hold true for functions with values in a Banach space.

Theorem 2.0.1. ***(Egorov's Theorem):***
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions converging pointwise a.e. to a (measurable) function f on a measurable set E of finite measure.Then given $\varepsilon > 0$, $\exists$ a measurable set $A \subset E$ with $\mu(A) < \varepsilon$ such that $f_n \to f$ uniformly on $E \setminus A$.

Proof. w.l.o.g. we can assume that

$$E_{n,k} = \bigcup_{m=n}^{\infty} \left\{ x \in E : |f_m(x) - f(x)| \geq \frac{1}{k} \right\}$$

for each $k \in \mathbb{N}$ and $n = 1, 2, 3, \ldots$.
Then note that

1. $E_{1,k} \supset E_{2,k} \supset E_{3,k} \supset \cdots$
2. $\bigcap_{n=1}^{\infty} E_{n,k} = \phi$,
   because if $x \in \bigcap_{n=1}^{\infty} E_{n,k}$
   $\implies x \in E_{n,k} \quad \forall n$ where k is fixed.
   $\implies |f_m(x) - f(x)| \geq \dfrac{1}{k} \quad \forall n$
   Then it would mean that at x the sequence of function cannot converge for infinitely many n. Which will contradict our hypothesis.
3. $E_{n,k}$ are measurable sets.(because of measurability of f_m and $f \implies |f_m - f|$ is measurable.)
   Moreover each $E_{n,k}$ has finite measure because E itself has finite measure.

$\therefore$ by Continuity property of measure, we have

$$\lim_{n\to\infty}\mu(E_{n,k}) = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = \mu(\phi) = 0$$

$\therefore$ for the given $\varepsilon > 0, \exists\, n_k$ s.t. $\forall n \geq n_k$

$$\mu(E_{n,k}) < \frac{\varepsilon}{2^k}$$

In particular, $\mu(E_{n_k,k}) < \dfrac{\varepsilon}{2^k}$
We can assume that $n_1 < n_2 < n_3 < \cdots$
Let $A = \bigcup_{k=1}^{\infty} E_{n_k,k}$
Then A is measurable and

$$\begin{aligned}\mu(A) &= \mu\left(\bigcup_{k=1}^{\infty} E_{n_k,k}\right)\\ &\leq \sum_{k=1}^{\infty}\mu(E_{n_k,k})\\ &< \sum_{k=1}^{\infty}\frac{\varepsilon}{2^k} = \varepsilon\end{aligned}$$

Next, we show that on $E\setminus A$, $f_n \to f$ uniformly.
For, $x \in E\setminus A \implies x \notin A \implies x \notin E_{n_k,k} \forall k = 1,2,3,\cdots$
$\implies |f_m(x) - f(x)| < \dfrac{1}{k} \quad \forall m \geq n_k.$ □

Corollary 2.0.2. *A function f is measurable $\iff$ f is the pointwise limit of a sequence of measurable simple function.*
$\iff$ *f is the a.e. pointwise limit of a sequence of measurable simple functions.*

The extension of the Theorem 2.0.1 can be given by the following statement.

Theorem 2.0.3. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions and let f be a measurable function. Then the following assertions are equivalent.*

1. *The sequence $(f_n)_{n=1}^{\infty}$ converges almost uniformly to f.*
2. *The sequence $(f_n)_{n=1}^{\infty}$ satisfies the vanishing restriction with respect to f.*
3. *The sequence $(f_n)_{n=1}^{\infty}$ satisfies the finiteness restriction with respect to f and $f(x) = \lim_{n\to\infty} f_n(x)$ almost everywhere*

To prove this statement let us first define some important terms and propositions.

2.1 The Finiteness and Vanishing Restrictions

Definition 2.1.1. Let (X ,$\boldsymbol{X}, \mu$) be a measure space. If (f_n) is a sequence of $\boldsymbol{X}$-measurable functions on X and $\alpha > 0, n \in \mathbb{N}$, then we define

$$E_n(\alpha) = \bigcup_{i,j=n}^{\infty} \{x \in X : |f_i(x) - f_j(x)| > \alpha\}. \tag{2.1}$$

If f is an $\boldsymbol{X}$-measurable function on X,we define

$$E_n^f(\alpha) = \bigcup_{i=n}^{\infty} \{x \in X : |f_i(x) - f(x)| > \alpha\}. \tag{2.2}$$

NOTE:

1. The sets $E_n(\alpha)$ and $E_n^f(\alpha)$ belong to $\boldsymbol{X}$ and if $m \leq n$, then $E_n(\alpha) \subseteq E_m(\alpha)$ and $E_n^f(\alpha) \subseteq E_m^f(\alpha)$.

2. If $0 < \alpha < \beta$, then we have $E_n(\beta) \subseteq E_n(\alpha)$ and $E_n^f(\beta) \subseteq E_n^f(\alpha)$.
3. In addition, we have $E_n(\alpha) \subseteq E_n^f(\alpha/2)$.
4. If the sequence (f_n) converges to f uniformly on X, then for all $\alpha > 0$,
   $\exists n_\alpha$ such that $E_{n_\alpha}(\alpha) = \phi$ [respectively, $E_{n_\alpha}^f(\alpha) = \phi$]

Lemma 2.1.2. *If $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x \in X$, then $E_n^f(\alpha) \subseteq E_n(\alpha)$, for all $\alpha > 0$, $n \in \mathbb{N}$.*

Proof. If $x_0 \notin E_n(\alpha)$, then we have $|f_i(x_0) - f_j(x_0)| \leq \alpha$ for all $i, j \geq n$.
Passing to the limit as $j \to \infty$,
we infer that $|f_i(x_0) - f(x_0)| \leq \alpha$.
Thus it follows that $x_0 \notin E_n^f(\alpha)$. □

Definition 2.1.3. Finiteness Restriction:
We say that a sequence (f_n) satisfies the finiteness restriction [w.r.t. f] if for all $\alpha > 0$, there exists a natural number such that the set $E_n(\alpha)$ [respectively $E_n^f(\alpha)$] has finite μ-measure.

Definition 2.1.4. Vanishing Restriction:
We say that a sequence (f_n) satisfies the vanishing restriction [w.r.t. f] if for all $\alpha > 0$ we have , $\lim_{n\to\infty} \mu(E_n(\alpha)) = 0$ [respectively $\lim_{n\to\infty} \mu(E_n^f(\alpha)) = 0$]

Theorem 2.1.5. *If the sequence (f_n) converges almost uniformly to f, then it satisfies the vanishing restriction[w.r.t. f]*

Proof. Let $\varepsilon > 0$ be arbitrary.

Since (f_n) is almost uniformly convergent to f, $\exists\, B_\varepsilon \subset X$ with $\mu(B_\varepsilon) < \varepsilon$ such that the sequence (f_n) is uniformly convergent to f on $X \setminus B_\varepsilon$.

Let $\alpha > 0$ be given.

Now as (f_n) converges to f uniformly on $X \setminus B_\varepsilon$. Consequently, there is an n_α such that $E_{n_\alpha}(\alpha) \subseteq B_\varepsilon$ [respectively, $E^f_{n_\alpha}(\alpha) \subseteq B_\varepsilon$]

which implies $\mu(E_{n_\alpha}(\alpha)) < \varepsilon$ [respectively, $\mu(E^f_{n_\alpha}(\alpha)) < \varepsilon$.]

Thus as $n \to \infty$, $\lim\limits_{n\to\infty} \mu(E_n(\alpha)) = 0$ [respectively, $\lim\limits_{n\to\infty} \mu(E^f_{n_\alpha}(\alpha)) = 0$]. □

Proposition 2.1.6. *(i) If the sequence $(f_n)_{n=1}^\infty$ satisfies the vanishing restrictions with respect to f then it converges almost uniformly to f*

(ii) If $(f_n)_{n=1}^\infty$ satisfies the vanishing restrictions, then there exists a measurable function f such that $(f_n)_{n=1}^\infty$ converges almost uniformly to f.

Proof. (i) Let $\varepsilon > 0$ be given.

Since (f_n) satisfies the vanishing restriction w.r.t. f we have,

for all $\alpha > 0, \exists n_k$ for each $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \mu(E^f_{n_k}(\alpha)) = 0$. In particular, for $\alpha = 1/k, \mu(E^f_{n_k}(1/k)) < \varepsilon/2^k$.

Let $B_\varepsilon \in \mathbb{X}$ be defined by

$$B_\varepsilon = \bigcup_{k=1}^{\infty} E^f_{n_k}(1/k)$$

so that $\mu(B_\varepsilon) < \varepsilon$.

Further, if $j \geq n_k$, then we have for all $x \notin E^f_{n_k}(1/k)$,

$$|f_j(x) - f(x)| \leq 1/k \qquad (*)$$

Since $E^f_{n_k}(1/k) \subseteq B_\varepsilon$, it follows that if $x \notin B_\varepsilon$, then $(*)$ holds provided $j \geq n_k$. But this implies that (f_n) converges to f uniformly on $X \setminus B_\varepsilon$. In this case we show, as in (a), that the sequence (f_n) is uniformly Cauchy on $X \setminus B_\varepsilon$.

Hence there is a function g_ε on $X \setminus B_\varepsilon$ to which (f_n) converges.

Let $B_0 = \bigcup B_{1/n}$, so that $\mu(B_0) = 0$ and let $f(x) = 0$ for $x \in B_0$ and $f(x) = g_{1/n}(x)$ for $x \notin B_{1/n}$.

It is readily seen that f is consistently defined and is the desired function. $\square$

Theorem 2.1.7. *If the sequence (f_n) satisfies the finiteness restriction with respect to f, and if*

$f(x) = \lim\limits_{k\to\infty} f_k(x)$ for almost all $x \in X$, then the sequence satisfies the vanishing restriction with respect to f.

Proof. For convenience let $C_n(\alpha) = X \setminus E^f_n(\alpha)$, so that

$$C_n(\alpha) = \bigcap_{i=n}^{\infty} \quad \{x \in X : |f_i(x) - f(x)| \leq \alpha\}$$

and let $C = \{x \in X : f(x) = \lim\limits_k f_k(x)\}$. We note that

$$C = \bigcap_{\alpha>0} \bigcup_{n=i}^{\infty} C_n(\alpha).$$

Therefore $C \subseteq \bigcup_{n=1}^{\infty} C_n(\alpha)$ for all $\alpha > 0$.

Now , by hypothesis, $X \setminus C$ is contained in a $\mu - null$ set. Therefore we have

$$\mu\left(\bigcap_{n=1}^{\infty} E^f_n(\alpha)\right) = 0$$

But $(E_n^f(\alpha))_n$ is a decreasing sequence of measurable sets and , since $\mu(E_n^f(\alpha)) < +\infty$ for $n \geq n_\alpha$, we deduce that $\lim_{n\to\infty} \mu(E_n^f(\alpha)) = 0$.
Therefore (f_n) satisfies the vanishing restriction with respect to f.

□

2.2 Domination Conditions

We shall be providing a sufficient condition for the finiteness restriction that is often applicable.

Definition 2.2.1. Finite Distribution Function Let g be a non-negative $\boldsymbol{X}$ -measurable function on X. We define the distribution function of g by

$$\omega_g(\alpha) = \mu(\{x \in X : g(x) > \alpha\}) \text{ for } \alpha > 0.$$

We say that g has a finite distribution function in case $\omega_g(\alpha) < +\infty$ for all $\alpha > 0$.
In general, ω_g is a decreasing function defined on $(0, +\infty)$ to $[0, +\infty]$

Example: 2.2.2. *As an example of finite distribution function we cite a non-negative integrable function that is integrable over* $(X, \boldsymbol{X}, \mu)$*. For such a function we have*

$$\alpha\omega_g(\alpha) \leq \int_X g < +\infty$$

Let $X = [0, \infty)$ *and with lesbegue measure and let* g_1 *be any nonnegative function such that* $\lim_{x\to\infty} g_1 = 0$*. Then* g_1 *has a finite distribution function.*

Theorem 2.2.3. *If g is a non-negative integrable function such that* $|f_n(x)| \leq g(x)$ *a.e.,* $n \in \mathbb{N}$ *and if* $f(x) = \lim_n f_n(x)$ *for almost all* $x \in X$*, then the convergence is almost uniform on X.*

Proof. We define a set ,

$$E_{kn} = \bigcap_{m=n}^{\infty} \quad \{x \in X : |f_m(x) - f(x)| < 1/k\}$$

note that each of these sets are measurable.

Let $H = \{x \in X : f(x) = \lim_n f_n(x)\} \subset X$

For convenience let $A_{kn} = X \setminus E_{kn}$.

It is enough to show that $\lim_n \mu(A_{kn}) = 0$

From definition of convergence, it follows that for each k,

$H \subset \bigcup_{n=1}^{\infty} E_{kn}$

So,

$$\mu\left(\bigcap_{n=1}^{\infty} A_{kn}\right) = 0$$

Now A_k is a decreasing sequence, so if we show that for some n , $\mu(A_{kn}) < \infty$ then by the continuity property of measure, we will have our desired result.

Given that $|f_n(x)| \leq g(x)$ a.e. $x \in X, n \in \mathbb{N}$

It follows that $|f| \leq g$ a.e.

Thus for each m we have,$|f_m - f| \leq 2g$ a.e.

$$\begin{aligned} A_{kn} &= \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq 1/k\} \\ &\subseteq \{x \in X : g(x) \geq 1/2k\} \end{aligned}$$

Since g is a non-negative integrable function we have $\int_X g < \infty$. So it follows that

$$\begin{aligned} &\mu(A_{kn}) \leq 1/k \int_X g \qquad \forall k \\ \implies &\mu(A_{kn}) < \infty \end{aligned}$$

for each k and n. Thus we have

$$\lim_n \mu(A_{kn}) = 0$$

Hence the theorem.

□

Proposition 2.2.4. *Suppose that g is a non-negative measurable function with finite distribution function. Suppose that the sequence (f_n) satisfies the condition*

$$|f_i(x) - f_j(x)| \leq g(x) \qquad [respectively, |f_i(x) - f(x)| \leq g(x)] \qquad (*)$$

for all $x \in X$ and $i, j \in \mathbb{N}$. Then the sequence (f_n) satisfies the finiteness restriction [w.r.t. f].

Proof. Indeed, if $(*)$ is satisfied for all $i, j \in \mathbb{N}$ then,

$$\{x \in X : |f_i(x) - f_j(x)| > \alpha\} \subseteq \{x \in X : g(x) > \alpha\}$$

Therefore, we have $\mu(E_n(\alpha)) \leq \omega_g(\alpha) < \infty$ for all $\alpha > 0, n \in \mathbb{N}$.
The above proposition directly yields the dominated form of Egorov's Theorem : If g is a nonnegative integrable function such that $|f_i(x)| \leq g(x)$ for $x \in X, i \in \mathbb{N}$ and if $f(x) = \lim_{i\to\infty} f_i(x)$ for almost all $x \in X$, then the convergence is almost uniform on X. □

Definition 2.2.5. If (f_n) is a sequence of $\boldsymbol{X}$-measurable function on X and $n \in \mathbb{N}$, we define

$$\phi_n(x) = sup\{|f_i(x) - f_j(x)| : i, j \in \mathbb{N}, i \geq n, j \geq n\}.$$

If f is an $\boldsymbol{X}$ -measurable function on X. We define,

$$\phi_n^f(x) = sup\{|f_i(x) - f(x)| : i \in \mathbb{N}, i \geq n\}.$$

NOTE:

1. The functions ϕ_n and ϕ_n^f are $\boldsymbol{X}$-measurable and if $m \leq n$, then
   $\phi_n(x) \leq \phi_m(x)$ [respectively $\phi_n^f(x) \leq \phi_m^f(x)$] for all $x \in X$.
2. If $0 < \alpha < \beta$, then we have $E_n(\beta) \subseteq E_n(\alpha)$ and $E_n^f(\beta) \subseteq E_n^f(\alpha)$.
3. It is also evident that if $x \in X$, then the sequence $(f_i(x))$ is a cauchy sequence if and only if $\lim_{n\to\infty} \phi_n(x) = 0$ and that $\lim_{n\to\infty} f_n(x) = f(x)$ if and only if $\lim_{n\to\infty} \phi_n^f(x) = 0$

Lemma 2.2.6. *If $\alpha > 0$ and $n \in \mathbb{N}$, then*

$$E_n(\alpha) = \{x \in X : \phi_n(x) > \alpha\} \qquad and \qquad E_n^f(\alpha) = \{x \in X : \phi_n^f(x) > \alpha\}.$$

Proof. If $x \in E_n(\alpha)$ then there exists $i, j \in \mathbb{N}$ with $i \geq n, j \geq n$ such that

$$|f_i(x) - f_j(x)| > \alpha$$
$$\therefore \sup|f_i(x) - f_j(x)| > \alpha \qquad i, j \in \mathbb{N}, i \geq n, j \geq n$$
$$\implies \phi_n(x) > \alpha$$

and conversely.

Similarly we prove for $E_n^f(\alpha)$ $\square$

Corollary 2.2.7. *The sequence (f_n) satisfies the finiteness restriction [w.r.t. fA] if and only if for every $\alpha > 0$ there exists a natural number n_α such that the set $\{x \in X : \phi_{n_\alpha}(x) > \alpha\}$ [respectively ,$\{x \in X : \phi^f_{n_\alpha}(x) > \alpha\}$] has finite μ- measure.*

Proof. Since (f_n) satisfies the finiteness restriction for every $\alpha > 0$ there is a natural number n_α such that $\mu(E_{n_\alpha}(\alpha))$ [respectively , $\mu(E^f_{n_\alpha}(\alpha))$] has finite μ-measure.
Now by the previous Lemma we have,

$$E_n(\alpha) = \{x \in X : \phi_n(x) > \alpha\} \qquad \textit{and} \qquad E_n^f(\alpha) = \{x \in X : \phi_n^f(x) > \alpha\}.$$

Thus the set $\{x \in X : \phi_{n_\alpha}(x) > \alpha\}$ [respectively ,$\{x \in X : \phi^f_{n_\alpha}(x) > \alpha\}$] has finite μ-measure. $\square$

Corollary 2.2.8. *The sequence (f_n) satisfies the vanishing restriction [w.r.t. f] if and only if for every $\alpha > 0$,then*

$$\lim_{n\to\infty} \mu\{x \in X : \phi_n(x) > \alpha\} = 0 \qquad [\textit{respectively} \lim_{n\to\infty} \mu\{x \in X : \phi_n^f(x) > \alpha\} = 0$$

Theorem 2.2.9. *Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions and let f be a measurable function. Then the following assertions are equivalent.*

1. *The sequence $(f_n)_{n=1}^\infty$ converges almost uniformly to f.*
2. *For every $\alpha > 0$ there is a natural number n_α such that $\mu\{x \in X : \phi^f_{n_\alpha}(x) > \alpha\}$ and $\lim_{n\to\infty} \phi_n^f = 0$ for μ-almost all x.*
3. *The sequence (ϕ_n^f) converges in measure to 0.*

Chapter 3

CONVERGENCE IN MEASURE

Convergence in measure is a powerful concept in mathematics that
extends the idea of convergence from sequences of numbers to sequences of functions or more general measures. It's motivated by the desire to understand when one sequence of functions "approaches" another in some sense, even if not pointwise everywhere i.e. Suppose (f_n) is a sequence of integrable functions that converges in L_1 to some (integrable) function f. Then we cannot claim that (f_n) converges pointwise a.e. to f. But we can at least make this claim : Given $\varepsilon > 0$, Chebyshev's inequality tells us that

$$\mu\{|f_n - f| \geq \varepsilon\} \leq \frac{1}{\varepsilon}\int |f_n - f| \to 0$$

as $n \to \infty$. In other words, the seqeunce (f_n) cannot get too far away from f "in measure." Let's try to understand this new phenomenon.

Definition 3.0.1. Convergence in Measure: A sequence of functions $(f_n)_{n=1}^{\infty}$ converges in measure to f on the measurable set E if
for all $\sigma > 0$, $\lim_{n\to\infty}\mu(\{x \in E / |f_n(x) - f(x)| \geq \sigma\}) = 0$.
It is denoted by $f_n \xrightarrow{m} f$.

Definition 3.0.2. Convergence in Mean: A sequence of functions (f_n) converges in mean to f on a domain $\mathbb{R}$ if

$$\lim_n \int_{\mathbb{R}} |f_n - f| = 0$$

Result 3.0.3. *If the sequence (f_n) on $\mathbb{R}$ converges in mean, then (f_n) converges in measure.*

Proof. We prove this by the contrapositive method.
Suppose convergence in measure fails to hold, then there exists $\varepsilon, \eta > 0$ such that

$$\mu\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \varepsilon\} \geq \eta$$

for an infinite number of values of n. Now let

$$E_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\} \quad \textit{where} \quad \mu(E_n) \geq \eta.$$

Then we have $\int_{\mathbb{R}} |f_n(x) - f(x)| \geq \varepsilon\mu(E_n)$
$\therefore \int_{\mathbb{R}} |f_n(x) - f(x)| \geq \varepsilon\eta$ for all x, and for infinitely many values of n.
$\implies$ Convergence in mean fails to hold.
Hence the result. $\square$

Some basic properties of convergence in measure:

Result 3.0.4. *The limits in measure are unique upto equality a.e.*

Proof. Let f_n be a sequence of measurable functions and let $f_n \xrightarrow{m} f$ and $f_n \xrightarrow{m} g$
T.P.T $f = g$ a.e.

Let $\alpha > 0$ be arbitrary.

Now since $|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)|$

it follows that,

$$\{x : |f(x) - g(x)| \geq \alpha\} \subseteq \{x : |f(x) - f_n(x)| \geq \alpha/2\} \cup \{x : |f_n(x) - g(x)| \geq \alpha/2\} \quad \forall \alpha > 0$$

$$\therefore \mu(\{x : |f(x) - g(x)| \geq \alpha\}) \leq \mu(\{x : |f(x) - f_n(x)| \geq \alpha/2\}) + \mu(\{x : |f_n(x) - g(x)| \geq \alpha/2\})$$

then as $f_n \xrightarrow{m} f$ and $f_n \xrightarrow{m} g$ the left hand side tends to 0 as $n \to \infty$.

Thus choosing $\alpha = 1/n, n \in \mathbb{N}$, we get $\mu(\{x : |f(x) - g(x)| \geq \alpha\}) = 0$ $\square$

Result 3.0.5. *If $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$ then $f_n + g_n \xrightarrow{m} f + g$*

Proof. Let (f_n) and (g_n) be sequences of measurable functions such that $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$.

T.P.T $f_n + g_n \xrightarrow{m} f + g$

Let $\varepsilon > 0$ be arbitrary,

Note that $|(f_n + g_n) - (f + g)| \leq |f_n - f| + |g_n - g|$

Thus

$$\begin{aligned}\{x : |(f_n + g_n)(x) - (f + g)(x)| \geq \varepsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \varepsilon/2\} \cup \\ \{x : |g_n(x) - g(x)| \geq \varepsilon/2\}\end{aligned}$$

$$\begin{aligned}\therefore \mu(\{x : |(f_n + g_n)(x) - (f + g)(x)| \geq \varepsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon/2\}) + \\ \mu(\{x : |g_n(x) - g(x)| \geq \varepsilon/2\}) \\ \longrightarrow 0 \quad as \quad n \to \infty\end{aligned}$$

□

Result 3.0.6. *If* $f_n \xrightarrow{m} f$ *and* $g_n \xrightarrow{m} g$ *and if* $\mu(X) < \infty$ *then* $f_n g_n \xrightarrow{m} fg$

Proof. To prove this , let us first prove the required lemma.

Lemma 3.0.7. *(A) Suppose* $\mu(X) < \infty$ *and* f *is a real-valued measurable function. Then given* $\varepsilon > 0$ *there exists* M *such that* $\mu(x : |f(x)| > M) < \varepsilon$ *for all n.*

(B) Suppose $\mu(X) < \infty$ *and* f_n*,* f *are real-valued with* $f_n \to f$ *in measure. Then part (a) can be done uniformly in n: given* $\varepsilon > 0$ *there exists* M *such that* $\mu(x : |f_n(x)| > M) < \varepsilon$ *for all n.*

Proof:

(A) Considering only integers $M \geq 1$*, since f is real-valued we have* $\bigcap_{M\geq 1} \{x : |f(x)| > M\} = \phi$*. Now as* $\mu(X) < \infty$ *we can use continuity of f to conclude that*

$\lim_{M\to\infty} \mu(\{x : |f(x)| > M\}) = 0$*, then for given* $\varepsilon > 0$*, there exists M with* $\mu(\{x : |f(x)| > M\}) < \varepsilon$*.*

(B) Let $\varepsilon > 0$*. Then by (A),* $\exists M_0$ *such that* $\mu(\{x : |f(x)| > M_0 - 1\}) < \varepsilon/2$*. By convergence in measure,* $\exists n_0$ *such that for* $\forall n \geq n_0$ *we have* $\mu(\{x : |f_n(x) - f(x)| > 1\}) < \varepsilon/2$*, and hence also*

$$\mu(\{x : |f_n(x)| > M_0\}) = \mu(\{x : |f(x)| > M_0 - 1\}) + \mu(\{x : |f_n(x) - f(x)| > 1\}) < \varepsilon.$$

By (A) again, for each $1 \leq n < n_0$ *there exists* M_n *such that* $\mu(\{x : |fn(x)| > M_n\}) < \varepsilon$*.*

Letting $M = max(M_0, M_1, .., M_{n_0-1})$ *we then have* $\mu(\{x : |fn(x)| > M\}) < 2\varepsilon$ *for all* $n \geq 1$*.Since* ε *is arbitrary this completes the proof.*

Now to prove the result, we have

$$|f_n g_n - fg| \leq |f_n(g_n - g)| + |(f_n - f)g|$$

$$\begin{aligned}\therefore \mu(\{x : |(f_n g_n)(x) - (fg)(x)| > \varepsilon\}) &\leq \mu(\{x : |f_n(x)| > k\}) + \mu(\{x : |g_n(x) - g(x)| > \varepsilon/2k\}) \\ &+ \mu(\{x : |f_n(x) - f(x)| > \varepsilon/2k\}) + \mu(\{x : |g(x)| > k\}) \\ &\longrightarrow 0 \quad as \quad n \to \infty\end{aligned}$$

Thus $f_n g_n \xrightarrow{m} fg$.

□

An alternative proof for Result 3.0.6 can be given as follows.

Lemma 3.0.8. *If* $f_n \xrightarrow{m} 0$ *and* $g_n \xrightarrow{m} 0$ *then* $f_n g_n \xrightarrow{m} 0$

Proof. This can be proven by the fact that for $\varepsilon > 0$,

$\{x : |f_n(x)g_n(x)| \geq \varepsilon\} \subset \{x : |f_n(x)| \geq \sqrt{\varepsilon}\} \cup \{x : |g_n(x)| \geq \sqrt{\varepsilon}\}$

$\therefore \mu(\{x : |(f_n(x)g_n(x)| \geq \varepsilon\}) \leq \mu(\{x : |f_n(x)| \geq \sqrt{\varepsilon}\}) + \mu(\{x : |g_n(x)| \geq \sqrt{\varepsilon}\})$
The R.H.S $\longrightarrow 0$ as $n \to \infty$.

Thus $f_n g_n \xrightarrow{m} 0$ □

Lemma 3.0.9. *If* $f_n \xrightarrow{m} 0$ *and* g *is a measurable function and* $\mu(X) < \infty$ *then* $f_n g \xrightarrow{m} 0$

Proof. We tacitly assume that g is finite a.e. and given $\mu(X) < \infty$ then we have,

$$\lim_{k \to \infty} \mu(\{x : |g(x)| \geq k\}) = 0$$

So given $\eta > 0$, $\exists k_0 \ni \mu(\{x : |g(x)| \geq k_0\}) < \eta$ Thus for any $\varepsilon > 0$, we have

$\{x : |f_n(x)g(x)| \geq \varepsilon\} \subset \{x : |f_n(x)| \geq \varepsilon/k_0\} \cup \{x : |g(x)| \geq k_0\}$

Hence $\quad f_n g \xrightarrow{m} 0.$

Now to prove the result, we note that

$$f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f)$$

Then by using Lemma 3.0.8, 3.0.9 and Result 3.0.5 we are done i.e. $f_n g_n \xrightarrow{m} fg$ $\square$

A counter example for the Result 3.0.6 would be ,

Example: 3.0.10. *if $\mu(X) = \infty$, take $X = \mathbb{R}$ and $f = g$ unbounded, say $f(x) = g(x) = x^2$ on $\mathbb{R}$ and let $f_n = g_n = f + 1/n$*

Then for any $x \in [n, \infty)$ we have

$$|f_n(x)g_n(x) - f(x)g(x)| \geq 2(\frac{x^2}{n}) \geq 2n$$

So, $\mu(\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq 1\}) \geq \mu([n, \infty)) \not\to 0$ as $n \to \infty$

Thus we see that $f_n g_n \not\to fg$

Definition 3.0.11. Cauchy sequence in measure: We say that a sequence (f_n) is cauchy sequence in measure if given $\varepsilon > 0$,

$\exists N \in \mathbb{N} \ni \mu(\{|f_n - f_m| \geq \varepsilon\}) < \varepsilon$ whenever $m, n \geq N$

Result 3.0.12. *If $f_n \xrightarrow{m} f$ then (f_n) is cauchy sequence in measure.*

Proof. Let $\eta, \varepsilon > 0$ be given, as $f_n \xrightarrow{m} f$,

$\exists N \in \mathbb{N} \ni \forall n \geq N$ we have,

$\mu(\{|f_n - f| \geq \eta/2\}) < \varepsilon/2$
Now for any $m, n \geq N$, $\quad |f_n - f_m| \geq |f_n - f| + |f - f_m|$

$$\begin{aligned}\mu(\{|f_n - f_m| \geq \eta\}) &\leq \mu(\{|f_n - f| \geq \eta/2\}) + \mu(\{|f - f_m| \geq \eta/2\}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

$\therefore$ for given $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni \mu(\{|f_n - f_m| \geq \eta\}) < \varepsilon$ whenever $m, n \geq N$ $\square$

Result 3.0.13. *If (f_n) is cauchy sequence in measure and if some subsequence $(f_{n_k}) \xrightarrow{m} f$, then $(f_n) \xrightarrow{m} f$*

Proof. Let $\alpha, \varepsilon > 0$ be given, since (f_n) is cauchy $\implies$ $\exists N \in \mathbb{N} \ni \forall m, n \geq N$
$\mu(\{|f_n - f_m| \geq \alpha/2\}) < \varepsilon/2$

Also, $(f_{n_k}) \xrightarrow{m} f$, $\exists K \in \mathbb{N} \ni \forall k \geq K$,

$\mu(\{|f_{n_k} - f| \geq \alpha/2\}) < \varepsilon/2$
Thus for all $n \geq N$ and taking $k \geq \max\{N, K\}$

$$\begin{aligned}\mu(\{|f_n - f_m| \geq \alpha\}) &\leq \mu(\{|f_n - f_{n_k}| \geq \alpha/2\}) + \mu(\{|f_{n_k} - f| \geq \alpha/2\}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

Hence $f_n \xrightarrow{m} f$ $\square$

Theorem 3.0.14. *If the sequence $(f_n)_{n=1}^{\infty}$ converges almost uniformly on E, then it converges in measure on E.*

Proof. Let $\varepsilon > 0$ be arbitrary.
Since (f_n) is almost uniformly convergent to f, $\exists$ a measurable set $A \subset E$ with $\mu(A) < \varepsilon$ such that the sequence $f_n \rightrightarrows f$ on $E \setminus A$.

Given $\delta > 0, \exists N \in \mathbb{N} \ni |f_n(x) - f(x)| < \delta, \quad \forall x \in E \setminus A, n \geq \mathbb{N}.$
It follows that $n \geq \mathbb{N}$,

$$\mu([x \in E : |f_n(x) - f(x)| \geq \delta]) \leq \mu([x \in E \setminus A : |f_n(x) - f(x)| \geq \delta]) + \mu(A)$$
$$= \mu(A) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,
$\implies \lim_{n\to\infty} \mu([x \in E : |f_n(x) - f(x)| \geq \delta]) = 0$ □

But the converse need not hold true. For example,

Example: 3.0.15. *Consider the sequence of functions* $f_n = \dfrac{1}{n} \sin(nx)$.

Corollary 3.0.16. *If* (f_n) *converges pointwise a.e. to f on E, where E has finite measure, then* (f_n) *also converges in measure to f on E.*

Proof. Let $\varepsilon > 0$ be arbitrary.
Since (f_n) is almost everywhere pointwise convergent to f, $\exists$ a measurable set $D \subset E$ with $\mu(D) = 0$ such that the sequence $f_n \to f$ pointwise on $E \setminus D$.
Given $\delta > 0, \exists N \in \mathbb{N} \ni \forall x \in E \setminus A, \forall n \geq \mathbb{N}, \quad |f_n(x) - f(x)| < \delta,.$
In particular for any $n \geq \mathbb{N}$,

$$\mu([x \in E : |f_n(x) - f(x)| \geq \delta]) \leq \mu([x \in E \setminus A : |f_n(x) - f(x)| \geq \delta]) + \mu(A)$$
$$= \mu(A) = 0.$$

$\implies \mu([x \in E : |f_n(x) - f(x)| \geq \delta]) = 0$ □

Now we will see by an example that convergence in measure is not implied by pointwise convergence in general.

Example: 3.0.17. *Let* $f_n = \chi_{[n,n+1]}$.
Then for $x \in [0,\infty)$*, the sequence* (f_n) *converges to* 0 *pointwise.*
Now for $0 < \varepsilon < 1$ *we have* $\mu(\{|f_n - 0| \geq \varepsilon\}) = 1$
i.e. $\mu(\{|f_n| \geq \varepsilon\}) = 1$
Thus $f_n \not\to f$ *in measure.*

Proposition 3.0.18. *If* (f_n) *converges in measure (or in mean) to f, and if* (f_n) *satisfies the vanishing restriction, then the sequence* (f_n) *converges almost uniformly to f.*

Proof. The proof follows from the Proposition 2.1.6 , that there exists a measurable function g to which (f_n) converges almost uniformly, and therefore in measure.
So thus, we have $f_n \xrightarrow{m} f$ and $f_n \xrightarrow{m} g$ which implies that $f(x) = g(x)$ for almost all $x \in X$, and hence it follows that (f_n) converges almost uniformly to f . □

The next definition is generalization of the notion of monotone convergence of a sequence (f_n) to a limit function f.

Definition 3.0.19. A sequence (f_n) of measurable functions is said to be M-convergent to a measurable function f if, for all $\alpha > 0$, we have

$\{x \in X : |f_j(x) - f(x)| > \alpha\} \subseteq \{x \in X : |f_i(x) - f(x)| > \alpha\}$ whenever $i \leq j (i, j \in \mathbb{N})$

Proposition 3.0.20. *If the sequence* (f_n) *converges in measure [respectively, in mean] to f and is M-convergent, then the sequence* (f_n) *satisfies the vanishing restriction and the convergence to f is almost uniform.*

Proof. If the sequence is M-convergent to f, then
$E_n^f(\alpha) = \{x \in X : |f_n(x) - f(x)| > \alpha\}$
But since (f_n) converges in measure to f, we have $\lim_n \mu(E_n^f(\alpha)) = 0$.
Thus the vanishing restriction is satisfied. □

Following is the fundamental result which gives the connection between convergence in measure and pointwise convergence, due to F.Riesz.

Theorem 3.0.21. *Let (f_n) be a sequence of real-valued measurable functions, all defined on a common measurable domain D. If (f_n) is Cauchy in measure, then there is a measurable function $f : D \to \mathbb{R}$ such that (f_n) converges in measure to f. Moreover, there is a subsequence (f_{n_k}) of (f_n) that converges pointwise a.e. to f.*

Proof. We first establish the "moreover" claim by showing that (f_n) has a subsequence which is pointwise Cauchy. To achieve this we have : Since (f_n) is given to be Cauchy in measure, we can choose a subsequence (f_{n_k}) satisfying

$$\mu(\{x \in D : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\}) < 2^{-k} \text{ for all k.}$$

In other words, we set $E_k = \{|f_{n_{k+1}} - f_{n_k}| \geq 2^{-k}\}$ where $\mu(E_k) < 2^{-k}$ for all k.
Now , since $\sum_k \mu(E_k) < \infty$, by the Borel-Cantelli Lemma , which states that: If each E_n is measurable,and if $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then
$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} = \mu\left(\limsup_{n\to\infty} E_n\right) = 0\right)$ we have, the set

$$E = \limsup_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \text{ has measure zero.}$$

Also note that for any $x \notin E$ we have $x \notin \bigcup_{j=k}^{\infty} E_j$ for some sufficiently large k, and thus

$$|f_{n_{j+1}}(x) - f_{n_j}(x)| < 2^{-j} \quad \text{ for all } j \geq k.$$

In particular, we must have $\sum_j (f_{n_{j+1}}(x) - f_{n_j}(x)) < \infty$. Hence for any $x \notin E$, the limit

$$f(x) = \lim_{j\to\infty} f_{n_j}(x) = f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x)) \text{exists.} \tag{3.1}$$

If we define $f(x) = 0$ for $x \in E$, then we have defined a measurable function f for which $f_{n_k}(x) \to f(x)$ for any $x \notin E$; that is $f_{n_k} \to f$ a.e. Now all that remains to check is that $f_n \to f$ in measure. For this, note that for any $x \notin E$ we may write

$$f(x) - f_{n_k}(x) = \sum_{j=k}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x)),$$

and hence from equation (3.1), for any $x \notin \bigcup_{j=k}^{\infty} E_j$ we have

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}$$

[i.e.(f_{n_k}) converges almost uniformly to f.]
In particular, we must have

$$\mu\{|f - f_{n_k}| \geq 2^{-k+1}\} \leq \mu(\bigcup_{j=k}^{\infty} E_j) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}.$$

Thus, (f_{n_k}) converges in measure to f. Since (f_n) is Cauchy in measure it further implies that (f_n) itself converges in measure to f. Now , it follows that if (f_n) is a sequence of measurable functions and if, for some function f on D, we have

$$\mu\{x \in D : |f_n(x) - f(x)| \geq \varepsilon\} \to 0 \text{ as } n \to \infty$$

for every $\varepsilon > 0$, then f is measurable.
Hence the theorem. $\square$

The next result is a version of the above theorem.

Proposition 3.0.22. *If a sequence (f_n) is Cauchy in measure [respectively in mean,], then it has a subsequence (f_{n_k}) that satisfies the vanishing restriction, and which converges almost uniformly to a measurable function f. Moreover, (f_n) converges in measure to f.*

Proof. Since (f_n) is Cauchy in measure, we can choose a subsequence $(f_{n_k}) = (g_k)$ such that if $A_k = \{x \in X : |g_{k+1}(x) - g_k| > 2^{-k}\}$ then $\mu(A_k) < 2^{-k}$ for all k.
Let $F_k = \bigcup_{j=k}^{\infty} A_j)$ so that $F_k \in \boldsymbol{X}$ and $\mu(F_k) < 2^{-k+1}$.
If $i \geq j \geq k$ and if $x \notin F_k$, then

$$\begin{aligned}|g_i(x) - g_j(x)| &\leq |g_i(x) - g_{i-1}(x)| + |g_{i-1}(x) - g_{i-2}(x)| + \cdots + |g_{j+1}(x) - g_j(x)| \\ &\leq 2^{-i+1} + 2^{-i+2} + \cdots + 2^{-j} \\ &< 2^{-j+1} \leq 2^{-k+1}\end{aligned}$$

Hence if $\alpha > 0$ is given, let K_1 be such that if $k \geq K_1$, then $2^{-k+1} < \alpha$.
It follows that, if $i \geq j \geq k$, then

$$|g_i(x_0) - g_j(x_0)| > \alpha \text{ implies that } x_0 \in F_k.$$

Thus we have $E_n(\alpha) \subseteq F_k$ for all $n \geq k$.
Now let $\varepsilon > 0$ be given and let $K \geq K_1$ be chosen such that$2^{-K+1} < \varepsilon$. If $n \geq K$ and if $i \geq j \geq n$, then we have $E_n(\alpha) \subseteq F_k$ so that $\mu(E_n(\alpha)) \leq \mu(F_k) < 2^{-K+1} < \varepsilon$.
Since $\varepsilon > 0$ is arbitrary, it follows that (g_k) satisfies the vanishing restriction .
Therefore (g_k) converges almost uniformly (and therefore in measure) to a measurable function f. □

Proposition 3.0.23. *If the sequence (f_n) satisfies the condition $\sum_{n=1}^{\infty} ||f_n - f|| < \infty$, then the sequence (f_n) satisfies the vanishing restriction with respect to f and converges almost uniformly to f.*

Proof. If $\alpha > 0$ and if $B_n = \{x \in X : |f_n(x) - f(x)| > \alpha\}$,
then we have , $B_n \in \boldsymbol{X}$ and $\mu(B_n) \leq (1/\alpha)||f_n - f||$.

Moreover, we have $E_n{}^f(\alpha) = \bigcup_{i=n}^{\infty} B_i$ so that

$$\mu(E_n{}^f(\alpha)) \leq (1/\alpha)\sum_{i=n}^{\infty} ||f_i - f||$$

Since the series $\sum ||f_n - f||$ converges , we deduce that $\lim \mu(E_n{}^f(\alpha)) = 0$, so that (f_n) satisfies the vanishing restriction. □

Now we will see an analogue of Fatou's Lemma which holds for convergence in measure.

Lemma 3.0.24. *Let (f_n) be a sequence of non-negative measurable functions and let f be a measurable function such that $f_n \to f$ in measure; then $\int f \leq \liminf_{n\to\infty} \int f_n$*

Proof. ***Case(i)*:**Suppose that $\int f < \infty$ and that $\int f > \liminf_{n\to\infty} \int f_n$.
Then there exist $\delta > 0$ and a sequence (n_k) such that, for each k,$\int f_{n_k} < \int f - \delta$.
But $f_{n_k} \to f$ in measure, so by theorem (A) we can find a subsequence (n_{k_l}) of (n_k) such that
$f_{n_{k_l}} \to f$ a.e. But then by Fatou's Lemma,

$$\int f \leq \liminf_{n\to\infty} \int f_{n_{k_l}} \leq \int f - \delta,$$

giving a contradiction.

***Case(ii)*:** Now suppose that $\int f = \infty$ and that $\liminf_{n\to\infty} \int f_n < \infty$.
Then there exist $K > 0$ and a subsequence f_{n_k} such that, for each k, $\int f_{n_k} < K$. But again we can find a subsequence n_{k_l} of (n_k) such that $f_{n_{k_l}} \to f$ a.e. But then, by Fatou's Lemma, $\liminf_{n\to\infty} \int f_{n_{k_l}} = \infty$, giving a contradiction.
So $\liminf_{n\to\infty} \int f_n = \infty$ giving the result. □

We also have an analogue of the Lesbegue Dominated Convergence theorem, using convergence in measure.

Theorem 3.0.25. *Let (f_n) be a sequence of measurable functions such that $|f_n| < g$, an integrable function, and let $f_n \to f$ in measure, where f is measurable. Then f is integrable,* $\lim_{n\to\infty} \int f_n = \int f$ *and* $\lim_{n\to\infty} \int |f_n - f| = 0$

Proof. By Theorem(3.0.21), there exists a subsequence (f_{n_k}) of (f_n) with limit f a.e., so we have $|f| \leq g$ and so f is integrable.
Also , for each n, we have $|f_n| < g$, that implies $g + f_n \geq 0$ and $g - f_n \geq 0$. Thus $(g + f_n)$ and $(g - f_n)$ are sequences of non-negative integrable function and $(g + f_n) \to (g + f)$ in measure and $(g - f_n) \to (g - f)$ in measure.
Applying the analogue of Fatou's Lemma to both these sequences we have,

$$\int g + \int f \leq \liminf_{n\to\infty} \int (g + f_n) \quad \text{and} \quad \int g - \int f \leq \liminf_{n\to\infty} \int (g - f_n)$$

So that we have, $\int f \leq \liminf_{n\to\infty} \int f_n$ and $\int f \geq \limsup_{n\to\infty} \int f_n$
Thus we have $\lim_{n\to\infty} \int f_n = \int f$.
Also , it is clear from the definition of convergence in measure that $|f_n - f| \to 0$ in measure.
But $|f_n - f| \leq 2g$, so the second result follows from the first. □

The concept of convergence in measure can be applied in Probability Theory. One of the application of convergence in measure in Probability can be given by the following example. But before that we first recall some basic terms used in Probability.

Definition 3.0.26. (Sample space) A sample space is a collection or a set of possible outcomes of a random experiment. The sample space is represented using the symbol, "S".

Definition 3.0.27. (Event) The subset of possible outcomes of an experiment is called an event.

Definition 3.0.28. (Random Variable) In probability, a real-valued function, defined over the sample space of a random experiment, is called a random variable.

Definition 3.0.29. (Discrete Random Variable) A discrete random variable can take only a finite number of distinct values such as $0, 1, 2, 3, 4, \cdots$ and so on.

Definition 3.0.30. (Expectation) Expected value of a discrete random variable can be defined by

$$E[X] = \sum_{i=1}^{n} X_i P_i(X)$$

where $X_1, X_2, X_3, \cdots$ are discrete random variables and $P_i(X)$ is the probability of the outcome X

Definition 3.0.31. (Variance) The variance of a random variable X is given by

$$Var[X] = \sigma^2 = E[(X - \mu)^2]$$

where $\mu = E[X]$

We note that the concept of convegence in measure is equivalent to the concept of convergence in probability. Having said so we see what is meant by convergence in probability.

Definition 3.0.32. (Convergence in Probability) A sequence of random variables $X_1, X_2, X_3, \cdots$ converges in probability to a random variable X, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0, \text{ for all } \varepsilon > 0.$$

Example: 3.0.33. *Let X be a random variable, and $X_n = X + Y_n$, where*

$$EY_n = \frac{1}{n}, \qquad Var(Y_n) = \frac{\sigma^2}{n},$$

where $\sigma > 0$ is a constant. Show that $X_n \xrightarrow{p} X$.

Proof. First note that by the triangle inequality, for all $a, b \in \mathbb{R}$, we have $|a+b| \leq |a| + |b|$. Choosing $a = Y_n - EY_n$ and $b = EY_n$, we obtain

$$|Y_n| \leq |Y_n - EY_n| + \frac{1}{n}.$$

Now for any $\varepsilon > 0$, we have

$$\begin{aligned}
P(|X_n - X| \geq \varepsilon) &= P(|Yn| \geq \varepsilon) \\
&\leq P(|Y_n - EY_n| + \frac{1}{n} \geq \varepsilon) \\
&= P\left(|Y_n - EY_n| \geq \varepsilon - \frac{1}{n}\right) \\
&\leq \frac{Var(Y_n)}{(\varepsilon - \frac{1}{n})^2} \qquad \text{(by Chebychev's Inequality)} \\
&= \frac{\sigma^2}{n(\varepsilon - \frac{1}{n})^2} \longrightarrow 0 \\
&\text{as } n \longrightarrow \infty.
\end{aligned}$$

Therefore, we conclude that $X_n \xrightarrow{p} X$. □

Chapter 4

LEBESGUES'S DIFFERENTIATION THEOREM

Why Lebesgues differentiation?

1. For which f does the formula $\int_a^b f' = f(b) - f(a)$ hold? If f' is to be integrable, then at the very least we will need f' to exist almost everywhere in [a,b]. But this alone is not enough: the Cantor function $f : [0,1] \to [0,1]$ satisfies $f' = 0$ a.e., but $\int_0^1 f' = 0 \neq 1 = f(1) - f(0)$

2. Stated in slightly different terms: If g is integrable, is the function $f(x) = \int_a^x g$ differentiable? And, if so, is $f' = g$ in this case? For which f is it true that $f(x) = \int_a^x g$ for some integrable g?

3. Given α increasing, is α differentiable at enough points so as to have $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$ hold for, say, all continuous f? i.e., is every Riemann-Stieltjes integral a Lebesgue integral?

4. In particular, if f is of bounded variation, does f' exist? Is f' integrable? If so, is it the case that $V_a^b f = \int_a^b |f'|$?

Definition 4.0.1. **{Derived Number}** Given a function $f : \mathbb{R} \to \mathbb{R}$, an extended real number λ is called a derived number for f at the point x_0 if there exists a sequence $h_n \to 0 (h_n \neq 0)$ such that

$$\lim_{n\to\infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = \lambda$$

In other words, λ is a derived number for f at x_0 if some sequence of difference quotients for f at x_0 converges to λ (where $\lambda = \pm\infty$ possibilities are included.) We use the following abbreviation to denote the above statement, $\lambda = Df(x_0)$, with the understanding that $Df(x_0)$ denotes just one of possibly many different derived numbers for f at x_0.
Note that since we permit infinite derived numbers, it is clear that derived numbers exist at every point x_0.This is because, if the derivative $f'(x_0)$ exists (whether finite or infinite), then $f'(x_0)$ is a derived number for f at x_0.

Example: 4.0.2. *Consider the function* $f(x) = x\sin(1/x), x \neq 0, f(0) = 0$, *at the point* $x_0 = 0$. *If we set* $h_n^{-1} = (4n-3)\pi/2$, *then*

$$\frac{f(x_0 + h_n) - f(x_0)}{h_n} = \frac{h_n \sin(h_n^{-1})}{h_n} = \sin\frac{(4n-3)\pi}{2} = 1$$

for all $n = 1, 2, 3, \cdots$
Thus, $\lambda = 1$ *is a derived number for f at 0. Also, every number in [-1,1] is a derived number for f at 0.*
If we set $h_n^{-1} = (2n-3)\pi/2$ *then we get* $\lambda = \pm 1$

Before we start, note that to say a function f has finite derivative almost everywhere is the same as saying that the set of points x_0 at which f has two different derived numbers, say $D_1f(x_0) < D_2f(x_0)$, has measure zero. To address this we shall consider those derived numbers that satisfy $D_1f(x_0) \leq p < q \leq D_2f(x_0)$, where $p < q$ are real numbers. To circumvent occasional concerns about the domain of f we assume that every function $f : [a,b] \to \mathbb{R}$ has been extended to all of $\mathbb{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$.

Definition 4.0.3. **{Vitali Cover}** We say that a collection $\mathscr{C}$ of closed, nontrivial intervals in $\mathbb{R}$ forms a Vitali cover for a subset E of $\mathbb{R}$ if, for any $\varepsilon > 0$, there is an interval $I \in \mathscr{C}$ with $x \in I$ and $\mu(I) < \varepsilon$.In other words, $\mathscr{C}$ is a Vitali cover for E if, for every $\varepsilon > 0$,

$$E \subset \bigcup \{I : I \in \mathscr{C} \text{ and } \mu(I) < \varepsilon\}.$$

Lemma 4.0.4. ***Vitali's Covering Theorem:*** *Let E be a set of finite outer measure, and let $\mathscr{C}$ be a Vitali cover for E. Then, there exist countably many pairwise disjoint intervals (I_n) in $\mathscr{C}$ such that*

$$\mu\left(E \setminus \bigcup_{n=1}^{\infty} I_n\right) = 0$$

Proof. We can simplify things a bit by making two observations: First, since $\mu^*(E) < \infty$ there is an open set U containing E with $\mu(U) < \infty$. Next, given $x \in E \subset U$ and $\varepsilon > 0$ there is an interval $I \in \mathscr{C}$ such that $x \in I \subset U$ and $\mu(I) < \varepsilon$. Thus, the collection $\{I \in \mathscr{C} : I \subset U\}$ is still a Vitali cover for E.

Since it is enough to prove the theorem for this collection, we may simply suppose that each element of $\mathscr{C}$ is already contained in U.

To begin, let's choose any interval I_1 in $\mathscr{C}$. If $\mu(E \setminus I_1) = 0$ we are done; otherwise, we continue to choose intervals from $\mathscr{C}$. Next, choose interval I_2 such that $I_1 \cap I_2 = \phi$. If $\mu(E \setminus I_1 \cup I_2) = 0$ then we are done or if not then we continue the process: Suppose that pairwise disjoint, closed intervals $I_1, I_2, \cdots, I_n$ have been constructed with $\mu(E \setminus \bigcup_{k=1}^{n} I_k) > 0$. We want to choose I_{n+1} so that it is the "next biggest" interval in $\mathscr{C}$ that is disjoint from $I_1, I_2, \cdots, I_n$.

To accomplish this, consider the intervals in $\mathscr{C}$ that are completely contained in the open set $G_n = U \setminus \bigcup_{k=1}^{n} I_k$.

Since $E \setminus \bigcup_{k=1}^{n} I_k \neq \phi$, and since $\mathscr{C}$ is a Vitali cover for E, such intervals exist; note that any interval J of such type will satisfy $0 < \mu(J) \leq \mu(U)$ (since the intervals in $\mathscr{C}$ are nontrivial.) Setting

$$k_n = \sup\{\mu(J) : J \in \mathscr{C} \text{ and } J \subset G_n\},$$

it is clear that $0 < k_n < \infty$. We now choose $I_{n+1} \in \mathscr{C}$ with $\mu(I_{n+1}) > k_n/2$ and $I_{n+1} \subset G_n = U \setminus \bigcup_{k=1}^{n} I_k$. Obviously I_{n+1} is disjoint from $I_1, I_2, \cdots, I_n$. If $\mu(E \setminus \bigcup_{k=1}^{n+1} I_k) = 0$, the construction terminates and the theorem is proved; otherwise we continue, choosing I_{n+2}, and so on. If our construction does not terminate in finitely many steps, then it yields a sequence (I_k) of pairwise disjoint intervals in $\mathscr{C}$ with $\bigcup_{k=1}^{\infty} I_k \subset U$ and, of course, $\sum_{k=1}^{\infty} \mu(I_k) \leq \mu(U) < \infty$. It only remains to show that $\mu(E \setminus \bigcup_{k=1}^{\infty} I_k) = 0$. To this end, first notice that each $J \in \mathscr{C}$ must overlap some I_n. Indeed, if $J \cap (\bigcup_{k=1}^{n} I_k) = \phi$ for all n, then we would have $\mu(J) \leq k_n < 2\mu(I_{n+1}) \to 0$ as $(asn \to \infty)$, which contradicts the fact that $\mu(J) > 0$.

Finally, let $\varepsilon > 0$ and choose N so that $\sum_{k=N+1}^{\infty} \mu(I_k) < \varepsilon$. Given $x \in E \setminus \bigcup_{k=1}^{N} I_k \subset G_N$, choose an interval $J \in \mathscr{C}$ with $x \in J$ and $J \cap (\bigcup_{k=1}^{N} I_k) = \phi$. By our observation above, we know that there is a smallest n such that $J \cap I_n \neq \phi$.

Necessarily, $n > N$ and $\mu(J) < 2\mu(I_n)$. Thus, if we let J_n, be the closed interval having the same midpoint as I_n but with radius five times that of I_n, that is, with $\mu(J_n) = 5\mu(I_n)$, then $J \subset J_n$. In other words, what we have shown is that

$$E \setminus \bigcup_{k=1}^{\infty} I_k \subset E \setminus \bigcup_{k=1}^{N} I_k \subset \bigcup_{k=N+1}^{\infty} J_k$$

and so,

$$\mu^*\left(E\setminus\bigcup_{k=1}^{\infty}I_k\right)\leq\mu^*\left(E\setminus\bigcup_{k=1}^{N}I_k\right)\leq\sum_{k=N+1}^{\infty}\mu(J_k)$$
$$=5\sum_{k=N+1}^{\infty}\mu(I_k)<5\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get $\mu E\setminus\bigcup_{k=1}^{\infty}I_k=0$. $\square$

Lemma 4.0.5. *Let $f:[a,b]\to\mathbb{R}$ be strictly increasing, let $E\subset[a,b]$, and let $0\leq p<\infty$. If, for every $x\in E$, there exists at least one derived number for f satisfying $Df(x)\leq p$, then $\mu^*(f(E))\leq p\mu^*(E)$*

Proof. Let $\varepsilon>0$ be arbitrary, we choose a bounded open set $G\supset E$ such that $\mu(G)<\mu^*(E)+\varepsilon$.

For each $x_0\in E$, choose a null sequence (h_n), with $h_n\neq 0$ for all n, such that

$$\lim_{n\to\infty}\frac{f(x_0+h_n)-f(x_0)}{h_n}=Df(x_0)\leq p.$$

Now consider the intervals,

$$d_n(x_0)=\begin{cases}[x_0,x_0+h_n] & if h_n>0,\\ [x_0+h_n,x_0] & if h_n<0,\end{cases}\tag{4.1}$$

and

$$\Delta_n(x_0)=\begin{cases}[f(x_0),f(x_0+h_n)] & if h_n>0,\\ [f(x_0+h_n),f(x_0)] & if h_n<0.\end{cases}\tag{4.2}$$

The intervals $\{d_n(x_0):x_0\in E,n\geq 1\}$ forms a cover for E while the intervals $\{\Delta_n(x_0):x_0\in E,n\geq 1\}$ forms a cover for $f(E)$. Notice that since f is strictly increasing, we have $\mu(\Delta_n(x_0))>0$ for any x_0,n.

Since $h_n \to 0$, we may assume that $d_n(x_0) \subset G$ for all n. We may also suppose that

$$\frac{f(x_0+h_n)-f(x_0)}{h_n} < p+\varepsilon \text{ for all n.} \tag{4.3}$$

Since $\mu(d_n(x_0)) = |h_n|$ and $\mu(\Delta_n(x_0)) = |f(x_0+h_n)-f(x_0)|$, equation (4.3) can be written as

$$\mu(\Delta_n(x_0)) < (p+\varepsilon)\mu(d_n(x_0)) \text{ for all n}$$

In particular, we must have $\mu(\Delta_n(x_0)) \to 0$ as $h_n \to 0$. Thus, the intervals $\{\Delta_n(x_0) : x_0 \in E, n \geq 1\}$ actually form a Vitali cover for f(E).

By Vitali's Covering Theorem, we can find countably many pairwise disjoint intervals $\{\Delta_{n_i}(x_i)\}$ such that

$$\mu^*\left(f(E) \setminus \bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right) = 0$$

Thus,

$$\mu^*(f(E)) \leq \sum_{i=1}^{\infty} \mu(\Delta_{n_i}(x_i)) < (p+\varepsilon)\sum_{i=1}^{\infty} \mu(d_{n_i}(x_i)) \tag{4.4}$$

But the intervals $\{d_{n_i}(x_i)\}$ must also be pairwise disjoint since f is strictly increasing. Hence,

$$\sum_{i=1}^{\infty} \mu(d_{n_i}(x_i)) = \mu\left(\bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right) \leq \mu(G). \tag{4.5}$$

Combining equations (4.4) and (4.5) yields

$$\mu^*(f(E)) < (p+\varepsilon)\mu(G) < (p+\varepsilon)(\mu^*(E)+\varepsilon)$$

Letting $\varepsilon \to 0$, we get $\mu^*(f(E)) \leq p\mu^*(E)$. □

Lemma 4.0.6. *Let $f : [a,b] \to \mathbb{R}$ be strictly increasing, let $E \subset [a,b]$, and let $0 \leq q < \infty$. If, for every $x \in E$, there exists at least one derived number for f satisfying $Df(x) \geq q$, then $\mu^*(f(E)) \geq q\mu^*(E)$*

Proof. Let $\varepsilon > 0$.Since f(E) is bounded, we may choose a bounded open set $G \supset f(E)$ such that $\mu(G) < \mu^*(f(E)) + \varepsilon$. For each $x_0 \in E$, choose a null sequence (h_n) such that

$$\lim_{n\to\infty} \frac{f(x_0+h_n) - f(x_0)}{h_n} = Df(x_0) \geq q.$$

As in the earlier proof we may assume that

$$\frac{f(x_0+h_n) - f(x_0)}{h_n} > q - \varepsilon \text{ for all n.} \tag{4.6}$$

Thus, if we define the intervals $d_n(x_0)$ and $\Delta_n(x_0)$ exactly as in equation (4.1) and (4.2), then we have $\mu(\Delta_n(x_0)) > (q-\varepsilon)\mu(d_n(x_0))$ for all n and all $x_0 \in E$

We would like to argue that by reducing to countably many intervals we can compare the measures of E and f(E), by way of the open set G. If $x_0 \in E$ is a point of continuity of f, then $\Delta_n(x_0)$ will be completely contained in G for all n sufficiently large. This works at nearly every point $x_0 \in E$: If we let S denote the set of points in E at which f is continuous , then since f is monotone, the set $E \setminus S$ is at most countable. Thus we will assume that $\Delta_n(x_0) \subset G$ actually occurs for all n and all $x_0 \in S$.

The intervals $\{d_n(x_0) : x_0 \in E, n \geq 1\}$ forms a Vitali cover for S. Thus, there are countably many pairwise disjoint intervals $\{d_{n_i}(x_i)\}$ such that

$$\mu^*\left(S \setminus \bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right) = 0$$

Hence,

$$\mu^*(S) \leq \sum_{i=1}^{\infty} \mu(d_{n_i}(x_i)) < \frac{1}{q-\varepsilon} \sum_{i=1}^{\infty} \mu(\Delta_{n_i}(x_i)) \tag{4.7}$$

Now, since f is strictly increasing, the intervals $\{\Delta_{n_i}(x_i\}$ must also be pairwise disjoint. Consequently,

$$\sum_{i=1}^{\infty} \mu(\Delta_{n_i}(x_i)) = \mu\left(\bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right) \leq \mu(G). \tag{4.8}$$

Combining our observations in light of equations (4.7) and (4.8) yields

$$\mu^*(E) = \mu^*(S) < \frac{1}{q-\varepsilon}[(\mu^*(f(E)) + \varepsilon)$$

Letting $\varepsilon \to 0$, we get $\mu^*(f(E)) \geq q\mu^*(E)$. □

Corollary 4.0.7. *If $f : [a,b] \to \mathbb{R}$ is increasing, then the set of points at which at least one derived number for f is infinite has measure zero.*

Proof. This is trivially true if f is strictly increasing. In this case, Lemma (4.0.6) tells us that if the set $E = \{x : Df(x) = +\infty\}$ has nonzero measure, then the set f(E) would have infinite measure. This is clearly impossible since $f(E) \subset [f(a), f(b)]$.

If f is not strictly increasing, we consider instead the function $g(x) = f(x) + x$. Now since g is strictly increasing and satisfies

$$\frac{g(x+h) - g(x)}{h} = \frac{f(x+h) - f(x)}{h} + 1$$

it is clear that $\{x : Df(x) = +\infty\} = \{x : Dg(x) = +\infty\}$. The latter set has measure zero. □

Corollary 4.0.8. *Let $f : [a,b] \to \mathbb{R}$ be increasing and let $0 \leq p < q < \infty$. If at every point x in some set $E_{p,q} \subset [a,b]$ there exist two derived numbers for f satisfying $D_1 f(x) \leq p < q \leq D_2 f(x)$, then $\mu(E_{p,q}) = 0$.*

Proof. If f is strictly increasing, then Lemmas (4.0.5) and (4.0.6) imply that

$$q\mu^*(E_{p,q}) < \mu^*(f(E_{p,q})) < p\mu^*(E_{p,q}),$$

and hence that $\mu(E_{p,q}) = 0$.

When f is not strictly increasing, we simply apply the first part of the proof to the function $g(x) = f(x) + x$, replacing p by $p+1$ and q by $q+1$. $\square$

Theorem 4.0.9. ***Lebesgue's Differentiation Theorem:*** *If $f : [a,b] \to \mathbb{R}$ is increasing, then f has a finite derivative at almost every point in [a,b].*

Proof. Let E denote the set of points $x \in [a,b]$ at which $f'(x)$ does not exist.

Let $\{x : D_1 f(x) < D_2 f(x)\}$ denote the set of points x at which f has two different derived numbers $D_1 f(x) < D_2 f(x)$. Then,

$$E = \{x : D_1 f(x) < D_2 f(x)\} = \bigcup_{p<q_{p,q\in\mathbb{Q}}} \{x : D_1 f(x) \leq p < q \leq D_2 f(x)\}$$

where $E_{p,q} = \{x : D_1 f(x) \leq p < q \leq D_2 f(x)\}$ denotes the set of points x at which f has two different derived numbers satisfying $D_1 f(x) \leq p < q \leq D_2 f(x)$.

From Corollary (4.0.8), each $E_{p,q}$ has measure zero and there are at most countably many such sets for $p,q \in \mathbb{Q}$ and hence $\mu(E) = 0$; that is,

$f'(x)$ exists at almost every point in [a,b].

From Corollary (4.0.7), we know that the set of points at which $f'(x) = +\infty$ has measure zero;

thus, $f'(x)$ exists as a finite real number almost everywhere. $\square$

Corollary 4.0.10. *If $f \in BV[a,b]$, then f has a finite derivative at almost every point in [a,b].*

Theorem 4.0.11. *(i) If f is increasing on [a,b], then f' is measurable, and $\int_a^b f' \leq f(b) - f(a)$.*

(ii) If $f \in BV[a,b]$, then $f' \in L_1[a,b]$ and $\int_a^b |f'| \leq V_a^b f$.

Proof. In the beginning we had assumed that any function f on [a,b] has been extended to all of $\mathbb{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$.
The proof of (i) can be given as follows: Let

$$f_n(x) = n\left(f\left(x+\frac{1}{n}\right) - f(x)\right),$$

and thus ,

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \lim_{1/n\to 0} n\left(f\left(x+\frac{1}{n}\right) - f(x)\right) \\ &= f'(x) \text{ for almost every } x \in [a,b].\end{aligned}$$

Hence f' is measurable.
Next we make use of Fatou's Lemma to estimate $\int_a^b f'$.

$$\begin{aligned}\int_a^b f' &= \int_a^b \lim_{n\to\infty} n\left(f\left(x+\frac{1}{n}\right) - f(x)\right)dx \\ &\leq \liminf_{n\to\infty} n\left(\int_a^b f\left(x+\frac{1}{n}\right)dx - \int_a^b f(x)dx\right) \\ &= \liminf_{n\to\infty} n\left(\int_{a+(1/n)}^{b+(1/n)} f - \int_a^b f\right) \\ &= \liminf_{n\to\infty} n\left(\int_b^{b+(1/n)} f - \int_a^{a+(1/n)} f\right) \\ &\leq f(b) - f(a)\end{aligned}$$

since f is increasing and since $f(x) = f(b)$ for $x > b$. Also note that the "change of variable" is justified here; because f is monotone, each of the integrals above is actually a Riemann integral.
Now suppose that f is of bounded variation on [a,b], and we know that we can write f as the difference of two monotonic functions i.e., $f = v - (v - f)$, where $v(x) = V_a^x f$, and where v and v-f are both increasing.

Then $f' = v' - (v - f)'$ exists a.e. and is measurable. But, note that for $x < y$ we have $|f(y) - f(x)| \leq V_x^y f = v(y) - v(x)$,
and it follows that $|f'| \leq v'$ a. e. So, from the first part of the proof, f' is integrable and

$$\int_a^b |f'| \leq \int_a^b v' \leq v(b) - v(a) = V_a^b f.$$

$\square$

Theorem 4.0.12. *Let g be integrable on [a,b], and let $f(x) = \int_a^x g$. Then:*

(i) $f \in C[a,b] \cap BV[a,b]$ *and* $\int_a^b |f'| \leq V_a^x f \leq \int_a^x |g|$.

(ii) $f \equiv 0$ *if and only if g = 0 a.e.*

(iii) $f' = g$ *a.e.; hence,* $f(x) = \int_a^x f'$ *and* $V_a^x f = \int_a^x |f'|$.

Proof. By the corollary to Dominated Convergence Theorem which is given by: If $f \in L_1$, then $F(x) = \int_{-\infty}^x f$ is continuous. From this we know that indefinite integrals are continuous. Also f is of bounded variation.
Note that

$$f(x) = \int_a^x g = \int_a^x g^+ - \int_a^x g^-$$

and both $\int_a^x g^+$ and $\int_a^x g^-$ are increasing.
Hence, by the triangle inequality for variations,

$$V_a^x f \leq \int_a^x g^+ + \int_a^x g^- = \int_a^x |g|.$$

Now $\int_a^b |f'| \leq V_a^b f$ follows from Theorem(4.0.11).(ii).
Next, (ii) follows from considering $\int_a^x g$ as a measure. (which is as a result of the following Corollary: If $f \in L_1$, then the map $E \mapsto \int_E f$ is a signed measure on $\mathscr{M}$. In particular,

if(E_n) is any sequence of pairwise disjoint, measurable sets, and if $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\sum_{n=1}^{\infty} \left| \int_{E_n} f \right| \leq \int_E |f| < \infty \text{ and } \sum_{n=1}^{\infty} \int_{E_n} f = \int_E f)$$

If $f \equiv 0$, then

$$\begin{aligned}
\int_a^x g = 0 \text{ for all x } &\Longrightarrow \int_c^d g = 0 \text{ for all (c,d)} \subset \text{[a,b]} \\
&\Longrightarrow \int_U g = 0 \text{ for all open sets } U \subset [a,b] \\
&\Longrightarrow \int_G g = 0 \text{ for all open sets } G_\delta - setsG \subset [a,b] \\
&\Longrightarrow \int_E g = 0 \text{ for all measurable sets } E \subset [a,b],
\end{aligned}$$

since every measurable set is, up to a null set, a G_δ-set. Consequently, g=0 a.e.

Since g = 0 a.e. always forces $f \equiv 0$, this proves (ii).

Finally, to prove (iii). We consider g^+ and g^- separately, we may suppose that $g \geq 0$. In turn, this will make f increasing, and hence $f' \geq 0$ a.e.

Now, to simplify things further we assume that g is also bounded, say, $0 \leq g \leq K$.

In this case,

$$n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) = n\int_x^{x+(1/n)} g \leq K$$

and $n\left(f\left(x+\frac{1}{n}\right)-f(x)\right) \to f'(x)$ a.e. So, by the Dominated Convergence Theorem,

$$\begin{aligned}
\int_a^x f' &= \lim_{n\to\infty}\int_a^x n\left(f\left(t+\frac{1}{n}\right)-f(t)\right)dt \\
&= \lim_{n\to\infty}\left[n\int_x^{x+(1/n)} f - n\int_a^{a+(1/n)} f\right] \\
&= f(x)-f(a), \text{ because f is continuous },\\
&= \int_a^x g
\end{aligned}$$

And now, $\int_a^x f' = \int_a^x g$, for all x, implies that $f' = g$ a.e., from (ii).
In the general case (where g is integrable and nonnegative but not necessarily bounded),we truncate g by defining $g_n(x) = g(x)$ if $g(x) \le n$ and $g_n(x) = 0$ otherwise; that is, $g_n = g\chi_{\{g\le n\}}$. Note that $g_n \to g$ a.e.
Now set $f_n(x) = \int_a^x g_n$. Since $0 \le g_n \le g$, we have that $f = (f - f_n) + f_n$, and each of $f - f_n$ and f_n is evidently increasing. But g_n is bounded: $0 \le g_n \le n$; thus, by the case we just proved, $f_n{}' = g_n$ a.e. Hence,

$$f' = (f-f_n)' + f_n{}' \ge f_n{}' = g_n \to g \text{ a.e.}$$

It follows that $f' \ge g$ a.e., and this turns out to be enough. Since f is increasing, we get

$$f(x) = f(x) - f(a) \ge \int_a^x f' \ge \int_a^x g = f(x)$$

Hence, $f' = g$ a.e. □

Corollary 4.0.13. *Let E be a measurable subset of $\mathbb{R}$ with finite measure, and consider the "distribution " function $f(x) = \mu(E\cap(-\infty,x])$. Then, for almost every x in $\mathbb{R}$, the "density " $f'(x)$ exists and satisfies $f' = \chi_E$ a. e. That is, $f'(x) = 1$ for a.e. $x \in E$ and $f'(x) = O$ for a.e. $x \in E^c$.*

Proof. Note that $f(x) = \int_{-\infty}^{x} \chi_E$.

Thus, since χ_E is integrable, we have

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} \chi_E = \chi_E(x) \text{ a.e.}$$

$\square$

Chapter 5

ANALYSIS AND CONCLUSION

In **Chapter 2** we have seen the extension of Egorov's Theorem. The difference between Egorov's theorem and its extensions is that Egorov's theorem typically deals with uniform convergence of sequences of measurable functions on a measurable set with finite measure, while its extensions may generalize to other types of convergence, such as convergence in measure and also relaxes the condition of "pointwise almost everywhere convergence" of sequence. Extension of Egorov's Theorem provides us somewhat less stringent restrictions than usually required for almost uniform convergence.

The formulation in **Chapter 3** makes almost uniform convergence appear much like convergence in measure , which can be considered as an advantage . In this chapter we have seen that the results which hold for almost uniform convergence also hold true for convergence in measure with certain conditions. Convergence in measure is important for understanding the behavior of sequences of functions. Also, we note that it's a weaker form of convergence compared to pointwise or uniform convergence.

In the beginning of **Chapter 4** the questions we have put forth can now be answered.

- We have now seen that a function which is increasing and of bounded variation satisfies $\int_a^b f' = f(b) - f(a)$ and exists almost everywhere in [a,b].
- If f is a continuous function and a function of bounded variation then it is true that $f(x) = \int_a^x g$ gor some integrable function g and $f' = g$ a.e.
- Every Riemann Stieltjes integral is a Lebesgue Integral.
- If f is of bounded variation then f' exists and f' is integrable. Also, it satisfies $\int_a^b |f'| \leq V_a^b f$.

Thus the Lebesgue's Differentiation Theorem helps to give clarity on certain statements in analysis.

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