Multiplicative And Additive Arithmetic Functions And Formal Power Series

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by

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Declaration

I hereby declare that the data presented in this Dissertation / Internship report entitled, "Multiplicative and Additive functions and formal power series" is based on the results of investigations carried out by me in the Mathematics Discipline at the school of physical and applied sciences, Goa University under the Supervision of Mr Brandon Fernandes and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University or its authorities will be not be responsible for the correctness of observations / experimental or other findings given the dissertation
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This is to certify that the dissertation / internship report "Multiplicative and Additive Arithmetic functions and Formal Power Series" is a bonafide work carried out by Ms/Mr Caridi Lourdes Pereira under my supervision in partial fulfilment of the requirements for the award of the degree of Master's Degree in the Mathematics Discipline at the School Of Physical and Applied Sciences, Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " Multiplicative And Additive Arithmetic Functions And Formal Power Series" This survey is divided into three chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report we define arithmetic functions and we look at the different arithmetic function and some examples based on them. Also, We will study the Dirichlet Convolution and Unitary Convolution and theorems based on these topics.

SECOND CHAPTER:

This chapter will discuss new concepts of arithmetic functions.we will also give the characterization of Completely Multiplicative and Additive Arithmetic Functions.

THIRD CHAPTER:

In this chapter we will discus a relationship between formal power series and arithmetic functions. However, we will need to first define the concept of formal power series.

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ABSTRACT

The theory of arithmetic functions and the theory of formal power series are classical and active parts of mathematics. Algebraic operations on sets of arithmetic functions, called convolutions, have an important place in the theory of arithmetic functions. The theory of formal power series also has its place firmly anchored in abstract algebra.
A first goal of this thesis will be to present a parallelism of known characterizations of the concepts of multiplicative and additive for arithmetic functions on the one hand and for formal power series on the other.Later, the proofs of some main results on completely and specially multiplicative functions has been replaced with new proofs using Bell series. This was a second goal of giving new proofs using Bell series, and so we bring the two topics (arithmetic functions and formal power series) closer together. I found this topic interesting and intriguing. I dived deep into the paper,studied its concepts and theorems to gain thorough understanding. In this paper I observed that a many theorem that were previously solved can be solved in a simpler way using the bell series.

Keywords: Arithmetic functions; Completely multiplicative functions; Additive functions; Specially multiplicative functions; Bell series; Formal power series

Notations and Abbreviations

$\mathcal{A}$	The set of all arithmetic functions
τ	Is the number of positive divisors of n
σ	Is the sum of all positive divisors of
μ	The Möbius Function
ϕ	Euler's totient function
$f * g$	Dirichlet Convolution
$(\mathcal{A}, *)$	Set of arithmetic functions with binary operation
$f \oplus g$	Unitary Convolution
ζ	Unit function
i	Identity function
$\mathcal{M}$	Set of non-zero multiplicative arithmetic function
$\mathcal{U}(\mathcal{A})$	The set of units/ invertible elements
λ	Louiville Lambda function
ω	Is the sum of prime powers where p^{α} exactly divides $n
$\mathcal{R}$	Commutative ring with unity
$f_p(x)$	Is called the Bell Series of f modulo p

Contents

Chapter 1

Arithmetic Functions and Convolutions

Introduction

The theory of Arithmetic Functions has always been a vital part of Number Theory. In this chapter we will define arithmetic functions and look at the different arithmetic function and some examples based on them. Also, We will study the Dirichlet Convolution and Unitary Convolution and theorems based on these topics.

1.1 Arithmetic Functions

An arithmetic, arithmetical or number-theoretic function is any function defined on the set of positive integers (natural numbers) $\mathbb{N}$ =$\{1, 2, 3, \cdots\}$ with values in the set of complex numbers . We will focus on the ring of arithmetic functions with the standard addition of functions and the Dirichlet convolution or unitary convolution as the multiplicative operation.

The following definitions and arithmetic functions of this chapter can be found in Sivaramakrishnan [12], Burton [2], McCarthy [6] and Niven [8]. However, notations may be different.

Definition 1.1.0.1. By R. Shivaramakrishnan: A function $f : \mathbb{N} \to \mathbb{C}$ is said to be an **Arithmetic function**

Definition 1.1.0.2. By I. Niven: An **Arithmetic function** f is one whose domain is the positive integers and whose range is a subset of the complex numbers.

Notation 1.1.0.3. *The **set of all arithmetic functions** will be denoted by*

$$\mathcal{A} = \{f : \mathbb{N} \to \mathbb{C}\}$$

We give some examples of the arithmetic functions that will be used and discussed throughout this paper.

Example 1.1.0.4. ***Euler Totient function(ϕ)***
$\phi(n) = \sum_{k=1}^{n}{}' \, 1, \forall\, n \in \mathbb{N}$
where the $'$ indicates that the sum is extended over those k relatively prime to n.
Note That:When p is a prime number we have $\phi(p) = p - 1$

Example 1.1.0.5. ***Divisor Function(τ)***
$\tau(n) = \sum_{d|n} 1 \;\forall\, n \in \mathbb{N}$, *is the number of positive divisors of n.*

Example 1.1.0.6. ***Sum of Divisor Function(σ)***
$\sigma(n) = \sum_{d|n} d \;\forall\, n \in \mathbb{N}$, *is the sum of all positive divisors of n.*

Example 1.1.0.7. ***The mobius Function (μ)***

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1, \\ (-1)^k & \text{if} \quad n = p_1.p_2...p_k \text{ with distinct primes} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

1.2 Dirichlet Convolution and Unitary Convolution

1.2.1 Dirichlet Convolution

Definition 1.2.1.1. Let f and g be arithmetic functions, then the **Dirichlet Convolution** is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) \forall n \in \mathbb{N}$$

Before we discuss the properties of the structure $(\mathcal{A}, *)$, it would be quite helpful to illustrate this operation through example. So, if we take two arithmetic functions f, g More concretely, let us take μ and σ and the integer 12, we obtain

$(\tau * \sigma)(12) = \tau(1)\sigma(12) + \tau(2)\sigma(6) + \tau(3)\sigma(4) + \tau(4)\sigma(3) + \tau(6)\sigma(2) + \tau(12)\sigma(1)$

$= 128 + 29 + 27 + 34 + 43 + 121$

$= 96$

Also, we can notice,

$\tau(1) = 1$ and $\sigma(1) = 1$

Now, let us examine the properties of the structure $(\mathcal{A}, *)$

Notation 1.2.1.2. $(\mathcal{A}, *)$ - *is a set of all arithmetic fuctions with a binary operation.*

Definition 1.2.1.3. Monoid- A monoid is a set that is closed under an associative binary operation and has an identity element.

Theorem 1.2.1.4. *The structure* $(\mathcal{A}, *)$*, is a commutative monoid.*

Proof. It needs to be shown that the operation is commutative, associative, and has identity. To show that this structure is commutative, we need to verify this property

$$f * g = g * f$$

for all arithmetic functions f, g in $\mathcal{A}$.So,

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{d_1 d_2 = n} f(d_1)g(d_2)$$
$$= \sum_{d_1 d_2 = n} g(d_2)f(d_1) = \sum_{d|n} g(d)f(\frac{n}{d}) = (g * f)(n)$$

Now we will verify the associative property

$$(f * g) * h = f * (g * h)$$

for any $f,g \in \mathcal{A}$

so,

$$[(f*g)*h](n) = \sum_{dd_3=n} [(f*g)(d)]h(d_3)$$

$$= \sum_{dd_3=n} [\sum_{d_1d_2=n} f(d_1)g(d_2)]h(d_3)$$

$$= \sum_{d_1d_2d_3=n} f(d_1)g(d_2)h(d_3)$$

By a similar calculation, it can be shown

$$[f*(g*h)](n) = \sum_{d_1d_2d_3=n} f(d_1)g(d_2)h(d_3)$$

which implies this structure is associative.

To determine the identity element, we need to identify e $\in \mathcal{A}$ with the property

$$f*e = e*f = f$$

for all $f \in \mathcal{A}$

To do this, consider the arithmetic function

$$e(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

Then it follows

$$(f*e)(n) = \sum_{d|n} f(d)e(\frac{n}{d}) = f(1)e(n) + \cdots + f(n)e(1) = 0 + 0 + \cdots + f(n) = f(n)$$

This implies that the arithmetic function e is the Dirichlet identity. With this, we have shown that $(\mathcal{A}, *)$, is a commutative monoid. □

Now that the properties of this structure have been identified, it would be beneficial to determine what it's inverses are.

Notation 1.2.1.5. *The set of **units/ invertible** elements of the structure* $(\mathcal{A}, *)$ *will be denoted by*

$$\mathcal{U}(\mathcal{A}) = \{f \in \mathcal{A} | f isinvertible\}$$

Which are these elements?

Theorem 1.2.1.6. *The invertible elements of the structure* $(\mathcal{A}, *)$*, are exactly those arithmetic functions with the property* $f(1) \neq 0$ *i.e*

$$\mathcal{U}(\mathcal{A}) = \{f \in \mathcal{A} | f(1) \neq 0\}$$

Proof. To prove this, it must be shown that $f \in \mathcal{U}(\mathcal{A}) \iff$f(1)$\neq$ 0

So, let $f \in \mathcal{U}(\mathcal{A})$. This implies that there exists an arithmetic function $\tilde{f} \in \mathcal{A}$ with the property

$$(f * \tilde{f}) = e$$

then since $e(1) = 1$,

$$e(1) = (f * \tilde{f})(1) = f(1)\tilde{f}(1)$$

$\iff f(1) \neq 0$

Conversely, let us assume $f(1) \neq 0$. Now, we will define the following arithmetic function recursively such that

$$\tilde{f}(n) = \begin{cases} \frac{1}{f(1)} & \text{if } n = 1 \\ -\frac{1}{f(1)} \sum_{d|n} f(d)\tilde{f}(\frac{n}{d}) & \text{if } n > 1 \end{cases}$$

So, for $n = 1$ we have

$$(f * \tilde{f})(1) = \sum_{d|1} f(d)\tilde{f}(\frac{1}{d}) = f(1)\tilde{f}(1) = f(1) \cdot \frac{1}{f(1)} = 1 = e(1)$$

For $n > 1$ we have

$$(f * \tilde{f})(n) = \sum_{d|n} f(d)\tilde{f}(\frac{n}{d}) = f(1)\tilde{f}(n) + \sum_{d|n,n>1} f(d)\tilde{f}(\frac{n}{d})$$

Notice,

$$\tilde{f}(n) = -\frac{1}{f(1)} \sum_{d|n,n>1} f(d)\tilde{f}(\frac{n}{d})$$

Therefore,

$$\sum_{d|n,n>1} f(d)\tilde{f}(\frac{n}{d}) = -\tilde{f}(n)f(1)$$

Then this is what follows

$$f(1)\tilde{f}(n) + \sum_{d|n,n>1} f(d)\tilde{f}(\frac{n}{d}) = f(1)\tilde{f} - f(1)\tilde{f}(n) = 0 - e(n)$$

So, we can say

$$(f * \tilde{f})(n) = e(n)$$

for all natural numbers n.

This implies, f is the inverse of $\tilde{f}$.

Therefore, arithmetic functions with the property, $f(1) \neq 0$, are inverse elements of $(\mathcal{A}, *)$

□

Definition 1.2.1.7. Let f be an arithmetic function, then f is called **multiplicative** if

$$f(mn) = f(m)f(n) \qquad \textit{when } (m,n) = 1$$

Definition 1.2.1.8. Let f be an arithmetic function, then f is called **Completely multiplicative** if

$$f(mn) = f(m)f(n) \qquad \forall m,n$$

Notation 1.2.1.9. *The set of all* ***non-zero multiplicative arithmetic*** *functions will be denoted by*
$\mathcal{M} = \{f \in \mathcal{A} - \{o\} | \ f$ *is multiplicative*$\}$

Note that $o(n) = 0 \quad \forall n \in \mathbb{N}$ is the **Zero function**.

Theorem 1.2.1.10. *The structure* $(\mathcal{M}, *)$ *is an abelian group.*

Proof. It is sufficient to prove that $\mathcal{M}$ is a subgroup of $\mathcal{U}(\mathcal{A})$,
since $\mathcal{U}(\mathcal{A})$is an abelian group. It needs to be shown that
1. The set $\mathcal{M}$ is a nonempty subset of $\mathcal{U}(\mathcal{A})$.
2. If arithmetic functions f and g are multiplicative, then their convolution, $f * g$, is a multiplicative arithmetic function.
3. If f is a multiplicative arithmetic function, then its inverse, $\tilde{f}$, is a multiplicative arithmetic function. □

Definition 1.2.1.11. Let f and g be arithmetic functions, then

$$(f+g)(n) = f(n) + g(n) \quad \forall f, g \in \mathcal{A}$$

Theorem 1.2.1.12. *The algebraic structure* $(\mathcal{A}, +, *)$ *is an integral domain.*

Proof.
- It is trivial to verify that $(\mathcal{A}, +)$ is an abelian group.
- $(\mathcal{A}, +)$ is associative and commutative.
- The arithmetic function $o(n) = 0$ is the additive identity.
- For all arithmetical functions f, the additive inverse is $\tilde{f}$.
- We also know $(\mathcal{A}, +)$ is a commutative monoid.
- $*$ distributes over $+$.

□

1.2.2 Unitary Convolution

Definition 1.2.2.1. Let n be a positive integer. Then d, a divisor of n, with the property

$$(d, \frac{n}{d}) = 1$$

is called a **unitary divisor** of n.

Definition 1.2.2.2. Let f and g be arithmetic functions, then the **Unitary Convolution** is defined as

$$(f \oplus g) = \sum_{d||n} f(d)g(\frac{n}{d}) \forall n \in \mathbb{N}$$

where $d||n$ means that d runs through the unitary divisors of n.

Theorem 1.2.2.3. $(\mathcal{A}, +, \oplus)$ *is a commutative ring with unity.*

Proof. i. First we show that $(\mathcal{A}, +)$ is an abelian group.

$$(f+g)(n) = f(n)+g(n) = g(n)+f(n) = (g+f)(n)$$

Hence Abelian

ii. To show that $(\mathcal{A}, \oplus)$ is associative and commutative.

$$(f \oplus g)(n) = \sum_{d||n} f(d)g(\frac{n}{d})$$

now, $d = d_1, \frac{n}{d_1} = d_2, (d_1, d_2) = 1$ hence,

$$(f \oplus g)(n) = \sum_{d_1 d_2 = n} f(d_1)g(d_2)$$

$$= \sum_{d_1 d_2 = n} g(d_2)f(d_1)$$

$$= \sum_{d||n} g(d) f(\frac{n}{d})$$

$$= (g \oplus f)(n)$$

hence it is commutative.

To show associativity i.e TST $(f \oplus g) \oplus h = f \oplus (g \oplus h)$

$$[(f \oplus g) \oplus h](n) = \sum_{d \cdot d_3} [(f \oplus g)(d)] h(d_3)$$

$$= \sum_{d \cdot d_3} [\sum_{d_1 \cdot d_2 = d} f(d_1) g(d_2)] h(d_3)$$

$$= \sum_{d_1 \cdot d_2 \cdot d_3 = d} f(d_1) g(d_2) h(d_3)$$

Similarly calculating

$$[f \oplus (g \oplus h)] = \sum_{d_1 \cdot d_2 \cdot d_3 = d} f(d_1) g(d_2) h(d_3)$$

Hence implies associativity

iii. $\oplus$ distributes over $+$.

$$[f \oplus (g+h)](n) = \sum_{d||n} f(d)(g+h)(\frac{n}{d})$$

$$= \sum_{d_1 d_2} f(d_1)[g(d_2) + h(d_2)]$$

$$= \sum_{d_1 d_2} f(d_1) g(d_2) + \sum_{d_1 d_2} f(d_1) h(d_2)$$

$$= \sum_{d||n} f(d) g(\frac{n}{d}) + \sum_{d||n} f(d) h(\frac{n}{d})$$

$$= (f \oplus g)(n) + (f \oplus h)(n)$$

$$
\begin{aligned}
[(f+g)\oplus h)](n) &= \sum_{d\|n}(f+g)(d)(h)(\frac{n}{d}) \\
&= \sum_{d_1 d_2}[f(d_1)+g(d_1)]h(d_2) \\
&= \sum_{d_1 d_2} f(d_1)h(d_2)+g(d_1)h(d_2) \\
&= \sum_{d\|n} f(d)h(\frac{n}{d}) + \sum_{d\|n} g(d)h(\frac{n}{d}) \\
&= (f\oplus h)(n)+(f\oplus g)(n)
\end{aligned}
$$

iv. Dirichlet's identity is the unitary convolution.

$$
e(n) = \begin{cases} 1 & \text{if} \quad n=1 \\ 0 & \text{if} \quad n>1 \end{cases}
$$

Then it follows

$(f*e)(n)=\sum_{d\|n} f(d)e(\frac{n}{d})$

$=f(1)e(n)+\cdots+f(n)e(1)$

$=0+0+\cdots+f(n)=f(n)$

$(f*e) = f(n) = (e*f)$ □

We introduce this structure, because we will be using it later in Chapter 3.

However, let us now discuss some applications to the theorems we have introduced

1.3 More on $\sigma, \tau$and ϕ

First, let's introduce a few more arithmetic functions.

Unit Function

$\zeta(n) = 1 \quad \forall n \in N$

Identity Function

$i(n) = n \quad \forall n \in N$ It is important to note that:

- $\zeta(m.n) = 1 = 1.1 = \zeta(m)\zeta(n)$
- $i(m.n) = m.n = i(m)i(n)$

This implies that the arithmetic functions ζ and i are both multiplicative.

We introduce these functions here because they have a special relationship with some of the arithmetic functions we have already discussed.

Definition 1.3.0.1. Let f be an arithmetic function, then

$$F(n) = \sum_{d|n} f(d)$$

is called the **Summation** of f.

This summation function will allow us to verify some important properties concerning the arithmetic functions we have discussed. One of those properties is determining multiplicative functions.

Theorem 1.3.0.2. *If f is a multiplicative arithmetic function, then the summation of f is a multiplicative arithmetic function.*

Proof. Let $f \in \mathcal{M}$ and let F be the summation of f, then

$F(n) = \sum_{d|n} f(d) = \sum_{d|n} f(d).1 = \sum_{d|n} f(d)\zeta(\frac{n}{d})$

so,

$F = f * \zeta$

f and ζ are both multiplicative arithmetic functions This implies F is a multiplicative arithmetic function, because, as we have shown, $(\mathcal{M}, *)$ is closed. □

Theorem 1.3.0.3. *We have* $\sum_{d|n} \mu(d) = e(n)\ \forall n \geq 1$

Proof. let $n = 1$

$$\sum_{d|1} \mu(d) = 1 = e(1)$$

. Now,let $n = p$ be prime. Therefore,

$$\sum_{d|p} \mu(d) = \mu(1) + \mu(p) = 1 - 1 = e(p)$$

If $n = p^{\alpha}$, where $\alpha \geq 2$, then,

$$\sum_{d|p^{\alpha}} \mu(d) = \mu(1) + \mu(p) + \mu(p^{\alpha}) + \cdots + \mu(d) = 1 - 1 + 0 + \cdots + 0 = 0 = e(p^{\alpha})$$

.

So for the prime factorization,$n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$,we have

$$\sum_{d|n} \mu(d) = 0 = e(n)$$

Therefore for any $n \in \mathbb{N}$ we have $\sum_{d|n} \mu(d) = e(n)$. □

With this we come to a nice corollary.

Corollary 1.3.0.4. *μ is the Dirichlet inverse of ζ*

Proof. $e(n) = \sum_{d|n} \mu(d) = \sum_{d|n} \mu(d) \cdot 1 = \sum_{d|n} \mu(d) \cdot \zeta(\frac{n}{d}) = (\mu * \zeta)(n)$.
this implies

$$\mu = \tilde{\zeta}$$

. □

This corollary gives us the following theorem which is called **the Möbius Inversion Formula**

Theorem 1.3.0.5. *Let f be an arithmetic function, then*

$$F(n) = \sum_{d|n} f(d) \text{ if and only if } f(n) = \sum_{d|n} F(d)\, \mu(\tfrac{n}{d})$$

Proof. let $F = f * \zeta$

$$\iff F * \tilde{\zeta} = f * \zeta * \tilde{\zeta}$$

$$\iff F * \tilde{\zeta} = f * e$$

$$\iff F * \mu = f$$

□

Corollary 1.3.0.6. *An arithmetic function, f, is multiplicative if and only if the summation of f is multiplicative.*

Proof. let $f \in \mathcal{M}$ and F be the summation of f

Then,

$$F(n) = \sum_{d|n} f(d)$$

$$= \sum_{d|n} f(d) \cdot 1$$

$$= \sum_{d|n} f(d) \zeta(\frac{n}{d})$$

f and τ are multiplicative arithmetic functions.

$\Longrightarrow$we have shown $(\mathcal{M}, *)$ is a closed structure.

$\Longrightarrow$ F is a multiplicative arithmetic function.

Conversely,

$F(n) = \sum_{d|n} f(d)$

Then,

$f(n) = \sum_{d|n} F(d)\, \mu(\frac{n}{d}) \Longrightarrow f = F * \mu$

f and μ are multiplicative arithmetic functions.

$\implies$ f is a multiplicative arithmetic function ☐

Theorem 1.3.0.7. *We have* $\sum_{d|n} \phi(d) = n \ \forall n \geq 1$

Corollary 1.3.0.8. *$\sigma, \tau and \phi$ are multiplicative arithmetic functions.*

Proof. Let the equation $\sum_{d|n} \phi(d) = n \ \forall n \geq 1$ hold, then

$$n = i(n) = \sum_{d|n} \phi(d)\zeta(\frac{n}{d}) = (\phi * \zeta)(n)$$

and,

$$\phi(n) = (i * \tilde{\zeta}) = (i * \mu)(n)$$

. Which shows us that ϕ is multiplicative. Now let's look at the sum of divisors function

$$\tau(n) = \sum_{d|n} 1 = \sum_{d|n} 1.1 = \sum_{d|n} \zeta(d)\zeta(\frac{n}{d}) = (\zeta * \zeta)(n)$$

. This implies that τ is a multiplicative function.

Also,

$$\sigma(n) = \sum_{d|n} d = \sum_{d|n} d.1 = \sum_{d|n} i(d)\zeta(\frac{n}{d}) = (i * \zeta)(n)$$

Similarly, σ can be said to be a multiplicative function.

Hence σ, τ and ϕ are multiplicative arithmetic functions. ☐

Theorem 1.3.0.9. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then*

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1)$$

Proof. Let p be prime and $\alpha \geq 1$. The set

$$D(p^{\alpha}) = \{1, p, p^2, \cdots, p^{\alpha}\}$$

is the set of all positive divisors of p^α . Therefore

$$\tau(p^\alpha) = \alpha + 1$$

We will now consider the prime factorization, $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Since, we have just shown that τ is multiplicative, it follows that

$$\tau(n) = \tau(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \tau(p_1^{\alpha_1}) \cdots \tau(p_k^{\alpha_k}) = (\alpha_1 + 1) \cdots (\alpha_k + 1) = \prod_{i=1}^{k} (\alpha_i + 1)$$

□

Theorem 1.3.0.10. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then*

$$\phi(n) = n \prod_{i=1}^{k} (1 - \frac{1}{p_i})$$

Proof. Let $n = p$ be prime. Then

$$\phi(p) = p - 1 = p(1 - \frac{1}{p})$$

Now, let $n = p^\alpha$ where $\alpha \geq 1$

We desire those integers who are relatively prime to p^k. It can be seen that the integers who are not relatively prime are those of the form

$$p, 2p, 3p, \cdots, p^{\alpha-1} \cdot p = p^\alpha$$

Therefore, there are $p^{\alpha-1}$ integers who are not relatively prime to p^α ,so we can say

$$\phi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha(1 - \frac{1}{p})$$

If we let $n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, the prime factorization of n, then it follows from ϕ being multiplicative that

$$\phi(p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \phi(p_1^{\alpha_1})\phi(p_2^{\alpha 2}) \cdots \phi(p_k^{\alpha_k}) = p_1^{\alpha_1}(1 - \frac{1}{p_1}) \cdots p_k^{\alpha_k}(1 - \frac{1}{p_k})$$

$$n(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_k}) = n \prod_{i=1}^{k}(1 - \frac{1}{p_i})$$

$\square$

Theorem 1.3.0.11. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then*

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}$$

Proof. Let $n = p$ where p is prime and $\alpha \geq 1$, then

$$\sigma(n) = 1 + p + p^2 + \cdots + p^{\alpha} = \frac{p^{\alpha+1} - 1}{p - 1}$$

Therefore, for

$$n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

the prime factorization of n,we have

$$\sigma(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \sigma(p_1^{\alpha 1}) \cdots \sigma(p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

$\square$

Theorem 1.3.0.12. *If f and g are multiplicative arithmetic functions with positive values and $n > 1$, then n is prime if and only if*

$$(f * g)(n) = (f + g)(n)$$

.

Proof. Let n be prime, then

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = f(1)g(n) + f(n)g(1) = 1 \cdot g(n) + f(n) \cdot 1 = f(n) + g(n) = (f+g)(n)$$

Conversely,

let us suppose$(f * g)(n) = (f+g)(n)$ and n is not prime. Then

$$\sum_{d|n} f(d)g(\frac{n}{d}) = f(n) + g(n)$$

This implies

$$\sum_{d|n, d\neq 1, n} f(d)g(\frac{n}{d}) + f(1)g(n) + f(n)g(1) = f(n) + g(n)$$

Thus we can conclude

$$\sum_{d|n, d\neq 1, n} f(d)g(\frac{n}{d}) = 0$$

This leads to a contradiction, since it is assumed that $f, g > 0$ for any positive integer n.

So, we can conclude that n must be prime. □

Corollary 1.3.0.13. *Let $n > 1$, then n is prime if and only if*

$$\sigma(n) + \phi(n) = n.\tau(n)$$

Proof. Let n be prime. We also known

$$(\sigma * \phi)(n) = n \cdot \tau(n)$$

Notice,

$$\sigma * \phi = (i * \zeta) * \phi == i * (\zeta * \phi) = i * i$$

Now,

$$(i * i)(n) = \sum_{d|n} i(d)i(\frac{n}{d}) = \sum_{d|n} d \cdot \frac{n}{d} = n \cdot \sum_{d|n} 1 = n \cdot \tau(n)$$

□

Corollary 1.3.0.14. *Let $n > 1$, then n is prime if and only if*

$$\tau(n) + \phi(n) = \sigma(n)$$

Proof. Let n be prime, then

$$(\tau + \phi)(n) = (\tau * \phi)(n)$$

Therefore,

$$\tau * \phi = (\zeta * \zeta) * \phi = \zeta * (\zeta * \phi) = \zeta * i = \sigma$$

□

Chapter 2

Characterization of Completely Multiplicative and Additive Arithmetic Functions

As of now we have only seen arithmetic functions and multiplicative arithmetic functions. This chapter will discuss new concepts of arithmetic functions, those of which were studied by Carlitz and Niederreiter, Lambek, and Schwab.

2.1 Completely Multiplicative Functions

In the previous chapter we discussed the concept of multiplicative functions. However, our previous definition was only concerned with relatively prime elements of non-negative integers. Now we will expand this property to any two non-negative integers

Definition 2.1.0.1. An arithmetic function, f, is said to be **completely multiplicative** if

$$f(nm) = f(n)f(m)$$

for all n, m positive integers.

With this, we can show some properties that these types of functions will possess.

Theorem 2.1.0.2. *If f is an arithmetic function then the following statements are equivalent.*

1. *f is completely multiplicative*
2. *$f(g*h) = fg*fh$ for all arithmetic functions g and h*
3. *$f(g*g) = fg*fg$ for all arithmetic functions g*
4. *$f\tau = f*f$.*

Proof. (1) $\Longrightarrow$ (2)

Let f be completely multiplicative, then

$$[f(g*h)](n) = f(n)[\sum_{d|n} g(d)h(\frac{n}{d})] = \sum_{d|n} f(n)g(d)h(\frac{n}{d}) = \sum_{d|n} f(d\cdot\frac{n}{d})g(d)h(\frac{n}{d})$$

$$= \sum_{d|n} f(d)f(\frac{n}{d})g(d)h(\frac{n}{d}) = \sum_{d|n}[f(d)g(d)][f(\frac{n}{d})h(\frac{n}{d})] = [fg*fh](n)$$

(2) $\Longrightarrow$ (3)

Assume,

$$f(g*h) = fg*fh$$

for all $g, h \in \mathcal{A}$

Then, it immediately follows that

$$f(g*g) = fg*fg$$

(3) $\Longrightarrow$ (4)

Assume

$$f(g*g) = fg*fg$$

Then, for all g

$$f\tau = f(\zeta*\zeta) = f\zeta*f\zeta = f\cdot 1*f\cdot 1 = f*f$$

(4) $\Longrightarrow$ (1)

Suppose $f * f = f\tau$.

We will show inductively that f is completely multiplicative.

Now take $n = 1$, then

$$(f * f)(1) = f(1)f(1) = \tau(1)f(1) = 1.f(1)$$

Therefore, $f(1) = 1$ or $f(1) = 0$.

Now take $n \geq 2$ and let $n = p_1^{e_1}, p_2^{e_2} \cdots p_m^{e_m}$ and $\alpha(n) = e_1 + e_2 + \cdots + e_m$

Then, it is enough to show

$$f(n) = f(1)f(p_1)^{e_1} \cdots f(p_m)^{e_m}$$

So, let $\alpha(n) = 1$, then n is prime, say $n = p$, which implies

$$2f(p) = \tau(p)f(p) = f(1)f(p) + f(p)f(1) = 2f(1)f(p)$$

Suppose this is true for all n with $\alpha(n) \leq k$ and $k \geq 1$.

Now we take n with $\alpha(n) = k + 1$ which gives

$$\tau(n)f(n) = \sum_{d|n} f(d)f(\frac{n}{d}) = 2f(1)f(n) + \sum_{d|n, d \neq 1, n} f(d)f(\frac{n}{d})$$

Now, let $d = d_1$ and $\frac{n}{d} = d_2$, so $d_1 \cdot d_2 = n$.

Also, $\alpha(d_1) \cdot \alpha(d_2) \leq k$ Then,

$$\tau(n)f(n) = 2f(1)f(n) + \sum_{d|n, d \neq 1, n} f(d_1)f(d_2)$$

Now, this fulfils the inductive step, so

$$\tau(n)f(n) = 2f(1)f(n) + (\tau(n) - 2)f(n) = f(1)f(p_1)^{e_1} \cdots f(p_m)^{e_m}$$

Since n is not prime, it is clear to see that $\tau(n) > 2$. So, for both $f(1) = 1$ and $f(1) = 0$ we get the desired result. □

2.2 Completely Additive Functions

Now, we will introduce a set of functions which have a similar property to the multiplicative functions, however, the functions are not split by multiplication, but by addition.

Definition 2.2.0.1. An arithmetic function, f, is said to be completely additive if

$$f(n.m) = f(n) + f(m)$$

for all n, m positive integers.

Example 2.2.0.2. *A familiar example of a completely additive function is the logarithmic function, as it is well known that*

1. $log(n.m) = log(n) + log(m)\ \forall n, m\mathcal{N}$.
2. *An immediate consequence of this property is, if* $f \in \mathcal{S}$
   then $f(1) = f(1.1) = f(1) + f(1)$
   $\Longrightarrow f(1) = 0$

Definition 2.2.0.3. $\Omega(n) = \sum_{p^\alpha \| n} \alpha$ is the sum of prime powers where p^α exactly divides n

Example 2.2.0.4. $\Omega(12) = \Omega(2^2.3) = 2 + 1 = 3$

$\Omega(30) = \Omega(2.3.5) = 1 + 1 + 1 = 3$

when $n = 1$ *we get*

$$\Omega(1) = 0$$

because, 1 has no prime divisors.

If we take arbitrary $n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1}, q_2^{\beta_2} \cdots q_l^{\beta_l}$ both being the canonical factorization of natural numbers nm, then we have

$\Omega(n \cdot m) = \Omega(p_1^{\alpha_1}.p_2^{\alpha_2} \cdots p_k^{\alpha_k}.q_1^{\beta_1}.q_2^{\beta_2} \cdots q_l^{\beta_l}) = \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_l$

$= \Omega(n) + \Omega(m)$

This means that Ω is completely additive

Now with this we can define a function which we can show to be completely multiplicative

Louiville Lambda function

• $\lambda(n) = (-1)^{\Omega(n)}$

Following the fact that $\Omega(1) = 0$ we see that $\lambda(1) = 1$ and since Ω is completely additive we get for any natural number n,m,

$$\lambda(n,m) = (-1)^{\Omega(n.m)} = (-1)^{\Omega(n)+\Omega(m)} = (-1)^{\Omega(n)}.(-1)^{\Omega(n)} = \lambda(n)\lambda(m)$$

Before we discuss additional properties of this function, it would be beneficial to introduce another arithmetic function, but first we must add a restriction to our definition of the completely additive arithmetic function.

Definition 2.2.0.5. An arithmetic function, f, is said to be additive if

$$f(n.m) = f(n) + f(m)$$

when $(n,m) = 1$.

Additive arithmetic functions, much like multiplicative arithmetic functions, only satisfy this "splitting" property for relatively prime natural numbers. The following function gives an example of this property.

• $\omega(n) = \sum_{p'|n} 1$ is the number of distinct primes, p', which divide n.

Note the different values of ω.

Let p be prime, then

$$\omega(1) = \sum_{p'|1} 1 = 0, \omega(p) = \sum_{p'|p} 1 = 1$$

It follows that when $\alpha \geq 1$, we have

$$\omega(p^{\alpha}) = \sum_{p'|p^{\alpha}} 1 = 1$$

This implies that when $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ the prime factorization of n, that

$$\omega(n) = \omega(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \sum_{p'|p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}} 1 = k$$

Notice, if $n = p_1^{\alpha_1}, p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1}, q_2^{\beta_2} \cdots q_l^{\beta_l}$ the prime factorization of n and m where $(n,m) = 1$ then,

$$\omega(n) + \omega(m) = k + l = \omega(n.m)$$

This implies that ω is additive.

We will now discuss some of the properties of completely additive functions

Theorem 2.2.0.6. *If f is an arithmetic function then the following statements are equivalent.*

1. *f is completely additive*
2. *$f(g * h) = fg * h + g * fh$ for all arithmetic functions g and h*
3. *$f(g * g) = 2(fg * g)$ for all arithmetic functions g*
4. *$f\tau = 2(f * \zeta)$.*

Proof. (1) $\Longrightarrow$ (2)

Let f be completely additive, then

$$[f(g * h)](n) = f(n)[\sum_{d|n} g(d)h(\frac{n}{d})] = \sum_{d|n} f(n)g(d)h(\frac{n}{d})$$

$$= \sum_{d|n} f(d, \frac{n}{d})g(d)h(\frac{n}{d}) = \sum_{d|n} [f(d) + f(\frac{n}{d})]g(d)h(\frac{n}{d})$$

$$= \sum_{d|n} f(d)g(d)h(\frac{n}{d}) + \sum_{d|n} f(\frac{n}{d})g(d)h(\frac{n}{d})$$

$$= [fg * h + g * fh](n)$$

(2) $\Longrightarrow$ (3)

Let,

$$f(g * h) = fg * h + g * fh$$

for all $g, h \in \mathcal{A}$

Then we have,

$$f(g * g) = fg * g + g * fg = fg * g + fg * g = 2(fg * g)$$

(3) $\Longrightarrow$ (4)

Let

$$f(g * g) = 2(fg * g)$$

Then,

$$f\tau = f(\zeta * \zeta) = 2(f\zeta * \zeta) = 2(f \cdot 1 * \zeta) = 2(f * \zeta)$$

(4) $\Longrightarrow$ (1)

Suppose

$$f\tau = 2(f * \zeta)$$

and let $n = p$.Then

$$f(p)\tau(p) = 2f(p) = 2(f * \zeta) \Longrightarrow f(p) = f(1) + f(p) \Longrightarrow f(1) = 0$$

Now, let $n \in \mathbb{N}, n > 1$ and $n = p_1^{k_1} \cdots p_t^{k_t}$.Then, it will be shown,when $m = k_1 + \cdots + k_t$, that

$$f(n) = k_1 f(p_1) + \cdots + k_t f(p_t)$$

So, if $M_i = 0, 1, 2, \cdots, k_i for i = 1, 2, \cdots t$ and $M = M_1 \times M_2 \times \cdots \times M_t$, then

$$\frac{1}{2}f(n)\tau(n) = \sum_{(i_1 \cdots i_t) \in M} f(p_i^{i_1} \cdots p_t^{i_t}) = f(n) + \sum_{(i_1 \cdots i_t) \in M, i_1 + \cdots + i_t \neq m} f(p_i^{i_1} \cdots p_t^{i_t})$$

by induction

$$\frac{1}{2}f(n)\tau(n) = f(n) + \sum_{(i_1 \cdots i_t) \in M, i_1 + \cdots + i_t \neq m} [i_1 f(p_1) + \cdots + i_t f(p_t)]$$

Now,

$$\sum_{(i_1 \cdots i_t) \in M, i_1 + \cdots + i_t \neq m} [i_1 f(p_1) + \cdots + i_t f(p_t)] = \frac{1}{2}[\prod_{i=1}^{t}(k_i + 1)][\sum_{i=1}^{t} k_i f(p_i)] - \sum_{i=1}^{t} k_i f(p_i)$$

This implies

$$f(n) = \sum_{i=1}^{t} k_i f(p_i)$$

□

Chapter 3

Multiplicative and Additive Power Series

In this chapter we will discus a relationship between formal power series and arithmetic functions. However, we will need to first define the concept of formal power series.

3.1 The Formal Power Series

Definition 3.1.0.1. Let $\mathcal{R}$ be a commutative ring with unity and $\mathbb{N}_o = 0, 1, 2, ...$ with $f : \mathbb{N}_o \longrightarrow \mathcal{R}$ such that $f = (a_0, a_1, a_2, ..., a_i, ...) = (a_i)_{i\in\mathbb{N}_o}$ and $a_i \in \mathcal{R}$
Then define,

$$\mathcal{R}' = \{f | f = (a_i)_{i\in\mathbb{N}_o}\}$$

with the properties

1. $f + g$=$(a_i + b_i)_{i\in\mathbb{N}_o}$
2. $f \cdot g$= $(c_k)_{k\in\mathbb{N}_o}$ $\qquad c_k = \sum_{i+j=k} a_i b_j$

We will show that $\mathcal{R}'$ with addition and multiplication forms a commutative ring with unity,

Theorem 3.1.0.2. $(\mathcal{R}', +, \cdot)$ *is a commutative ring with unity*

Let us formalize this concept by introducing some notation.

Notation 3.1.0.3.

$$(1,0,0,0.....) = x^0$$

$$(0,1,0,0.....) = x^1$$

Also,

$$(0,0,0,...1...) = x^k$$

where there are k many terms before 1.

Example 3.1.0.4. $(a,b,0,0,...) = ax^0 + bx^1 = a(1,0,0,0.....) + b(0,1,0,0.....)$

We will call these $a, b \in \mathcal{R}$ coefficients of x.

With this we can say

$$\mathcal{R}' = \mathcal{R}[[x]] = \{f = \sum_{k=0}^{\infty} a_k x^k\}$$

Now we can define the following

Definition 3.1.0.5. The ring $\mathcal{R}'$ is called the **formal power series** in x with coefficients in $\mathcal{R}$ is denoted by $\mathcal{R}[[x]]$.The elements of $\mathcal{R}[[x]]$ are infinite expressions of the form

$$f(x) = a_0x^0 + a_1x^1 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

and addition and multiplication are defined as

$$\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$\sum_{k=0}^{\infty} a_k x^k . \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (\sum_{i+j=k} a_i b_j) x^k$$

Example 3.1.0.6. *A few well known examples of formal power series*

1. ***Geometric Series***

$$S(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

2. ***Exponential Series***

$$exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

where $exp(x)$ *represents the traditional exponential function and* $k! = k.(k-1)\ldots 1$.

3. ***Logarithmic Series***

$$log(\frac{1}{1-x}) = \sum_{k=0}^{\infty} \frac{x^k}{k}$$

A known property of the exponential function is sort of a reverse additive property

$$exp(z+w) = exp(z).exp(w) \forall z, w \in \mathbb{C}$$

So, if we have $exp(ax)$,where $a \in \mathbb{N}_o$, then we obtain

$$\sum_{k=0}^{\infty} \frac{(ax)^k}{k!} = exp(ax) = exp(x + \cdots + x) = exp(x).exp(x)\ldots exp(x)$$

$$= \sum_{k=0}^{\infty} \frac{(x)^k}{k!} \cdots \sum_{k=0}^{\infty} \frac{(x)^k}{k!} = (\sum_{k=0}^{\infty} \frac{(x)^k}{k!})^a$$

Let us introduce a type of power series which gives an immediate connection to arithmetic functions

3.2 Bell Series

In this section and the following, we will be addressing the known results found in Apostol regarding the concept of Bell Series and their connection to arithmetic functions. Then, we will come to results proposed by McCarthy. However, we will verify them using Bell Series

Definition 3.2.0.1. Let f be an arithmetic function and p be a prime. Then the formal power series

$$f_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k$$

is called the **Bell Series** of f modulo p.

This concept was first studies by E. T. Bell in order to observe multiplicative properties of arithmetic functions with power series

Example 3.2.0.2. ***The mobius Function (μ)***

To illustrate an example of this type of series, recall the Möbius function. It can be observed that

$$\mu_p(x) = \sum_{k=0}^{\infty} \mu(p^k)x^k$$

Remember, the Möbius function is defined as follows

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n=1, \\ (-1)^k & \text{if} \quad n=p_1.p_2...p_k \text{ with distinct prime} \\ 0 & \text{if} \quad \text{if there exists a prime such that } p^2|n \end{cases}$$

$Therefore,$

$\mu_p(x) = \sum_{k=0}^{\infty} \mu(p^k)x^k = 1 \cdot x^0 + (-1) \cdot x^1 + 0 \cdot x^2 + \cdot + 0 \cdot x^k + \cdots = 1 - x$

Example 3.2.0.3. ***Dirichlet identity function***

Also, we can see the Bell series representation of the Dirichlet identity function by

$$e_p(x) = \sum_{k=0}^{\infty} e(p^k)x^k e_p(x) = e(p^0)x^0 + e(p^1)x^1 + e(p^2)x^2 + \cdots$$

$$= 1 + 0 + 0 + \cdots = 1$$

This gives us a good representation of the mobius and identity function, but how would we define the other arithmetic functions we have discussed?

Let us recall the completely multiplicative function, then this result and proof from Apostol follows immediately.

Theorem 3.2.0.4. *If f is a completely multiplicative arithmetic function, then*

$$f_p(x) = \frac{1}{1 - f(p)x}$$

Proof. Let f be completely multiplicative and p prime with $K \geq 1$, then

$$f(p^k) = f(p) \cdots f(p) = f(p)^k$$

so,

$$f_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty} f(p)^k x^k = \sum_{k=0}^{\infty} 1 \cdot (f(p)x)^k$$

□

Note, that the above expression yields a geometric power series, meaning $\sum_{k=0}^{\infty} 1 \cdot x^k = \frac{1}{1-x}$

$$\sum_{k=0}^{\infty} 1 \cdot (f(p)x)^k = f_p(x) = \frac{1}{1 - f(p)x}$$

Example 3.2.0.5. *We have studied quite a few completely multiplicative functions in this paper so their power series representations are the following*

1. $\zeta_p(x) = \frac{1}{1-\zeta(p)x} = \frac{1}{1-x}$

   LHS-

$$\zeta(p) = 1 \Longrightarrow \frac{1}{1 - \zeta(p)x} = \frac{1}{1-x}$$

   RHS-

$$\zeta_p(x) = \sum_{k=0}^{\infty} \zeta(p^k)x^k = \zeta(p^0)x^0 + \zeta(p^1)x^1 + \zeta(p^2)x^2 + \cdots$$

$$= 1 + 1.x + 1.x^2 + \cdots = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

2. $i_p(x) = \frac{1}{1-i(p)x} = \frac{1}{1-p\cdot x}$

LHS-

$$i(p) = p \Longrightarrow \frac{1}{1-i(p)x} = \frac{1}{1-px}$$

RHS-

$$i_p(x) = \sum_{k=0}^{\infty} i(p^k)x^k = i(p^0)x^{+}i(p^1)x^1 + i(p^2)x^2 + \cdots$$

$$= 1 + p.x + p^2.x^2 + \cdots = 1 + (px) + (px)^2 + \cdots = \frac{1}{1-px}$$

3. $i_p^\alpha(x) = \frac{1}{1-i(p^\alpha)x} = \frac{1}{1-p^\alpha\cdot x}$

LHS-

$$i(p^\alpha) = p^\alpha \Longrightarrow \frac{1}{1-i(p^\alpha)x} = \frac{1}{1-p^\alpha x}$$

RHS-

$$i_p^\alpha(x) = \sum_{k=0}^{\infty} i^\alpha(p^k)x^k = i^\alpha(p^0)x^0 + i^\alpha(p^1)x^1 + i^\alpha(p^2)x^2 + \cdots$$

$$= 1 + p^\alpha.x + p^{2\alpha}.x^2 + \cdots = \frac{1}{1-p^\alpha x}$$

4. $\lambda_p(x) = \frac{1}{1-\lambda(p)x} = \frac{1}{1+x}$

LHS-

$$\lambda(p) = (-1) \Longrightarrow \frac{1}{1-\lambda(p)x} = \frac{1}{1-(-1)x} = \frac{1}{1+x}$$

RHS-

$$\lambda_p(x) = \sum_{k=0}^{\infty} \lambda(p^k)x^k = \lambda(p^0)x^+ \lambda(p^1)x^1 + \lambda(p^2)x^2 + \cdots$$

$$= 1 + (-1).x + 1.x^2 + (-1)x^3 + \cdots = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

Before we continue it is important to discuss the following theorem.

Theorem 3.2.0.6. *If f and g are multiplicative arithmetic functions, then $f = g$ if and only if*

$$f_p(x) = g_p(x)$$

for all primes p

Proof. First, let us assume that $f = g$. Then we see that

$$f(p^k) = g(p^k)$$

for any prime,p, and $k \geq 1$ Therefore, it is clear to see

$$f_p(x) = g_p(x)$$

for all primes p conversely, let

$$f_p(x) = g_p(x)$$

for all primes p. Then,

$$f_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty} g(p^k)x^k$$

This means,

$$f_p(x) = g_p(x)$$

for any power$k$. Also, f and $g$are assumed to multiplicative, therefore, we can say for any prime p

$$f = g$$

□

Theorem 3.2.0.7. *If f,g and h are arithmetic functions and $h = f * g$, then*

$$h_p(x) = f_p(x)g_p(x)$$

Proof. Let$p$be prime and$k \geq 1$. Recall that the divisors of p^k are

$$D = 1, p, p^2, \cdots, p^k$$

so,

$$h(p^k) = (f * g)(p^k) = \sum_{d^k} f(d) \cdot g(\frac{p^k}{d}) = \sum_{i+j=k} f(p^i)g(p^j)$$

Then, following our definition of formal power series multiplication, we can say

$$h_p(x) = \sum_{k=o}^{\infty}(\sum_{i+j=k} f(p^i)g(p^j))x^k = f_p(x) \cdot g_p(x)$$

□

With this result we can determine the Bell series representation of some arithmetic functions.

Application 3.2.0.8. *Recall*

$$\phi = i * \mu$$

therefore, we can say

$$\phi_p(x) = i_p(x) \cdot \mu_p(x) = \frac{1}{1 - p \cdot x} \cdot (1 - x) = \frac{1 - x}{1 - p \cdot x}$$

This is quite significant, since with this we can determine the formula representation of the Euler totient function

$$\phi_p(x) = \frac{1 - x}{1 - p \cdot x} = (1 - x) \cdot \sum_{k=0}^{\infty} p^k x^k = \sum_{k=0}^{\infty} p^k x^k - x \sum_{k=0}^{\infty} p^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} p^k x^k - \sum_{k=1}^{\infty} p^{(k-1)} x^k = 1 + \sum_{k=1}^{\infty} (p^k - p^{(k-1)}) x^k$$

Thus, we can say for $k \geq 1$

$$\phi(p^k) = p^k - p^{k-1} = p^k(1 - \frac{1}{p})$$

Extending this to any natural number $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ multiplicative we obtain

$$\phi(n) = \phi(p_1^{\alpha_1}.p_2^{\alpha_2}...p_k^{\alpha_k}) = \phi(p_1^{\alpha_1}).\phi(p_2^{\alpha_2})...\phi(p_k^{\alpha_k})$$

$$= p_1^{\alpha_1}(\frac{1}{1-p_1}).p_2^{\alpha_2}(\frac{1}{1-p_2})...p_k^{\alpha_k}(\frac{1}{1-p_k}) = n\prod_{i=1}^{k}(1 - \frac{1}{p_i})$$

This gives us a new way of determining the formula for the totient function, using what was learned from Apostol.

This gives us reason to believe that there are potentially more formulas of arithmetic functions which can be derived using Bell Series.

Application 3.2.0.9. *Another application of this theorem comes from the arithmetic function*

•$\sigma_\alpha = i^\alpha * \zeta \Longrightarrow \sigma_{\alpha p}(x) = i_p^\alpha(x).\zeta_p(x) = \frac{1}{1-p^\alpha x}.\frac{1}{1-x} = \frac{1}{(1-p^\alpha x).(1-x)}$

•$\tau = \zeta * \zeta \Longrightarrow \tau_p(x) = \zeta_p(x).\zeta_p(x) \Longrightarrow \tau_p(x) = \frac{1}{(1-x)^2}$

Theorem 3.2.0.10. *If f is a multiplicative arithmetic function, then f is completely multiplicative if and only if*

$$\tilde{f} = \mu f$$

Proof. First we will assume that f is completely multiplicative. Then, we can say

$$f_p(x) = \frac{1}{1 - f(p)x}$$

Now,

$$(\mu f)_p(x) = \sum_{k=0}^{\infty}(\mu f)(p^k)x^k = \sum_{k=0}^{\infty}\mu(p^k)f(p^k)x^k$$

Recalling a multiplicative property, we can say $f(1) = 1$. Therefore

$$= \sum_{k=0}^{\infty}\mu(p^k)f(p^k)x^k = 1 - f(p)x$$

Also, we can clearly see

$$e_p(x) = (\mu f)_p(x) \cdot f_p(x)$$

This is only the case if μf is Dirichlet inverse of f

Conversely, assume $\tilde{f} = \mu f$ then we have,

$$\tilde{f}_p(x) = (\mu f)_p(x) = 1 - f(p)x$$

If f is the inverse, it must be that

$$\mu f * f = e$$

which implies

$$1 = (1 - f(p)x) \cdot f_p(x)$$

This implies

$$f_p(x) = \frac{1}{1 - f(p)x}$$

meaning, f must be completely multiplicative $\square$

Theorem 3.2.0.11. *If f is a multiplicative arithmetic function, then f is completely multiplicative if and only if*

$$\tilde{f}(p^\alpha) = 0$$

$\forall\alpha \geq 2$

Proof. Let f be completely multiplicative.Then we have $\tilde{f} = \mu f$

$$\tilde{f}_p(x) = (\mu f)_p(x) = 1 - f(p)x$$

This implies

$$\tilde{f}(p^\alpha) = 0$$

Conversely, let, $\tilde{f}(p^\alpha) = 0 \ \forall \alpha \geq 2$

Then,

$$\tilde{f}_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k = 1 + f(p)x$$

$$f_p(x) = \frac{1}{1 + f(p)x}$$

This implies

$$f_p(x) = \frac{1}{1 - (-f(p)x)}$$

meaning, f must be completely multiplicative. □

Application 3.2.0.12. *Since*

$$\tilde{\lambda} = \mu\lambda$$

Then we have,

$$\tilde{\lambda}_p(x) = (\mu\lambda)_p(x) = \sum_{k=0}^{\infty} \mu(p^k)\lambda(p^k)x^k = 1 + x = \sum_{k=0}^{\infty} \mu(p^k)\mu(p^k)x^k = \mu_p^2(x)$$

So we can see that

$$\tilde{\lambda} = \mu^2$$

Application 3.2.0.13. *Let*

$$f(n) = 2^{\omega(n)}$$

then,

$$f_p(x) = \sum_{k=0}^{\infty} 2^{\omega(p^k)} x^k = 1 + \sum_{k=1}^{\infty} 2x^k = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

Therefore, we have

$$f_p(x) = \mu_p^2(x).\zeta_p(x)$$

Meaning we have a formalization for this function

$$2^{\omega(n)} = \mu^2 * \zeta = \sum_{k=0}^{\infty} \mu^2(d)$$

Therefore, the arithmetic function 2^{ω} *is the summation of* μ^2.

3.3 Bell Series and Specially Multiplicative Functions

In this section we will use Bell Series to verify results shown in the work of McCarthy, in regards to the concept of specially multiplicative arithmetic functions.

Definition 3.3.0.1. Let f be a multiplicative arithmetic function, then f is said to be **specially multiplicative** if

$$f = g * h$$

where g and h are completely multiplicative arithmetic functions.

The following result comes from McCarthy

Theorem 3.3.0.2. *If f is a multiplicative arithmetic function, then it is specially multiplicative if and only if*

$$f_p(x) = \frac{1}{1 - bx + cx^2}$$

Proof. Let f be specially multiplicative and p be prime, then we have

$$f = g * h$$

where g, h are completely multiplicative.
we also have,
$g_p(x) = \frac{1}{1-g(p)x}$ and $h_p(x) = \frac{1}{1-h(p)x}$
It is also known, that

$$f_p(x) = g_p(x) \cdot h_p(x) = \frac{1}{1-g(p)x}\frac{1}{1-h(p)x} = \frac{1}{1-[g(p)+h(p)]x+[g(p)h(p)]x^2}$$

Notice that $[g(p)+h(p)]$and$[g(p)h(p)]$ are elements of $\mathbb{C}$, so we can see the condition is satisfied.
Conversely,assume

$$f_p(x) = \frac{1}{1 - bx + cx^2}$$

where $b, c \in \mathbb{C}$. Then we have,

$$f_p(x) = \frac{1}{1 - bx + cx^2} = \frac{1}{1 - a_1 \cdot x} \cdot \frac{1}{1 - a_2 \cdot x}$$

with a_1 and a_2 being the roots of quadratic equation,

$$1 - bx + cx^2 = 0$$

Now,we can say there exists two arithmetic functions g and h where $g(p) = a_1$ and $h(p) = a_2$. Therefore
$g_p(x) = \frac{1}{1-g(p)x}$ and $h_p(x) = \frac{1}{1-h(p)x}$
meaning, g and h are completely multiplicative.With this,we can conclude

$$f_p(x) = g_p(x).h_p(x)$$

which implies

$$f = g * h$$

$\Longrightarrow f$ is specially multiplicative. □

Illustration 3.3.0.3. $g(n) = 2^{\Omega(n)}$ *and* $h(n) = 3^{\Omega(n)}$
Both functions are completely multiplicative,so the function

$$f(n) = 2^{\Omega(n)} * 3^{\Omega(n)}$$

is a specially multiplicative function

Illustration 3.3.0.4. *Also,recall* ζ *and* i *are completely multiplicative,and*
$\tau = \zeta * \zeta$ *and* $\sigma = \zeta * i$
Therefore we can say τ *and* σ *are specially multiplicative.*

Now we should recall the property of completely multiplicative functions,that being if f is completely multiplicative then
$\tilde{f}(p^{\alpha}) = 0 \ \forall \alpha \geq 2$

Theorem 3.3.0.5. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if*

$$\tilde{f}(p^{\alpha}) = 0 \quad \forall \alpha \geq 3$$

Proof. Let f be specially multiplicative.Then we have

$$f = g * h$$

where g,h are completely multiplicative. Then, it is the case that

$$\tilde{f} = \tilde{g} * \tilde{h} = \mu g * \mu h$$

which gives us

$$\tilde{f}_p(x) = (\sum_{k=0}^{\infty} \mu(p^k) g(p^k) x^k)(\sum_{k=0}^{\infty} \mu(p^k) h(p^k) x^k) = (1 - g(p)x)(1 - h(p)x)$$

$$\Longrightarrow \tilde{f}(p^{\alpha}) = 0 \quad \forall \alpha \geq 3$$

Conversely we will say

$$\tilde{f}(p^{\alpha}) = 0 \quad \forall \alpha \geq 3$$

It follows that

$$\tilde{f}_p(x) = \sum_{k=0}^{\infty} \tilde{f}(p^k) x^k = 1 + \tilde{f}(p)x + \tilde{f}(p^2)x^2$$

Therefore we have,

$$f_p(x) = \frac{1}{1 + \tilde{f}(p)x + \tilde{f}(p^2)x^2} = \frac{1}{1 - (-\tilde{f}(p))x + \tilde{f}(p^2)x^2}$$

So by the previous theorem we see that it must be the case that f is specially multiplicative.

□

Theorem 3.3.0.6. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if*

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

for all primes, p, and for all ≥ 1.

Proof. Let f be specially multiplicative.Then we have

$$f = g * h$$

where g,h are completely multiplicative. Then,

$$f_p(x) = g_p(x)h_p(x)$$

This gives us,

$$\sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty}(\sum_{i+j=k} g(p^i)h(p^j))x^k$$

So for $k = 1$ we obtain

$$f(p)x = [g(p)h(1) + g(1)h(p)]x = x$$

and for $k = 2$ we obtain

$$f(p^2)x^2 = \sum_{i+j=k} g(p^i)h(p^j))x^2 = [h(p2) + g(p)h(p) + g(p2)]x^2$$

Notice, since g and h are completely multiplicative we see

$$f(p^2) - f(p)^2 = [h(p2) + g(p)h(p) + g(p2)] - ([g(p) + h(p)])^2$$

$$= [h(p^2) + g(p)h(p) + g(p^2)] - [g(p^2) + 2g(p)h(p) + h(p^2) = -g(p)h(p)$$

Then RHS,

$$f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

$$= [g(p) + h(p)][(\sum_{i+j=\alpha} g(p^i)h(p^j))] + [(\sum_{i+j=\alpha-1} g(p^i)h(p^j))][-g(p)h(p)]$$

$$= [(\sum_{i+j=\alpha} g(p^i + 1)h(p^j))] + [(\sum_{i+j=\alpha} g(p^i)h(p^{j+1}))] - [(\sum_{i+j=\alpha-1} g(p^i + 1)h(p^{j+1}))]$$

$$= [(\sum_{i+j=\alpha} g(p^{i+1})h(p^j)) - (\sum_{i+j=\alpha-1} g(p^{i+1})h(p^{j+1}))] + [(\sum_{i+j=\alpha} g(p^i)h(p^{j+1}))]$$

$$= g(p^{i+1})h(1) + \sum_{i+j=\alpha} g(p^i)h(p^{j+1})$$

$$= \sum_{i+j=\alpha} g(p^{i+1})h(p^{j+1})$$

$$= \sum_{i+j=\alpha+1} g(p^i)h(p^j)$$

$$f(p^{\alpha+1}) = LHS$$

Conversely, let us assume that, for all $\alpha \geq 1$, we have

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

Now, $f \in mathcalM$,so $f(1) = 1$. Also, for any prime p we have,

$$0 = e(p) = (f * \tilde{f})(p) = f(1)\tilde{f}(p) + f(p)\tilde{f}(1)$$

This gives us

$$\tilde{f}(p) = -f(p)$$

Following for p^2 we obtain

$$0 = e(p^2) = (f * \tilde{f})(p^2) = f(1)\tilde{f}(p^2) + f(p)\tilde{f}(p) + f(p^2)\tilde{f}(1)$$

which implies

$$\tilde{f}(p^2) = f(p)^2 - f(p^2)$$

Also, for p^3 we obtain

$$0 = e(p^3) = (f * \tilde{f})(p^3) = f(1)\tilde{f}(p^3) + f(p)\tilde{f}(p^2) + f(p^2)\tilde{f}(p) + f(p^3)\tilde{f}(1)$$

This implies the following

$$\tilde{f}(p^3) = -f(p)\tilde{f}(p^2)f(p^2)\tilde{f}(p) - f(p^3)$$

$$) = -f(p)(f(p))^2 - f(p^2)) - f(p^2)(-f(p)) - f(p^3)$$

$$= -f(p)^3 + 2f(p)f(p^2) - f(p^3)$$

we have RHS

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

for all $\alpha \geq 1$

Then we can say for $\alpha = 2$

$$f(p^3) = f(p)f(p^2) + f(p)[f(p^2) - f(p)^2] = 2f(p)f(p^2) - f(p^3)$$

so,

$$\tilde{f}(p^3) = -f(p)^3 + 2f(p)f(p^2) - f(p^3)$$

$$= -f(p)^3 + 2f(p)f(p^2) - [2f(p)f(p^2) - f(p^3)] = 0$$

Next,

let us assume that it is the case

$$\tilde{f}(p^{\alpha}) = 0$$

when $3 \leq \alpha \leq n$

If this is true, then what follows is

$$0 = e(p^{\alpha+1}) = (f * \tilde{f})(p^{\alpha+1}) = (\tilde{f} * f)(p^{\alpha+1}) = \sum_{k=0}^{n+1} f(p^i)f(p^{n+1-i})$$

$$= \tilde{f}(1)f(p^{n+1}) + \tilde{f}(p)f(p^n) + \tilde{f}(p^2)f(p^{n-1}) + \tilde{f}(p^{n+1})f(1)$$

$$= f(p^{n+1}) - f(p)f(p^n) + [f(p)^2 - f(p^2)]f(p^{n-1}) + \tilde{f}(p^{n+1})$$

Therefore,

$$\tilde{f}(p^{n+1}) = -f(p^{n+1}) + f(p)f(p^n) - [f(p)^2 - f(p^2)]f(p^{n-1})$$

Since $f(p^{\alpha+1}) = (f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$

$$\tilde{f}(p^{n+1}) = -(f(p)f(p^n) + f(p^{n-1})[f(p^2) - f(p)^2]) + f(p)f(p^n) - [f(p)^2 - f(p^2)]f(p^{n-1}) = 0$$

So f must be specially multiplicative □

Theorem 3.3.0.7. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if there exists a multiplicative function, F,*
such that for all m and n

$$f(mn) = \sum_{d|(m,n)} f(\frac{m}{d})f(\frac{n}{d})F(d)$$

Proof. Let us assume f is specially multiplicative.

If $(mn, m'n') = 1$, then $((m,n),(m',n')) = 1$ and $(mm',nn') = (m,n)(m',n')$.

It must be shown that there exists some multiplicative function F which satisfies

$$f(p^{\alpha+\beta}) = \sum_{i=0}^{min(\alpha,\beta)} f(p^{\alpha-i})f(p^{\beta-i})F(p^i)$$

for all $\alpha,\beta \geq 1$

Now, let $F = \mu G$, where G is a completely multiplicative function and for each prime p

$$G(p) = f(p)^2 - f(p^2)$$

Using induction

Then, for $\beta \leq \alpha$ and $\beta = 1$ we have since, f is specially multiplicative.

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2] = f(p)f(p^{\alpha})G(1) - f(p^{\alpha-1})G(P)$$

$$= f(p)f(p^{\alpha})\mu(1)G(1) + f(p^{\alpha-1})\mu(p)G(p) = f(p)f(p^{\alpha})F(1) + f(p^{\alpha-1})F(p)$$

which satisfies the sum.

Now assume for $\beta > 1$ that the equation holds for $\beta - 1$ for all $\beta \leq \alpha$. Also,

$$F(p^2) = F(p^3) = \cdots = 0$$

Therefore we obtain the following

$$f(p^{\alpha+\beta}) = f(p^{\alpha+1+\beta-1}) = f(p^{\alpha+1})f(p^{\beta-1}) + f(p^{\alpha})f(p^{\beta-2})F(p)$$

$$= [f(p)f(p^{\alpha}) - f(p^{\alpha-1})G(P)]f(p^{\beta-1}) + f(p^{\alpha})f(p^{\beta-2})\mu(p)G(p)$$

$$= f(p^{\alpha})[f(p)f(p^{\beta-1}) - f(p^{\beta-2})G(P)] - f(p^{\alpha-1})f(p^{\beta-1})G(p)$$

using for β what we have

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha})G(1) - f(p^{\alpha-1})G(P)$$

We get,

$$f(p^{\alpha+\beta}) = f(p^{\alpha})[f(p)f(p^{\beta-1}) - f(p^{\beta-2})G(P)] - f(p^{\alpha-1})f(p^{\beta-1})G(p)$$

$$= f(p^{\alpha})f(p^{\beta})F(1) + f(p^{\alpha-1})f(p^{\beta-1})\mu(p)G(p)$$

$$= f(p^{\alpha})f(p^{\beta})F(1) + f(p^{\alpha-1})f(p^{\beta-1})F(p)$$

This is what we needed to show.

Conversely assume the equation defined above holds.

Let p be a prime with$m = p^{\alpha}$ and $n = p$ where $\alpha \geq 1$. Then

$$f(p^{\alpha+1}) = f(p^1)f(p^{\alpha})F(1) + f(p^1)f(p^{\alpha-1})F(p)$$

If we let $\alpha = 1$, then

$$f(p^2) = f(p)^2 + F(p)$$

which implies

$$F(p) = f(p^2) - f(p)^2$$

With the equation from theorem satisfied and hence we can say f is specially multiplicative.

□

Theorem 3.3.0.8. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if there exists a completely multiplicative function, G, such that for all m and n*

$$f(m)f(n) = \sum_{d|(m,n)} f(\frac{mn}{d^2})G(d)$$

Proof. Assume f is specially multiplicative. Then the equation from previous theorem holds.

So RHS-

$$\sum_{d|(m,n)} f(\frac{mn}{d^2})G'(d) = \sum_{d|(m,n)} f(\frac{m}{d}\cdot\frac{n}{d})G'(d)$$

$$= \sum_{d|(m,n)} \sum_{D|(\frac{m}{d}),\frac{n}{d})} f(\frac{\frac{m}{d}}{D})f(\frac{\frac{n}{d}}{D})\mu(D)G'(D)G'(d)$$

Continuing we obtain

$$= \sum_{d|(m,n)} \sum_{k|(m,n)(d|k)} f(\frac{m}{k})f(\frac{n}{k})\mu(\frac{k}{d})G'(\frac{k}{d})G'(d) = \sum_{d|(m,n)} \sum_{k|(m,n)(d|k)} f(\frac{m}{k})f(\frac{n}{k})\mu(\frac{k}{d})G'(k)$$

$$= \sum_{k|(m,n)} f(\frac{m}{k})f(\frac{n}{k})G'(k) \sum_{(d|k)} \mu(\frac{k}{d}) = f(m)f(n)$$

=LHS

Since here, $\mu(\frac{k}{d}) = 1$ and $k = 1$

Conversely,

assume the above equation holds.

Let$p$ be a prime and $p = m = n$, then

$$f(m)f(n) = f(p)f(p) = \sum_{d|(m,n)} f(\frac{mn}{d^2})G(d) = f(\frac{p^2}{1})G(1) + f(\frac{p^2}{p^2})G(p)$$

$$f(m)f(n) = f(p)^2 = f(p^2) + G(p)$$

Therefore,

$$G(p) = f(p)^2 - f(p^2)$$

If $m = p^\alpha$ and $n = p$ with $\alpha \geq 1$, we obtain

$$f(p^\alpha)f(p) = \sum_{d|(p^\alpha,p)} f(\frac{p^{\alpha+1}}{d^2})G(d)$$

$$= f(\frac{p^{\alpha+1}}{1})G(1) + f(\frac{p^{\alpha+1}}{p^2})G(p)$$

$$= f(p^{\alpha+1}) + f(p^{\alpha-1})[f(p)^2 - f(p^2)]$$

Since $G(p) = f(p)^2 - f(p^2)$ Therefore,

$$f(p^\alpha)f(p) = f(p^{\alpha+1}) + f(p^{\alpha-1})[-f(p^2) + f(p)^2]$$

$$f(p^{\alpha+1}) = f(p^{\alpha})f(p) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

This means, by Theorem, f is specially multiplicative □

3.4 Multiplicative and Additive Power Series

This section we will discuss the embedding of the formal power into the unitary ring of arithmetic functions.

Recall $(\mathcal{A}, +, \oplus)$ is the unitary ring and let us consider the formal power series ring $C[[x]]$

Theorem 3.4.0.1. *The ring $C[[x]]$ can be embedded in the unitary ring of arithmetic functions.*

Proof. Consider the map $\eta : C[[x]] \longrightarrow \mathcal{A}$ such that

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(n) = \omega(n)! a_{\omega(n)}$$

Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$,where $p_1 \cdots pk$ are distinct primes and $\alpha \geq 1$, then

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(n) + \eta(\sum_{k=0}^{\infty} b_k x^k)(n) = \omega(n)! a_{\omega(n)} + \omega(n)! b_{\omega(n)} = k! a_k + k! b_k$$

$$= k!(a_k + b_k) = \eta(\sum_{k=0}^{\infty}(a_k + b_k)x^k)(n) = \eta(\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k)(n)$$

Also,

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(n) \oplus \eta(\sum_{k=0}^{\infty} b_k x^k)(n) = \omega(n)! a_{\omega(n)} \oplus \omega(n)! b_{\omega(n)} = k! a_k ! b_k$$

$$= (\sum_{d|k}(d)! a_d \cdot (\frac{k}{d})! b_{\frac{k}{d}} = k! \sum_{d|k} a_d b_{\frac{k}{d}} = \eta(\sum_{k=0}^{\infty}(\sum_{i+j=k} a_i b_j) x^k)(n) = \eta(\sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} b_k x^k)(n)$$

Therefore, η is a homomorphism.

We can also show that this mapping is injective.

Let,

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(n) = \eta(\sum_{k=0}^{\infty} a_k x^k)(m)$$

This implies,

$$\omega(n)!a_{\omega(n)} = \omega(m)!a_{\omega(m)}$$

Now this is only the case when $\omega(n) = \omega(m)$. Therefore, n and m must a product of primes, both with k factors, meaning η is injective.

So, $\mathbb{C}[[x]]$ can be embedded A $\square$

we can determine the characteristics of a formal power series in $C[[x]]$ as if it were an arithmetic function.

Definition 3.4.0.2. A formal power series $\sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ is called **multiplicative** if arithmetic function $\eta(\sum_{k=0}^{\infty} a_k x^k)$ is multiplicative.

Definition 3.4.0.3. A formal power series $\sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ is called **additive** if arithmetic function $\eta(\sum_{k=0}^{\infty} a_k x^k)$ is additive.

The binary operation

$$\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} k! a_k b_k x^k$$

will give us the opportunity to create analogues for the properties studied in chapter 2 with multiplicative and additive formal power series.

Theorem 3.4.0.4. *Let* $\sum_{k=0}^{\infty} a_k x^k \in C[[x]]$ *be a non-zero power series.Then the following are equivalent*

1. $\sum_{k=0}^{\infty} a_k x^k$ *is multiplicative*

2. $a_k = \frac{a_1^k}{k!}\ \forall k \in \mathbb{N}$

3. $\sum_{k=0}^{\infty} a_k x^k \odot (\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k) = (\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} b_k x^k)\ (\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} c_k x^k)$

for all $\sum_{k=0}^{\infty} b_k x^k, \sum_{k=0}^{\infty} c_k x^k \in C[[x]]$

4. $\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} a_k x^k$

Proof. (1) $\Longrightarrow$ (2)

Let $\sum_{k=0}^{\infty} a_k x^k$ be multiplicative, then

$$1 = \eta(\sum_{k=0}^{\infty} a_k x^k)(1) = \omega(1)!a_\omega(1) = a_0$$

$$a_1 = \eta(\sum_{k=0}^{\infty} a_k x^k)(p^\alpha) = \omega(p^\alpha)!a_\omega(P^\alpha) = a_0$$

So we can say,

$$a_1^k = \eta(\sum_{k=0}^{\infty} a_k x^k)(p_1^{\alpha_1}) \cdots \eta(\sum_{k=0}^{\infty} a_k x^k)(p_k^{\alpha_k})$$

$$= \eta(\sum_{k=0}^{\infty} a_k x^k)(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = k!a_k$$

(2) $\Longrightarrow$ (1)

Let $a_k = \frac{a_1^k}{k!}$ $\forall n \in \mathbb{N}$ and let $n, m \in \mathbb{N}$ st. $(m, n) = 1$ then,

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(m.n) = \eta(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k)(m \cdot n) = a_1^{\omega(m \cdot n)}$$

$$a_1^{\omega(m)+\omega(n)} = a_1^{\omega(m)} \cdot a_1^{\omega(n)} = \eta(\sum_{k=0}^{\infty} a_k x^k)(m) \cdot \eta(\sum_{k=0}^{\infty} a_k x^k)(n)$$

(2) $\Longrightarrow$(3)

Let $a_k = \frac{a_1^k}{k!}$ $\forall n \in \mathbb{N}$ then,

$$(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} b_k x^k)(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} c_k x^k) = (\sum_{k=0}^{\infty} k!\frac{a_1^k}{k!} b_k x^k)(\sum_{k=0}^{\infty} k!\frac{a_1^k}{k!} c_k x^k)$$

$$= (\sum_{k=0}^{\infty} a_1^k b_k x^k)(\sum_{k=0}^{\infty} a_1^k c_k x^k) = \sum_{k=0}^{\infty} a_1^k (\sum_{i+j=k} b_i c_j) x^k)$$

$$= \sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} (\sum_{i+j=k} b_i c_j) x^k = \sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot (\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k)$$

(3) $\Longrightarrow$(4)

Let the distributive property hold.Then

$$\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$$

Recall,

$$exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

This would imply

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} (2x)^k = exp(x) \cdot exp(x) = (\sum_{k=0}^{\infty} \frac{1}{k!} x^k)(\sum_{k=0}^{\infty} \frac{1}{k!} x^k)$$

$$= \sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} a_k x^k \odot (\sum_{k=0}^{\infty} \frac{1}{k!} x^k \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k)$$

$$= (\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{1}{k!} x^k) \cdot (\sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k) = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} a_k x^k$$

(4) $\Longrightarrow$ (2)

Assume $\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} a_k x^k$ holds

$$\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{r,s=0}^{\infty} a_r a_s x^{r+s}$$

$$= \sum_{k=0,0 \le r \le k}^{\infty} a_r a_{k-r} x^k$$

$$= \sum_{k=0}^{\infty} (\sum_{r=0}^{\infty} a_r a_{k-r}) x^k$$

Thus,

$$2^k a_k = \sum_{r=0}^{\infty} a_r a_{k-r}$$

$$\implies 2^k a_k = a_0 a_k + a_1 a_{k-1} + a_2 a_{k-2} + \cdots + a_{k-1} a_1 + a_k a_0$$

$$\implies 2^k a_k = 2a_k + (a_1 a_{k-1} + a_2 a_{k-2} + \cdots + a_{k-1} a_1)$$

$$\implies (2^k - 2)a_k = (a_1 a_{k-1} + a_2 a_{k-2} + \cdots + a_{k-1} a_0) \qquad \cdots\cdots\cdots(1)$$

To prove $a_k = \frac{a_1^k}{k!}\ \forall k \in \mathbb{N}$

we prove by induction

For $k = 1$;

$$a_1 = \frac{a_1^1}{1!}$$

assume

$$a_k = \frac{a_1^k}{k!}$$

For $k+1$ by (1) above

$$a_{k+1} = \frac{1}{(2^{k+1}-2)} \cdot (a_1 a_k + a_2 a_{k-1} + \cdots + a_k a_1)$$

using induction

$$a_{k+1} = \frac{1}{(2^{k+1}-2)} \cdot \left(\frac{a_1 a_1^k}{1!k!} + \frac{a_1^2 a_1^{k-1}}{2!(k-1)!} + \cdots \frac{a_1^k a_1^1}{k!1!}\right)$$

$$= \frac{a_1^{k+1}}{(2^{k+1}-2)} \cdot \left(\frac{1}{1!k!} + \frac{1}{2!(k-1)!} + \cdots \frac{1}{k!1!}\right)$$

$$= \frac{a_1^{k+1}}{(k+1)!} \cdot \frac{1}{(2^{k+1}-2)}\left(\frac{(k+1)!}{1!k!} + \frac{(k+1)!}{2!(k-1)!} + \cdots \frac{(k+1)!}{k!1!}\right)$$

$$= \frac{a_1^{k+1}}{(k+1)!} \cdot \frac{1}{(2^{k+1}-2)}[(k+1) + (k)(k+1) + \cdots + (k+1))$$

using binomial expansion

$$a_{k+1} = \frac{a_1^{k+1}}{(k+1)!}$$

Hence shown that (4) $\Longrightarrow$ (2) $\square$

Theorem 3.4.0.5. *Let* $\sum_{k=0}^{\infty} a_k x^k \in C[[x]]$ *be a non-zero power series.Then the following are equivalent*

1. $\sum_{k=0}^{\infty} a_k x^k$ *is additive*

2. $a_0 = 0$ *and* $a_k = \frac{a_1}{(k-1)!}$ $\forall n \in \mathbb{N}$

3. $\sum_{k=0}^{\infty} a_k x^k \odot (\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k) =$
$[(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} b_k x^k) \cdot \sum_{k=0}^{\infty} c_k x^k] + [(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} c_k x^k) \cdot \sum_{k=0}^{\infty} b_k x^k]$
for all $\sum_{k=0}^{\infty} b_k x^k, \sum_{k=0}^{\infty} c_k x^k \in C[[x]]$

4. $\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} \frac{2}{k!} x^k \cdot \sum_{k=0}^{\infty} a_k x^k$

Proof. (1) $\Longrightarrow$ (2)

Let $\sum_{k=0}^{\infty} a_k x^k$ be additive, then

$$0 = \eta(\sum_{k=0}^{\infty} a_k x^k)(1) = \omega(1)! a_\omega(1) = a_0$$

$$a_1 = \eta(\sum_{k=0}^{\infty} a_k x^k)(p^\alpha) = \omega(p^\alpha)! a_\omega(P^\alpha) = a_0$$

So we can say,

$$ka_1 = \eta(\sum_{k=0}^{\infty} a_k x^k)(p_1^{\alpha_1}) + \cdots + \eta(\sum_{k=0}^{\infty} a_k x^k)(p_k^{\alpha_k})$$

$$= \eta(\sum_{k=0}^{\infty} a_k x^k)(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = k! a_k$$

Therefore

$$a_k = \frac{a_1}{(k-1)!}$$

(2) $\Longrightarrow$ (1)

Let $a_k = \frac{a_1}{(k-1)!}$ $\forall n \in \mathbb{N}$ and let $n, m \in \mathbb{N}$ st. $(m, n) = 1$ then,

$$\eta(\sum_{k=0}^{\infty} a_k x^k)(m.n) = \eta(\sum_{k=0}^{\infty} \frac{k \cdot a_1}{k!} x^k)(m \cdot n) = \omega(m \cdot n) a_1$$

$$= \omega(m) a_1 + \omega(n) a_1 = \eta(\sum_{k=0}^{\infty} \frac{k \cdot a_1}{k!} x^k)(m) + \eta(\sum_{k=0}^{\infty} \frac{k \cdot a_1}{k!} x^k)(n)$$

$$= \eta(\sum_{k=0}^{\infty} a_k x^k)(m) + \eta(\sum_{k=0}^{\infty} a_k x^k)(n)$$

(2) $\Longrightarrow$(3)

Let $a_k = \frac{a_1}{(k-1)!}$ $\forall n \in \mathbb{N}$ then,

$$[(\sum_{k=0}^{\infty} \frac{k \cdot a_1}{k!} x^k \odot \sum_{k=0}^{\infty} b_k x^k) \cdot \sum_{k=0}^{\infty} c_k x^k] + [(\sum_{k=0}^{\infty} \frac{k \cdot a_1}{k!} x^k \odot \sum_{k=0}^{\infty} c_k x^k) \cdot \sum_{k=0}^{\infty} b_k x^k]$$

$$= \sum_{k=0}^{\infty} k a_1 b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k + \sum_{k=0}^{\infty} k a_1 c_k x^k \cdot \sum_{k=0}^{\infty} b_k x^k$$

$$= \sum_{k=0}^{\infty} (\sum_{i+j=k} i a_1 b_i c_j) x^k + \sum_{k=0}^{\infty} (\sum_{i+j=k} i a_1 c_i b_j) x^k = \sum_{k=0}^{\infty} [\sum_{i+j=k} i a_1 (b_i c_j + c_i b_j)] x^k$$

$$= \sum_{k=0}^{\infty} [k a_1 \sum_{i+j=k} (b_i c_j)] x^k = \sum_{k=0}^{\infty} \frac{k \cdot a_1^k}{k!} x^k \odot (\sum_{k=0}^{\infty} b_k c_k x^k)$$

$$= \sum_{k=0}^{\infty} \frac{k \cdot a_1^k}{k!} x^k \odot (\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k)$$

(3) $\Longrightarrow$(4)

Let the distributive property hold.Then

$$\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$$

Recall,

$$exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

This would imply

$$\sum_{k=0}^{\infty}\frac{2^k}{k!}x^k=\sum_{k=0}^{\infty}\frac{1}{k!}(2x)^k=exp(x)\cdot exp(x)=(\sum_{k=0}^{\infty}\frac{1}{k!}x^k)(\sum_{k=0}^{\infty}\frac{1}{k!}x^k)$$

therefore,

$$=\sum_{k=0}^{\infty}a_kx^k\odot\sum_{k=0}^{\infty}\frac{2^k}{k!}x^k=\sum_{k=0}^{\infty}a_kx^k\odot(\sum_{k=0}^{\infty}\frac{1}{k!}x^k\cdot\sum_{k=0}^{\infty}\frac{1}{k!}x^k)$$

$$=[(\sum_{k=0}^{\infty}a_k\odot\sum_{k=0}^{\infty}\frac{1}{k!}x^k)\cdot\sum_{k=0}^{\infty}\frac{1}{k!}x^k]+[(\sum_{k=0}^{\infty}a_k\odot\sum_{k=0}^{\infty}\frac{1}{k!}x^k)\cdot\sum_{k=0}^{\infty}\frac{1}{k!}x^k)]$$

$$=\sum_{k=0}^{\infty}a_kx^k\cdot\sum_{k=0}^{\infty}\frac{2}{k!}x^k$$

(4) $\Longrightarrow$ (2)

Assume $\sum_{k=0}^{\infty}2^ka_kx^k=\sum_{k=0}^{\infty}\frac{2}{k!}x^k\cdot\sum_{k=0}^{\infty}a_kx^k$ holds

$$\sum_{k=0}^{\infty}2^ka_kx^k=\sum_{r,s=0}^{\infty}\frac{2}{s!}a_rx^k$$

$$=\sum_{k=0,0\leq s\leq k}^{\infty}\frac{2}{s!}a_{k-s}x^k$$

$$=\sum_{k=0}^{\infty}(\sum_{s=0}^{\infty}\frac{2}{s!}a_{k-s})x^k$$

Thus,

$$2^ka_k=\sum_{s=0}^{\infty}\frac{2}{s!}a_{k-s}$$

$$\Longrightarrow 2^ka_k=2a_k+\frac{2}{1!}a_{k-1}+\frac{2}{2!}a_{k-2}+\frac{2}{3!}a_{k-3}+\cdots+\frac{2}{(k-1)!}a_{k-k+1}+\frac{2}{k!}a_{k-k}$$

$$\Longrightarrow 2^ka_k=2a_k+\frac{2}{1!}a_{k-1}+\frac{2}{2!}a_{k-2}+\frac{2}{3!}a_{k-3}+\cdots+\frac{2}{(k-1)!}a_1+\frac{2}{k!}a_0$$

$$\Longrightarrow (2^k-2)a_k=\frac{2}{1!}a_{k-1}+\frac{2}{2!}a_{k-2}+\frac{2}{3!}a_{k-3}+\cdots+\frac{2}{(k-1)!}a_1 \qquad \cdots\cdots\cdots(1)$$

To prove $a_k = \frac{a_1}{(k-1)!}\ \forall k \in \mathbb{N}$

we prove by induction

For $k = 1$;

$$a_1 = \frac{a_1}{0!}$$

assume

$$a_k = \frac{a_1}{(k-1)!}$$

For $k+1$ by (1) above

$$a_{k+1} = \frac{1}{(2^k-2)}(\frac{2}{1!}a_k + \frac{2}{2!}a_{k-1} + \frac{2}{3!}a_{k-2} + \cdots + \frac{2}{(k-1)!}a_2 + \frac{2}{(k)!}a_1)$$

using induction

$$a_{k+1} = \frac{2}{(2^k-2)}(\frac{a_1}{1!(k-1)!} + \frac{a_1}{2!(k-2)!} + \frac{a_1}{3!(k-3)!} + \cdots + \frac{a_1}{(k-1)!1!} + \frac{a_1}{(k)!})$$

$$= \frac{2}{(2^k-2)}\frac{a_1}{k!}(\frac{k!}{1!(k-1)!} + \frac{k!}{2!(k-2)!} + \frac{k!}{3!(k-3)!} + \cdots + \frac{k!}{(k-1)!1!} + \frac{k!}{(k)!})$$

using binomial expansion

$$= \frac{1}{(2^k-1)}\frac{a_1}{k!}(2^k-1)$$

$$a_{k+1} = \frac{a_1}{k!}$$

Hence shown that (4) $\Longrightarrow$ (2) $\square$

Chapter 4

Analysis and Conclusions

In **Chapter 1** we defined arithmetic functions and looked at the different arithmetic function and some examples based on them. Also, We will study the Dirichlet Convolution and Unitary Convolution and theorems based on these topics. we stated and proved some basic theorems that we would require in the later part.

In **Chapter 2** we discussed some new concepts of arithmetic functions.we gave the characterization of Completely Multiplicative and Additive Arithmetic Functions. we also defined some more arithmetic functions.

In **Chapter 3** we looked at the multiplicative and additive power series. The main interest of the author was to see the relationship between formal power series and arithmetic functions. We first defined the concept of formal power series and the Bell Series, then we discussed the Bell series and specially multiplicative functions. we verified the Bell series expansion for the different arithmetic functions. we also proved some theorems and parts of the theorem that were left unsolved.

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