Convergence of Fourier series

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by

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Convergence of Fourier series" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. M. Kunhanandan and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report “Convergence of Fourier series” is a bonafide work carried out by Ms.Pranita Ravi Khanolkar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This project report was produced in part to meet the requirements for the M.Sc. in Mathematics program's Subject: MAT - 651 Discipline Specific Dissertation in the academic year 2023–2024.

The research report is on the topic "Convergence of Fourier series." The background needed for this report is Functional Analysis , Measure Theory , Several Variable Calculus and Elementary theory of Linear Differential Equations. There are a total of five chapters . Every chapter is relevant and significant in its own right. The chapters address important aspects of the subject and are organized logically, methodically, and scientifically to cover every angle.

CHAPTER 1:

The Introductory chapter of this Project report has a brief description about the Fourier series and its convergence subsequently prerequisites are mentioned, easier formulation of Fourier series is derived along with some examples and some properties of Fourier coefficients are proved.

CHAPTER 2:

In this chapter we study Fubinis theorem for two variables. Fubinis theorem will be used in later part of the report to solve convolution properties and prove results of approximation to identity in chapter 4.

CHAPTER 3:

In this chapter convolution properties of integrable periodic functions are proved along with approximation lemma which is used to prove that convolution of two integrable functions is continuous.

CHAPTER 4:

In this chapter we first see some basic definitions and proofs concerning Dirichlet kernel and Fejer kernel. Then we prove density of continuous functions in L^p spaces, define the concept of approximation to identity, show Dirichlet kernel is not an approximation to the identity whereas Fejer kernel is approximation to identity. Finally we prove sufficient condition for almost everywhere pointwise convergence of Fourier series.

CHAPTER 5:

In this chapter we will see L^2 and L^p convergence of Fourier series and define Hilbert transform, use Hilbert transform to prove convergence of Fourier series in L^p space. Furthermore we see uniform convergence, an example of divergent Fourier series and almost everywhere convergence in L^p space.

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ABSTRACT

The dissertation revolves around Fourier series which is a sum of sine and cosine waves that represents a periodic function. The main topic of this dissertation is convergence of Fourier series in different modes .The convergence of a Fourier series is essential because it determines the accuracy of approximating a periodic function. Without convergence, the approximation may fail to capture the essential features of the function, leading to inaccuracies in modeling physical phenomena or processing signals in engineering applications. The dissertation includes the concepts like necessary or/and sufficient conditions for L^p convergence, pointwise convergence, uniform convergence, almost everywhere convergence in L^p space and specially L^2 spaces under L^p spaces. It will also observe some examples of divergent Fourier series. It illustrates a variety of concepts, including the uniqueness theorem, the characteristics of convolution of periodic integrable functions , approximation to the identity, Fubinis theorem and Hilbert transform.

Keywords: Fourier coefficient; trigonometric polynomial; Dirichlet kernel; Fejer kernel

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Notations and Abbreviations

$f * g$	Convolution of f and g
D_N	Dirichlet kernel
F_N	Fejer kernel
$\sigma_N f$	Fejer mean wrt f
Hf	Hilbert transform of f
$\hat{f}(n)$	n^{th} Fourier coefficient of f
L(I)	Set of Lebesgue integrable functions

Chapter 1

INTRODUCTION

Fourier series, introduced by the French physicist Joseph Fourier (1768-1830), is one of the most powerful tools in mathematics, finding mathematical applications in solving differential equations and technological applications in signal processing, image processing, and electrical engineering, among others. A Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines. It is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Joseph Fourier, when developed Fourier series, did not have any formal definition of function, and hence he did not investigate about the convergence of these series. In 1876, DuBois Reymond showed the existence of continuous function with divergent Fourier series, this issues of convergence took precedence, leading to many of the results investigating the question of convergence of Fourier series in many senses, such as convergence in the L^p norm for $1 \leq p < \infty$, uniform convergence, and almost everywhere pointwise convergence.

1.1 PREREQUISITE

Definition 1.1.0.1. Periodic function
A function f is said to be periodic with period L, if for x $\in \mathscr{D}(f)$, where $\mathscr{D}(f)$is the domain of f and if $x+L \in \mathscr{D}(f)$ and , L$\geq$ 0 then

$$f(x+L) = f(x)$$

The smallest value of L is called the fundamental period of f. If L is the period of f and m is any integer then mL is also a period of f.
Example, $\sin\frac{m\pi x}{L}; \cos\frac{m\pi x}{L}; e^{\frac{im\pi x}{L}}$ are periodic functions with period $\frac{2L}{m}$.

Definition 1.1.0.2. Even and Odd functions
Assuming the domain of function f is symmetric wrt 0 i.e. if x $\in \mathscr{D}(f)$ then -x $\in \mathscr{D}(f)$ the function is called even if

$$f(-x) = f(x) \qquad \forall x \in \mathscr{D}(f)$$

and the function is odd if

$$f(-x) = -f(x) \qquad \forall x \in \mathscr{D}(f)$$

Definition 1.1.0.3. Orthogonal functions
Two functions u and v are said to be orthogonal on an interval [L', L] if their product is integrable and

$$\int_{L'}^{L} u(x)\overline{v(x)}dx = 0$$

where $\overline{v(x)}$ indicates the complex conjugate.

We list some standard results

Lemma 1.1.0.4. *If f is periodic function with period $L \geq 0$ and if f is integrable on every finite interval then*

$$\int_0^L f(x)dx = \int_a^{L+a} f(x)dx \qquad \forall a \in \mathbb{R}$$

Lemma 1.1.0.5. *If f is integrable on every finite interval then*

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \forall a \geq 0 \text{ if } f \text{ is even} \\ 0 & \forall a \geq 0 \text{ if } f \text{ is odd} \end{cases}$$

Lemma 1.1.0.6.

$$\int_{-L}^{L} \left(\cos\frac{m\pi x}{L}\right)\left(\cos\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

Lemma 1.1.0.7.

$$\int_{-L}^{L} \left(\cos\frac{m\pi x}{L}\right)\left(\sin\frac{n\pi x}{L}\right) dx = 0$$

Lemma 1.1.0.8.

$$\int_{-L}^{L} \left(\sin\frac{m\pi x}{L}\right)\left(\sin\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

Lemma 1.1.0.9.

$$\int_{-L}^{L} \left(\sin\frac{m\pi x}{L}\right) = \int_{-L}^{L} \left(\cos\frac{m\pi x}{L}\right) = 0$$

Lemma 1.1.0.10.

$$\int_{-L}^{L} e^{inx-imx}dx \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

Lemma 1.1.0.6, 1.1.0.7and 1.1.0.8 implies that the functions $\sin\frac{m\pi x}{L}; \cos\frac{m\pi x}{L}$, m = 1,2... are mutually orthogonal set on the interval [-L,L].

1.2 FORMATION OF FOURIER SERIES AND FOURIER COEFFICIENT

Let us consider the series of the form

$$\frac{a_0}{2} + \sum_{m=0}^{\infty} \left(a_m \left(\cos\frac{m\pi x}{L} \right) + b_m \left(\sin\frac{m\pi x}{L} \right) \right) \tag{1.1}$$

The series consists of 2L periodic function therefore if (1.1) converges for all x then the function to which it converges is also a 2L periodic function.
Let us denote the limiting function by $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{m=0}^{\infty} \left(a_m \left(\cos\frac{m\pi x}{L} \right) + b_m \left(\sin\frac{m\pi x}{L} \right) \right) \tag{1.2}$$

We have to determine a_m and b_m.
For that we assume that the integration can legimately be carried out, this is possible when $\sum_{m=1}^{\infty} |a_n| + |b_m| < \infty$.
Multiplying $\cos\frac{m\pi x}{L}$ and integrating wrt x in equation (1.2) and then using lemma (1.1.0.7) ,(1.1.0.6),(1.1.0.9). We get

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \left(\cos\frac{m\pi x}{L} \right) dx \quad m = 1, 2\ldots \tag{1.3}$$

Similarly, multiplying $\sin\frac{m\pi x}{L}$ and integrating wrt x in equation (1.2) and then using (1.1.0.7),(1.1.0.8) ,(1.1.0.9) we get

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\left(\sin\frac{m\pi x}{L}\right)dx \quad m = 1,2\ldots \tag{1.4}$$

From(1.3) we have

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{1.5}$$

Let f be integrable function (not necessarily periodic) on the interval [-L, L] the Fourier series of f is trigonometric series (1.1) in this case we write

$$f(x) \sim \frac{a_0}{2} + \sum_{m=0}^{\infty}\left(a_m\left(\cos\frac{m\pi x}{L}\right) + b_m\left(\sin\frac{m\pi x}{L}\right)\right) \tag{1.6}$$

where $a_m,\ b_m$ and a_0 can be calculated by (1.3),(1.4)and (1.5)

Now using Eulers formula we get

$$f(x) \sim \sum_{m=-\infty}^{\infty} c_m e^{imx} \tag{1.7}$$

where the value of c_m is given by the following equation.

$$c_m = \begin{cases} \frac{a_m}{2} + \frac{b_m}{2i} & ,m = 1,2,\ldots \\ \frac{a_0}{2} & ,m = 0 \\ \frac{a_{-m}}{2} - \frac{b_{-m}}{2i} & ,m = -1,-2,\ldots \end{cases} \tag{1.8}$$

Now we take the value of L= π and $x \in \mathbb{T}$ where $\mathbb{T}$ is a unit circle (compact metric space) throughout this report and the integral of the function f is defined to be

$$\int_{\mathbb{T}} f = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

This is true because the map $[0,2\pi) \to \mathbb{T}$, $x \to e^{ix}$ is a bijection using which we can transfer the Lebesgue measure on $[0,2\pi)$ onto the unit circle.

c_m can also be given in an integral form

For m positive

$$\begin{aligned}
c_m &= \frac{a_m}{2} + \frac{b_m}{2i} \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos\frac{m}{\pi} - \frac{i}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin\frac{m}{\pi}\right) \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx
\end{aligned}$$

Similarly, we get same integral form for m=0 and for negative values of m

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx \text{ for } m \in \mathbb{Z} \tag{1.9}$$

c_m is called the Fourier coefficient of f and is denoted by $\hat{f}(m)$

For example, let us find the Fourier series of signum function on interval $[-\pi,\pi]$ is

$$sgn(x) = \begin{cases} -1 & -\pi \leq x \leq 0 \\ 0 & x = 0 \\ 1 & 0 < x \leq \pi \end{cases}$$

$$\begin{aligned} a_0 = &\frac{1}{2\pi}\int_{\pi}^{\pi} sgn(x)dx \\ &\frac{1}{2\pi}(\int_{-\pi}^{0} -1dx + \int_{0}^{\pi} 1dx) = 0 \end{aligned}$$

Since the function is odd the value of $a_m = 0$

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} sgn(x)\sin(mx)dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(mx)dx) + \int_{0}^{\pi} \sin(mx)dx\right) \\ &= \frac{1}{\pi}(\left[\frac{\cos mx}{m}\right]_{-\pi}^{0} - \left[\frac{-\cos mx}{m}\right]_{0}^{\pi}) \\ &= \begin{cases} \frac{2}{m\pi} & for\ odd\ m \\ 0 & for\ even\ m \end{cases} \end{aligned}$$

hence $f(x) = \sum\limits_{m=1}^{\infty} \frac{2}{(2m-1)\pi}\sin(2m-1)x$. For $f(x) = x$ for $x \in [-\pi, \pi]$
$a_0 = 0$ and $a_m = 0$ as f(x)is odd.

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_{\pi}^{\pi} x\sin(mx)dx \\ &= \left[-x\frac{\cos mx}{m} + \frac{\sin mx}{m^2}\right]_{-\pi}^{\pi} \\ &= \frac{(-1)^{m+1}2}{m} \end{aligned}$$

hence $f(x) = \sum\limits_{m=1}^{\infty} \frac{2(-1)^{m+1}}{(m)}\sin(2m-1)x$.

1.3 PROPERTIES OF FOURIER COEFFICIENT

Let f and g be integrable functions

1. $(f \hat{+} g)(n) = \hat{f}(n) + \hat{g}(n)$
2. $(\hat{cf})(n) = c(\hat{f}(n))$
3. $\hat{\tau_\delta f}(n) = e^{-in\delta}\hat{f}(n), where\ \tau_\delta f(x) = f(x-\delta)$
4. $\hat{g}(n) = \hat{f}(n-m)$; if $g(t) = f(t)e^{imt}$
5. $(\hat{f'})(n) = in\hat{f}(n)$ if f is continuously differentiable.
6. Suppose if f is integrable function $(i.e. \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)| < \infty)$ where the norm is defined by
$$\|f\| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|dx$$
then $\sup\limits_{n\in\mathbb{Z}}|\hat{f}(n)| \leq \|f\|$.
7. If f_m converges uniformly to f then $\hat{f_m}(n) \to \hat{f}(n)$ as m $\to \infty$.

Proof. Properties 1 and 2 is proved by linearity of functions.

For 3, we use change of variables

$$\begin{aligned}
\hat{\tau_\delta f}(n) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau_\delta f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x-\delta)e^{-inx}dx \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(u)e^{-in(u+\delta)}dx \\
&= e^{-in\delta}\hat{f}(x)
\end{aligned}$$

For 4, we directly substitute

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i(n-m)x}dx \\
&= \hat{f}(n-m)
\end{aligned}$$

For 5, we use integration by parts

$$\begin{aligned}
(\hat{f'})(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx \\
&= f(x)e^{-inx} - \int_{-\pi}^{\pi} -inf(x)e^{-inx}dx \\
&= in\hat{f}(n)
\end{aligned}$$

For 6, we get the result as $|e^{ix}| = 1$

$$\begin{aligned}
|\hat{f}(n)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|dx \\
&\leq \|f\|_1
\end{aligned}$$

For 7 we have $|f_m(x) - f(x)| < \varepsilon$ for some $\varepsilon > 0$

$$
\begin{aligned}
|\hat{f}_m(n) - \hat{f}(x)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi} f_m(x)e^{-inx} - f(x)e^{-inx}dx| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_m(x) - f(x)|dx \\
&< \frac{1}{2\pi}\varepsilon\int_{-\pi}^{\pi} 1dx = \varepsilon
\end{aligned}
$$

Hence the result $\hat{f}_m(n) \rightarrow \hat{f}(n)$ as m $\rightarrow \infty$ $\square$

Chapter 2

FUBINIS THEOREM

Definition 2.0.0.1. A general n-dimentional interval I in $\mathbb{R}^n$ is of the form $I = I_1 \times I_2 \times \dots I_n$ where $I_k \ \forall k = 1, 2, \dots$ is an interval in $\mathbb{R}^1$

Definition 2.0.0.2. The bounded interval I in $\mathbb{R}^n$ is given by $I = I_1 \times I_2 \times \cdots \times I_n$ where $I_k \ \forall k \in \{1, 2, \dots, n\}$ is a bounded interval in $\mathbb{R}^1$

Definition 2.0.0.3. I is a compact interval in $\mathbb{R}^n$ if I is of the form $I = I_1 \times I_2 \times \cdots \times I_n$ where $I_k \ \forall k \in \{1, 2, \dots, n\}$ is a compact interval in $\mathbb{R}^1$.

Definition 2.0.0.4. Measure of an interval I in $\mathbb{R}^1$ is the absolute difference between the end points of the interval.

Definition 2.0.0.5. The n-measure I is denoted by $\mu(I)$ which is given by $\mu(I) = \mu(I_1) \times \mu(I_2) \times \cdots \times \mu(I_k)$ where $\mu(I_k)$ is the one dimensional measure of $I_k \ \forall k \in \{1, 2, \dots, n\}$

Definition 2.0.0.6. A subset T of $\mathbb{R}^n$ is said to be n-measure 0 if for all $\varepsilon > 0$, if T can be covered by a countable collection of n- dimensional intervals the sum of whose n-measure is less then ε.

Definition 2.0.0.7. A property P is said to hold almost everywhere if $M = \{x : \text{P does not hold for } x\}$ is of measure 0.

Definition 2.0.0.8. The partition of an interval I in $\mathbb{R}$ having end points a and b, is the set

$$P = \{x_i : a = x_0 < x_1 < x_2 < \dots x_n = b\}$$

If P_k is the partition of I_k into m_k parts then the Cartesian product $P = P_1 \times P_2 \times \cdots \times P_n$ partitions I into $m = m_1.m_2 \dots m_n$ parts.

Definition 2.0.0.9. Let $J_1, J_2 \dots J_m$ be n-dimensional partitions of I.
A step function f defined on I is a function where the value on the interior of J_k $(i.e.\ (J_k)^\circ) \forall k \in \{1, 2, \dots n\}$ is same. i.e.

$$f(x) = c_k \quad \forall x \in \ (J_k)^\circ$$

Definition 2.0.0.10. The integral of f on I is defined by the equation

$$\int_I f = \sum_{j=1}^{n} c_j \ \mu(I_j)$$

Definition 2.0.0.11. Let G be a general n dimensional interval in $\mathbb{R}^n$ which need not be compact. A function f is called the step function of G if there is a compact n dimensional sub-interval I of G such that f is step function on I and $f(x) = 0$ for $x \in (G - I)$.
The integral f over G is defined by

$$\int_I f = \int_G f$$

Definition 2.0.0.12. A real valued function g defined on I in $\mathbb{R}^n$ is called upper function if there is an increasing sequence of step functions f_n such that

a) $f_n \to g$ almost everywhere on I

b) $\lim_{n\to\infty} \int_I f_n$ exists.

The sequence $\{f_n\}$ is said to generate g

The integral of g over I is defined by equation

$$\int_I g = \lim_{n\to\infty} \int_I f_n$$

The function g is upper function then it is denoted as $g \in U(I)$

Definition 2.0.0.13. We say that g is a Lebesgue integrable function if it is of the form

$$g = u - v$$

where u and v are upper functions.

The set of all Lebesgue integrable function is denoted by L(I).

The integral of Lebesgue integrable function g is given by

$$\int_I g = \int_I u - \int_I v$$

We first prove Fubinis theorem for step functions

Theorem 2.0.0.14. *Let f be a step function on $\mathbb{R}^2$ then for each y in $\mathbb{R}^1$ the integral $\int_{\mathbb{R}^1} f(x,y)dx$ exists and, as the function of y is Lebesgue integrable on $\mathbb{R}^1$, and moreover we have*

$$\int\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x,y)dx\right] dy \tag{2.1}$$

Similarly for each x in $\mathbb{R}^1$ the integral $\int_{\mathbb{R}^1} f(x,y)dy$ exists and, as the function of x is Lebesgue integrable on $\mathbb{R}^1$, and moreover we have

$$\int\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x,y)dy\right] dx \tag{2.2}$$

Proof. There is a compact interval I= $[a,b]\times[c,d]$ such that f is a step function on I and

$$f(x,y)=0 \qquad if\ (x,y)\in(\mathbb{R}^2-I) \tag{2.3}$$

If I is partitioned into mn sub-rectangles $I_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$ such that f is constant in the interior of I_{ij} say
$f(x,y)=c_{ij}$ if $(x,y)\in(I_{ij})^\circ$
then

$$\int\int_{I_{ij}} f(x,y)d(x,y)=c_{ij}(x_i-x_{i-1})(y_j-y_{j-1})=\int_{y_{j-1}}^{y_j}\left[\int_{x_{j-1}}^{x_j} f(x,y)dy\right]dx$$

summing on i and j we get

$$\int\int_I f(x,y)d(x,y)=\int_c^d\int_a^b f(x,y)dxdy$$

since f vanishes outside I , (2.1) is proved. Similarly we prove (2.2). □

We see some of the theorems for set measure zero.

Theorem 2.0.0.15. *Let S be a subset of $\mathbb{R}^n$, S has n-measure 0 if and only if there exists a countable collection of n-dimensional intervals $J_1,J_2,\ldots$, the sum of whose n-measures is finite, such that each point in S belongs to J_k for infinitely many k.*

Proof. Assume that S is set of measure zero. Then S can be covered by a countable collection of n-dimensional intervals $\{I_{m1},I_{m2},\ldots\}$ such that $\sum_{k=1}^{\infty}\mu(I_{mk})<\frac{1}{2^m}$.
Let set A consists of all interval I_{mk} for $m=1,2,...\ and\ k=1,2,..$ then A is a countable collection which covers S.Let us rename it as $J_1,J_2,\ldots$
we see that

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\mu(I_{mk})\leq\sum_{m=1}^{\infty}\frac{1}{2^m}=1$$

therefore if $a \in S$ then $a \in I_{mk}$ for each m and some k we observe that a is in J_k for infinitely many k.

Conversely , Assume that there is countable collection of n dimensional interval $\{J_1, J_2, \dots\}$ such that $\sum_{k=1}^{\infty} \mu(J_k)$ converges at each point in S belongs to J_k for infinitely many k.

given $\varepsilon > 0$ $\exists$N $\in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \mu(I_{mk}) < \varepsilon$

Each point in S lies in the set $\bigcup_{k=N}^{\infty} J_k$ and thus S is covered by countable a collection of intervals sum of whose measure is $< \varepsilon$. So S has measure 0. □

Definition 2.0.0.16. A real valued function is called measurable on an interval I in $\mathbb{R}^n$ if there is a sequence of step functions $\{f_n\}$ on I such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ almost everywhere on I} \quad .$$

We denote this f by $f \in M(I)$.

Definition 2.0.0.17. A subset S of $\mathbb{R}^n$ is called measurable if the characteristic function χ_S is measurable.

Levi's theorems for Lebesgue integrable functions:

Levi's theorem for sequence of Lebesgue integrable function:

Let $\{f_n\}$ be a sequence of functions in L(I) in $\mathbb{R}$ such that

a) $\{f_n\}$ converges a.e. on I

b) $\lim\limits_{n\to\infty} \int_I f_n$ exists.

Then $\{f_n\}$ converges a.e. on I to a limit function f such that f $\in L(I)$ and

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

Levi's theorem for series of Lebesgue integrable function:

Let $\{f_n\}$ be a sequence of functions in L(I) in $\mathbb{R}$ such that

a) f_n is non negative on almost everywhere on I

b) the series $\sum_{n=1}^{\infty} \int_I f_n$converges then the series $\sum_{n=1}^{\infty} f_n$ converges a.e on I to g in $L(I)$, and we have

$$\int_I g = \int_I \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_I f_n$$

Definition 2.0.0.18. If S is an arbitrary subset of $\mathbb{R}^2$ and if $(x,y) \in \mathbb{R}^2$ we denote
$S_y = \{x : x \in \mathbb{R}^1 and (x,y) \in S\}$
$S^x = \{y : y \in \mathbb{R}^1 and (x,y) \in S\}$

Theorem 2.0.0.19. *If S is a subset of $\mathbb{R}^2$ with 2-measure 0, then S_y has 1-measure 0 for almost all y in $\mathbb{R}^1$, and S^x has 1-measure 0 for almost all x in $\mathbb{R}^1$.*

Proof. We will prove that S_y has 1-measure 0 for almost all y in $\mathbb{R}^1$
Since S has 2-measure 0, by Theorem 2.0.0.15 there is a countable collection of rectangles I_k such that the series $\sum_{k=1}^{\infty} \mu(I_k)$ converges (i)
and such that every point (x, y) of S belongs to I_k for infinitely many k.
Write $I_k = X_k \times Y_k$, where X_k and Y_k are sub-intervals of $\mathbb{R}^1$'. Then,

$$\mu(I_k) = \mu(X_k)\mu(Y_k) = \mu(X_k) \int_{\mathbb{R}^1} \chi_{Y_k} = \int_{\mathbb{R}^1} \mu(X_k)\chi_{Y_k}$$

Where χ_{Y_k} is the characteristic function of the interval Y_k.
Let $g_k = \mu(X_k)\chi_{Y_k}$
Then (i) implies that the series $\sum_{k=1}^{\infty} \int_{\mathbb{R}^1} g_k$ converges.

Now g_k is a sequence of non-negative functions in $L(\mathbb{R}^1)$ such that the series $\sum_{k=1}^{\infty} \int_{\mathbb{R}^1} g_k$ converges. Therefore, by the Levi theorem for series of lebesgue integrable function , the series $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on $\mathbb{R}^1$.
In other words, there is a subset T of $\mathbb{R}^1$ of 1-measure 0 such that the series
$\sum_{k=1}^{\infty} \mu(I_k)\chi_{Y_k}$ converges for all y in $\mathbb{R}^1 - T$. (ii)
Take a point y in $\mathbb{R}^1 - T$, keep y fixed and consider the set S_y.

We will prove that S_y has 1-measure zero.We can assume that S_y is nonempty; otherwise the result is trivial.

Let $A(y) = \{X_k : y \in Y_k, k = 1, 2, ...\}$.

Then A(y) is a countable collection of one-dimensional intervals which we relabel as $J_1, J_2, ...$ The sum of the lengths of all the intervals J_k converges because of (ii). If $x \in S_y$, then (x, y) ∈ S so (x, y) $\in I_k = X_k \times Y_k$ for infinitely many k, and hence $x \in J_k$. for infinitely many k. By the one-dimensional version of Theorem 2.0.0.15 it follows that S_y has 1-measure zero for almost all y in $\mathbb{R}^1$.

A similar argument proves that S^x has 1-measure zero for almost all x in $\mathbb{R}^1$. □

Following is a proof for Fubinis theorem for double integral

Theorem 2.0.0.20. *Assume f is Lebesgue-integrable on $\mathbb{R}^2$. Then we have:*

a) There is a set T of 1-measure 0 such that the Lebesgue integral $\int_{\mathbb{R}^1} f(x,y)dx$ exists for all y in $\mathbb{R}^1$ - T.

b) The function G defined on $\mathbb{R}^1$ by the equation

$$G(y) = \begin{cases} \int_{\mathbb{R}^1} f(x,y)dx; & if\ y \in \mathbb{R}^1 - T. \\ 0; & if\ y \in T \end{cases}$$

is Lebesgue-integrable on $\mathbb{R}^1$.

c)

$$\int\int_{\mathbb{R}^2} f = \int_{\mathbb{R}^1} G(y)dy$$

$$i.e. \int\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x,y)dx \right] dy$$

correspondingly

$$\int\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x,y)dy \right] dx$$

Proof. We prove the theorem for upper functions. If f $\in$U($\mathbb{R}^2$) there is an increasing sequence of step functions such thatf_n(x, y) $\rightarrow$ f(x, y) for all (x, y) in ($\mathbb{R}^2$ - S), where S is a set of 2-measure 0;

Also,

$$\lim_{n\to\infty}\int\int_{\mathbb{R}^2} f_n(x,y)d(x,y) = \int\int_{\mathbb{R}^2} f(x,y)dydx$$

Now (x, y)$\in$ ($\mathbb{R}^2$ - S) if, and only if, x$\in$ $\mathbb{R}^1$ - S_y.

Hence

$f_n(x,y) \rightarrow f(x,y)$ if x $\in \mathbb{R}^1 - S_y$. (i)

Let $t_n(y) = \int_{R^1} f_n(x,y)dx$. This integral exists for each real y and is an integrable function of y.

Moreover, by Theorem 2.0.0.14 we have

$$\begin{aligned}\int_{R^1} t_n(y)dy &= \int_{\mathbb{R}^1}[\int_{\mathbb{R}^1} f_n(x,y)dx]dy \\ &= \int_{\mathbb{R}^1}[\int_{\mathbb{R}^1} f_n(x,y)dy]\,dx \\ &= \int_{\mathbb{R}^2} f_n(x,y)d(x,y) \\ &\leq \int_{\mathbb{R}^2} f\end{aligned}$$

Sequence t_n is increasing and $\lim\limits_{n\to\infty}\int_{R^1} t_n(y)dy$ exists then we can use the Levis theorem for sequence of Lebesgue integrable function.

We get function t $\in L(\mathbb{R}^1)$ such that $\lim\limits_{n\to\infty} t_n \rightarrow t$ for all y on $\mathbb{R}^1 - T_1$ where T_1 is a subset of real numbers and has 1-measure 0.

Moreover,

$$\int_{R^1} t(y)dy = \lim_{n\to\infty}\int_{R^1} t_n(y)dy$$

Again t_n is an increasing function and

$$t_n(y) = \int_{\mathbb{R}^1} f_n dx \leq t(y)$$

if y $\in \mathbb{R}^1 - T_1$

Applying Levis theorem again to $\{f_n\}$. There exists a Lebesgue integrable function g such that ,

$$f_n(x,y) \to g(x,y) \ \forall y \in \ \mathbb{R}^1 - A$$

Where A has 1-measure 0. (ii)

From (i) and (ii) we have

$g(x,y) = f(x,y) \ \ \forall x \in \mathbb{R}^1 - (A \cup S_y)$ (iii)

S_y has 1-measure 0 for a.e. y . I.e there is a T_2 of 1-measure 0 such that S_y has 1-measure 0 for all y $\in \mathbb{R}^1 - T_2$

Let $T = T_1 + T_2$ then T has 1-measure 0 . Then if u $\in \mathbb{R}^1 - T$ then $A \cup S_y$ has measure 0 hence (iii) holds. Since $\int_{\mathbb{R}^1} g(x,y)$ exists if y $\in \mathbb{R}^1 - T$ it follows that $\int_{\mathbb{R}^1} f(x,y)$ exists if y $\in \mathbb{R}^1 - T$

hence (a)

If $y \in \mathbb{R}^1 - T$. We have

$$\int_{\mathbb{R}^1} f(x,y)dx = \int_{\mathbb{R}^1} g(x,y)dx = \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x,y)dx = t(y)$$

Since t is a Lebesgue integrable function we get (b).

To prove (c)

$$\begin{aligned}
\int_{\mathbb{R}^1} t(y)dy &= \int_{\mathbb{R}^1} \lim_{n\to\infty} t_n(y)dy \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} t_n(y)dy \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \left\{ \int_{\mathbb{R}^1} f_n(x,y)dx \right\} dy \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} f_n(x,y)d(x,y) \\
&= \int \int_{\mathbb{R}^2} f(x,y)d(x,y)
\end{aligned}$$

By (b) we get (c). $\square$

If f $\in L(\mathbb{R}^1)$ then f is of the form u-v where u and v are upper functions then

$$
\begin{aligned}
\int\int_{\mathbb{R}^2} f &= \int\int_{\mathbb{R}^2} u - \int\int_{\mathbb{R}^2} v \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^1}\left\{\int_{\mathbb{R}^1} u(x,y)dx\right\}dy - \lim_{n\to\infty}\int_{\mathbb{R}^1}\left\{\int_{\mathbb{R}^1} v(x,y)dx\right\}dy \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^1}\left\{\int_{\mathbb{R}^1} u(x,y) - v(x,y)dx\right\}dy \\
&= \int_{\mathbb{R}^1}\left\{\int_{\mathbb{R}^1} f(x,y)dx\right\}dy
\end{aligned}
$$

Similarly we can prove

$$\int\int_{\mathbb{R}^2} f = \int_{\mathbb{R}^1}\left\{\int_{\mathbb{R}^1} f(x,y)dy\right\}dx.$$

Chapter 3

CONVOLUTION

Let g and f be periodic integrable functions the convolution $f*g$ on $[-\pi,\pi]$ is given by

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$$

Product of two intergable fuctions is integrable hence the convolution makes sense. The following are some of the properties of convolution of periodic integrable functions

1. $f*(g+h) = (f*g)+(f*h)$
2. $(cf)*g = c(f*g) = f*(cg)$
3. $(f*g) = (g*f)$
4. $(f*g)*h = f*(g*h)$
5. $(f*g)$ is continuous
6. $(f\hat{*}g)(n) = \hat{f}(n)\hat{g}(n)$

Proof of 1 and 2 is by linearity of integrable functions.

To prove 3

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$$

let (x-y) = u then we use the lemma (1.1.04) and we get

$$(f*g) = (f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)g(x-u)du = (g*f)(u) = (g*f)$$

To prove 4

$$\begin{aligned}
(f*g)*h &= ((f*g)*h)(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(y)(h(x-y))dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)g(y-z)dz)\right)(h(x-y))dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(y-z)h(x-z-(y-z))dy)\right)dz
\end{aligned}$$

In the above step we used Fubinis theorem, now let y -z = u

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(u)h(x-z-(u))du)\right)dz \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)(g*h)(x-z)dz \\
&= f*(g*h)(x)
\end{aligned}$$

To prove 6

$$\begin{aligned}
(f \hat{*} g)(n) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(f*g)(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}f(y)g(x-y)dy)\right)e^{-inx}dx \\
&= \left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{1}{2\pi}f(y)e^{-iny}dy\right)\left(\int\limits_{-\pi}^{\pi}g(x-y)e^{-in(x-y)}dx)\right) \\
&= \hat{f}(n)\hat{g}(n)
\end{aligned}$$

To prove 5 we need to prove a lemma because of which the property of convolution holds for all integrable periodic function.

Lemma 3.0.0.1. *(Approximation Lemma):*
Suppose f is integrable on a circle and bounded by B the there is a sequence of continuous functions $\{f_k\}_{k=1}^{\infty}$ *on the circle such that*

$$\sup_{x\in[-\pi,\pi]}|f_k(x)| \leq B \ \ \forall k = 1,2,\ldots and$$

$$\int_{-\pi}^{\pi}|f(x)-f_k(x)|dx \to 0 \ as \ k \to \infty$$

Proof. When f is real, given $\varepsilon > 0, \exists$ partition P of the interval $[-\pi,\pi]$ such that $U(P,f)-L(P,f) \leq \varepsilon$ as f is integrable.
We define step function g on $[-\pi,\pi]$ as

$$g(x) = \sup_{x_{j-1}\leq y<x_j} f(y) \quad if \ x \in [x_{j-1},x_j) \ for \ j = 1,2,\ldots,N$$

Function g is a bounded function. and we have

$$\int_{-\pi}^{\pi} |g(x) - f(x)|dx = \int_{-\pi}^{\pi} g(x) - f(x)dx < \varepsilon$$

To modify the step function g to make it continuous we will take small $\delta > 0$, and define $g^*(x) = g(x)$ when distance between x and partition points in P is more than δ, $g^*(x) = 0$ for $x \in [-\pi - \delta, -\pi + \delta] \cup [\pi - \delta, \pi + \delta]$ and $g^*(x)$ is a linear function from $g(x - \delta)\ to\ g(x + \delta)$ of corresponding partitions we see that g^* is a continuous function Since g^* differs by g in N intervals and g^* is bounded and length of the N intervals can be written as scalar times δ we have,

$$\int_{-\pi}^{\pi} |g(x) - g^*(x)|dx \leq 4BN\delta$$

when δ become very small

$$\int_{-\pi}^{\pi} |g(x) - g^*(x)|dx < \varepsilon$$

By triangle inequality we have ,

$$\int_{-\pi}^{\pi} |f(x) - g^*(x)|dx \leq \int_{-\pi}^{\pi} |g(x) - f(x)|dx + \int_{-\pi}^{\pi} |f(x) - g^*(x)|dx < 2\varepsilon$$

Let $2\varepsilon = \frac{1}{k}$ and denote $g^*\ as\ f_k$ then

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)|dx \to 0\ as\ k \to \infty$$

If f is a complex function we apply the above proof for real and imaginary part separately to get the result. $\square$

Now for property 5

$$
\begin{aligned}
(f*g)-(f_k*g_k) &= (f*g)-(f_k*g)+(f_k*g)-(f_k*g_k) \\
&= (f-f_k)*g+f_k(g*g_k)
\end{aligned}
$$

$$
((f-f_k)*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f-f_k)(x-y)g(y)dy
$$

$$
\begin{aligned}
|((f-f_k)*g)(x)| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|(f-f_k)(x-y)||g(y)|dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|(f(x,y)-f_k(x-y))||g(y)|dy \\
&\leq \frac{1}{2\pi}\sup_y|g(y)|\int_{-\pi}^{\pi}|(f-f_k)(x-y)|dy \\
&\to 0 \; as \; k \to \infty
\end{aligned}
$$

The convergence is uniform.

Similarly $f_k*(g-g_k)$ converges uniformly to 0 as k tends to infinity.Hence f_k*g_k converges uniformly to $f*g$ as k tends to infinity.

By continuity of each f_k*g_k we have $f*g$ continuous.

Chapter 4

TOWARDS CONVERGENCE

4.1 BASIC DEFINITIONS AND RESULTS

Definition 4.1.0.1. Let f be an integrable function then the N^{th} **partial sum of Fourier series is f** is given by

$$S_N f(t) = \sum_{m=-N}^{N} \hat{f}(x) e^{imx}$$

Definition 4.1.0.2. Trigonometric polynomial is a function of the form

$$P(t) = \sum_{m \in \mathbb{Z}} a_n e^{imt}$$

such that $\exists$ N $\in \mathbb{N}$ st $\forall m > N$ a_m and a_{-m} vanishes and this N is called the degree of the polynomial

proposition 4.1.0.3. *Fourier series of any trigonometric polynomial converges back to itself*

Proof. we can do this by proving $\hat{P(n)} = a_n$ where P is a trigonometric polynomial of degree p.

$$\begin{aligned}
\hat{P(n)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m\in\mathbb{Z}} a_m e^{imx}e^{-inx}dx \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} a_m \int_{-\pi}^{\pi} e^{ix(m-n)} = a_n
\end{aligned}$$

Hence $||S_N P - P|| = ||\sum_{m=-N}^{N}\hat{P}(x)e^{imx} - \sum_{n=-p}^{p} a_n e^{inx}||$
as $N \to \infty$ $||S_N P - P|| \to 0$ □

Definition 4.1.0.4. The N^{th}Dirichlet kernel is defined to be

$$D_N(t) = \sum_{m=-N}^{N} e^{imx}$$

we can deduce the property that

$$S_N f = D_N * f$$

$$\begin{aligned}
S_N f(t) &= \sum_{n=-N}^{N} \hat{f}(n)e^{int} \\
&= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-int}dx\right)e^{int} \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)(\sum_{n=-N}^{N} e^{in(t-x)})dx\right) \\
&= (f * D_N)(t)
\end{aligned}$$

Definition 4.1.0.5. The N^{th} Fejer kernel is defined to be

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$$

If f is an integrable function the Nth **Fejer mean of f** is defined to be

$$\sigma_N f(x) = (F_N * f)(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(x)$$

$$\begin{aligned}
(F_N * f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(y) f(x-y)dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N}\sum_{n=0}^{N-1} D_n(y) f(x-y)dy \\
&= \frac{1}{2\pi}\frac{1}{N}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_N(y) f(x-y)dy \\
&= \frac{1}{N}\sum_{n=0}^{N-1}(D_N * f)(x) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} S_n(x)
\end{aligned}$$

For N $\in \mathbb{Z}$ we have values of Dirichlet kernel and Fejer kernel as

$$D_N f(x) = \frac{\sin\left(\frac{2N+1}{2}x\right)}{\sin(\frac{x}{2})}$$

$$F_N f(x) = \frac{1}{N}\left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin(\frac{x}{2})}\right)^2$$

$$\begin{aligned}
D_N(x) &= \sum_{n=-N}^{N} e^{inx} \\
&= e^{-iNx} \sum_{n=0}^{2N} e^{inx} \\
&= \frac{e^{-iNx}}{e^{\frac{1}{2}}} \frac{e^{i(2N+1)x} - 1}{(e^{\frac{ix}{2}} - e^{-\frac{-ix}{2}})} \\
&= e^{-i(\frac{2N+1}{2})x} \frac{e^{i(2N+1)x} - 1}{(e^{\frac{ix}{2}} - e^{-\frac{-ix}{2}})} \\
&= \frac{2i}{2i} \frac{e^{i(\frac{2N+1}{2})x} - e^{-i(\frac{2N+1}{2})x}}{(e^{\frac{ix}{2}} - e^{-\frac{-ix}{2}})} \\
&= \frac{\sin((\frac{2N+1}{2})x)}{\sin(\frac{x}{2})}
\end{aligned}$$

We use the result $\frac{e^{ix}-e^{-ix}}{2i} = \sin x$

$$\begin{aligned}
F_N(x) &= \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \frac{\sin((\frac{2n+1}{2})x)}{\sin(\frac{x}{2})} \\
&= \frac{1}{N} \frac{1}{\sin(\frac{x}{2})} \sum_{n=0}^{N-1} \Im(e^{\frac{2n+1}{2}ix}) \\
&= \frac{1}{N\sin(\frac{x}{2})} \Im(\sum_{n=0}^{N-1} e^{\frac{2n+1}{2}ix}) \\
&= \frac{1}{N\sin(\frac{x}{2})} \Im(e^{\frac{ix}{2}}(\frac{e^{iNx}-1}{e^{ix}-1})) \\
&= \frac{1}{N\sin(\frac{x}{2})} \Im(\frac{e^{iNx}-1}{e^{\frac{ix}{2}} - e^{\frac{-ix}{2}}}) \\
&= \frac{1}{N\sin(\frac{x}{2})} \Im(\frac{e^{iNx}-1}{2i\sin(\frac{x}{2})})
\end{aligned}$$

$$
\begin{aligned}
F_N(x) &= \frac{1}{N\sin^2(\frac{x}{2})}\Re(\frac{e^{iNx}-1}{-2}) \\
&= \frac{1}{N}\frac{1-\cos Nx}{2\sin^2(\frac{x}{2})} \\
&= \frac{1}{N}\left(2\frac{\sin(\frac{Nx}{2})}{2\sin(\frac{x}{2})}\right)^2 \\
&= \frac{1}{N}\left(\frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})}\right)^2
\end{aligned}
$$

Definition 4.1.0.6. For $1 \leq p < \infty$,$L^p(\mathbb{T})$ is the space all measurable function $f : (\mathbb{T}) \to \mathbb{C}$ such that

$$\|f\|_p = (\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^p)^{\frac{1}{p}} < \infty$$

Definition 4.1.0.7. $L^\infty(\mathbb{T})$ is the set of measurable function $f : (\mathbb{T}) \to \mathbb{C}$ st

$$\|f\|_\infty = \inf\{C \geq 0 : m(\{x : f(x) > C\}) = 0\} < \infty$$

Since the map $[0,2\pi) \to \mathbb{T}, x \to e^{ix}$ is.a bijection we can naturally identify $L^p[0,2\pi)$ with $L^p(\mathbb{T})$.
$(L^p(\mathbb{T}), \|.\|_p)$ and $(L^\infty(\mathbb{T}), \|.\|_\infty)$ are Banach spaces is well known.

4.2 DENSITY OF CONTINUOUS FUNCTIONS IN L^p

We state some basic definitions and results that require

Definition 4.2.0.1. Let X be a normed linear space with norm $\|.\|$. Given two subsets $\mathscr{F}$ and $\mathscr{G}$ of X with $\mathscr{F} \subseteq \mathscr{G}$. We say that $\mathscr{F}$ is dense in $\mathscr{G}$ provided for each function g in $\mathscr{G}$ s.t. $\forall\, \varepsilon > 0\ \exists$ a function f in $\mathscr{F}$ s.t. $||f-g|| < \varepsilon$.

Also if $\mathscr{F}$ is dense in $\mathscr{G}$ and $\mathscr{G}$ is dense in $\mathscr{H}$ then $\mathscr{F}$ is dense in $\mathscr{H}$

Definition 4.2.0.2. Let $E_1, E_2, \ldots, E_n$ be subsets of $\mathbb{R}$ then the function of the form $\phi(x) = a_1\chi_{E_1}(x) + a_2\chi_{E_2}(x) + \cdots + a_n\chi_{E_n}(x)$ where $a_1, a_2, \ldots, a_n$ are real numbers is called a simple function.

Theorem 4.2.0.3 (Simple Approximation Theorem). *Let E be a measurable set and $f : E \to [0,\infty]$ be a non-negative, extended real valued measurable function. Then $\exists$ a sequence $(\phi_n)_{n=0}^{\infty}$ of measurable simple functions on E s.t.*

(a) $0 \leq \phi_1 \leq \phi_2 \leq, \ldots, \leq f$

(b) $\phi_n(n) \to f$ *pointwise everywhere on E (Uniformly convergent if f is bounded)*

Corollary 4.2.0.4. *Let E be measurable set and $f : E \to [-\infty,\infty]$ be extended real valued measurable function then $\exists$ a sequence $(\phi_n)_{n=1}^{\infty}$ of simple function s.t.*

(i) $|\phi_n| \leq |f| \ \forall\, n$

(ii) $\phi_n \to f$ *converges pointwise everywhere on E*

Proposition 4.2.0.5. Let E be a measurable set and $1 < p \leq \infty$ then the subspace of simple function in $L^p(E)$ is dense in $L^p(E)$.

Proof. Let $g \in L^p(E)$. First consider $p = \infty$. There is a subset E_0 of E s.t. g is bounded on $E \backslash E_0$ where $m(E_0) = 0$

We infer from Simple Approximation theorem that there is a sequence of simple function on $E \backslash E_0$ that converges uniformly on $E \backslash E_0$ to g and therefore converges w.r.t. L^∞ norm. Hence simple functions are dense in $L^\infty(E)$

Now suppose $1 < p < \infty$

Since g is measurable by Simple Approximation Theorem $\exists$ a sequence of simple functions on E s.t. $\{\phi_n\} \to g$ pointwise on E and $|\phi_n| \leq |g|$ on E for all $n \in \mathbb{N}$. Each ϕ_n belongs to L^p

We claim $\phi_n \to g$ in L^p

$$|\phi_n - g|^p \leq 2^p \{|\phi_n|^p + |g|^p\} = 2^{p+1}|g|^p \; on \; E$$

Now, $|g|^p$ is integral of E . From Dominated Convergence Theorem $\{\phi_n\} \to g$ in L^p

Since $|\phi_n| < |g|$ and $|\phi_n| \to |g|$

We have $|\phi_n - g|^p \leq |g|^p$ and ϕ_n and g are Lebesgue integrable.

$\Rightarrow |\phi_n - g|^p$ is sequence Lebesgue integrable

$$|\phi_n - g|^p \to 0 \; as \; |\phi_n - g| \to 0 \; as \; n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} \int |\phi_n - g|^p = \int \lim_{n\to\infty} |\phi_n - g|^p = 0$$

$\Rightarrow \{\phi_n\} \to g$ in L^p □

Proposition 4.2.0.6. Let $[a,b]$ be a closed, bounded interval and $1 \leq p < \infty$ then the subspace of step function on $[a,b]$ is dense in L^p.

Proof. The previous proposition tells us that the simple functions are dense in $L^p([a,b])$. Therefore it suffices to show that the step function are dense in simple function w.r.t. the $||\cdot||_p$ norm.

Each simple function is a linear combination of characteristic functions of measurable sets. Therefore if each characteristic function can be arbitrarily closely approximated step function on $||\cdot||_p$ norm.

Since linear combination of step function is step function, so any simple function can be approximated arbitrarily w.r.t. L^p norm by step function.

Let $g = \chi_A$ where A is measurable subset of $[a,b]$ and let $\varepsilon > 0$ and seek a step function f on $[a,b]$ for which $||f-g||_p < \varepsilon$

Now since A is measurable $\exists$ finitely many disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which we define $\mathscr{U} = \bigcup_{k=1}^{n} I_k$ then symmetric difference $A\Delta\mathscr{U} = (A\backslash\mathscr{U}) \cup (\mathscr{U}\backslash A)$ has the property that $m^*(A \cup \mathscr{U}) < \varepsilon^p$

Since $\mathscr{U}$ is a finite disjoint union of open interval. Then $\chi_{\mathscr{U}}$ is a step function

Moreover, $||\chi_A - \chi_{\mathscr{U}}||_p = \left\{\int_{[a,b]} |\chi_A - \chi_{\mathscr{U}}|^p\right\}^{\frac{1}{p}}$

$\Rightarrow \left\{\int_{[a,b]} |\chi_A - \chi_{\mathscr{U}}|^p\right\}^{\frac{1}{p}} \leq \{\int_{A\Delta\mathscr{U}} 1\}^{\frac{1}{p}} = (m(A\Delta\mathscr{U}))^{\frac{1}{p}} = (\varepsilon^p)^{\frac{1}{p}} = \varepsilon$

So step function $\chi_{\mathscr{U}}$ approximates characteristic function χ_A to within $\varepsilon > 0$ w.r.t. L^p norm. □

Proposition 4.2.0.7. Continuous functions are dense in L^p.

Proof. Approximating continuous function to step function.

Let f be continuous function on compact interval $[a,b]$ then f is uniformly continuous in [a,b].

Let $\varepsilon > 0$ be given then $\exists\, \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ where $|x-y| < \delta$

Let P be a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ such that $|x_i - x_{i-1}| < \delta$ for $i = 1, 2, \ldots, n$

$\forall\, x \in [x_i, x_{i-1})$ let $g(x) = \sup_{x\in[x_i,x_{i-1})} f(x)$ then g is a step function

then $|g(x) - f(x)| < \varepsilon \;\forall\, x \in [a,b]$

$\Rightarrow \sup_{x\in[a,b]} |g(x) - f(x)| \leq \varepsilon$

$\Rightarrow\; ||g-f||_p^p = \frac{1}{(b-a)} \int_a^b |g-f|^p \leq \varepsilon^p(b-a)$

$\Rightarrow ||g-f||_p < \varepsilon$ Hence continuous functions are dense in L^p

as step functions are dense in L^p hence continuous functions are dense in L^P □

4.3 APPROXIMATION TO THE IDENTITY

A family of functions $\{K_n\}_{n\in\mathbb{N}}$ is an approximation to the identity if the following three properties hold

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1\ \forall n$

(b) $\sup_{n\in\mathbb{N}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(x)|dx < \infty$

(c) $\int_{\delta\le x\le\pi} |K_n(x)|dx \to 0$ as $n\to\infty\ \forall\delta>0$

Theorem 4.3.0.1. *Let $\{K_n\}_{n\in\mathbb{N}}$ is an approximation to the identity then for any continuous function f we have as n$\to\infty$*

$$\|K_n * f - f\|_\infty \to 0$$

furthermore for any $f\in L^p(\mathbb{T})$ with $1\le p<\infty$ we have as n$\to\infty$

$$\|K_n * f - f\|_p \to 0$$

Proof. Suppose f is continuous. Since $\mathbb{T}$ is compact, f is uniformly continuous in $\mathbb{T}$. So let $x,y\in\mathbb{T}$ and let $\varepsilon>0$ and $\delta>0$ satisfy
$|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon$
f is a function on a compact space hence f is bounded

$$\Rightarrow |f(x)| < M\ \forall x\in\mathbb{T}.$$

Then

$$
\begin{aligned}
|(K_n * f)(t) - f(t)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t-x)f(x)dx - f(t)| \\
&= |\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t-x)f(x)dx - f(t)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t-x)dx| \\
&= |\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t-x)(f(x)-f(t))dx| \\
&\leq \frac{1}{2\pi}\int_{|x-t|<\delta} |K_n(t-x)||(f(x)-f(t))|dx \\
&\qquad + \frac{1}{2\pi}\int_{\delta\leq|x-t|<\pi} |K_n(t-x)||(f(x)-f(t))|dx \\
&\leq \frac{1}{2\pi}\int_{|x-t|<\delta} |K_n(t-x)|\varepsilon dx + 2M\frac{1}{2\pi}\int_{\delta\leq|x-t|<\pi} |K_n(t-x)|dx \\
&\leq (\varepsilon)M' + 2M\frac{1}{2\pi}\int_{\delta\leq|x-t|<\pi} |K_n(t-x)|dx \\
&= \varepsilon M' \ as\ n \to \infty
\end{aligned}
$$

We get the second term tends to zero as n tends to infinity by the second property of approximation to the identity. Now since ε is arbitrary and above is true for all t.

$$||K_n * f - f||_\infty \to 0 \ as\ n \to \infty$$

Now for the second part when $f \in L^p$ then by density of continuous functions in L^p $\exists g \in \mathscr{C}(\mathbb{T})$ such that $||f-g||_p < \varepsilon$ for some $\varepsilon > 0$. Next, we use Minikowskis integral inequality $(||\int f||_p \leq \int ||f||_p)$.

We have

$$\begin{aligned}
||K_n*(f-g)||_p &= ||\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)(f-g)(x-y)dy||_p \qquad ,(x\in\mathbb{T})\\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} ||K_n(y)(f-g)(x-y)||_p dy\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(y)(f-g)(x-y)|^p dx\right)^{\frac{1}{p}} dy\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(y)|dy\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|(f-g)(x-y)|^p dx\right)^{\frac{1}{p}}\\
&\leq ||K_n||_1||f-g||_p\\
&< \varepsilon\sup_{n\in\mathbb{N}}||K_n||_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
||K_n*f-f||_p &\leq ||K_n*f-K_n*g||_p+||K_n*g-g||_p+||g-f||_p\\
&< \varepsilon\sup_{n\in\mathbb{T}}||K_n||_1+||K_n*g-g||_\infty+\varepsilon\\
&< \varepsilon(n\sup_{n\in\mathbb{T}}||K_n||_1+2)
\end{aligned}$$

Hence we get the second result. □

Next we show that Dirichlet kernel in not an approximation to the identity.This result will be useful to show that for some conditions the Fourier series does not converge to the functions in L^1 and L^∞ space.

Proposition 4.3.0.2. $\|D_N\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(x)|dx \geq \frac{4}{\pi^2}\log(2N+2)$.
In particular $\|D_N\| \to \infty$ as $N \to \infty$

Proof.

$$|D_N(t)| = |\frac{\sin\frac{(2N+1)t}{2}}{\sin\frac{t}{2}}| \geq |\frac{\sin\frac{(2N+1)t}{2}}{\frac{t}{2}}|$$

As $|sinx| \leq |x|, \ \forall x > 0$ Now for $(\frac{k\pi}{2} \leq t \leq \frac{(k+1)\pi}{2})$ where $k = 0, 1, 2, \ldots 2N$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|dt &= \frac{1}{2\pi}\int_{=0}^{\pi} 4\frac{|\sin(\frac{(2N+1)t}{2})|}{t}dt \\
&= \frac{1}{2\pi}\int_{0}^{\frac{(2N+1)\pi}{2}} 4\frac{|\sin t|}{t}dt \\
&\geq \frac{4}{\pi}\sum_{k=0}^{2N}\int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} \frac{|\sin t|}{(k+1)\pi}dt \\
&= \frac{4}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1} \\
&\geq \frac{4}{\pi^2}\log(2N+2) \geq \frac{4\log N}{\pi^2}
\end{aligned}$$

Hence we get the result. □

Proposition 4.3.0.3. Fejer kernel form an approximation to the identity

Proof. To prove first condition of approximation to the identity

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N}\sum_{n=0}^{N-1} D_n(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=-n}^{n} e^{imx}dx \\
&= \frac{1}{2\pi}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=-n}^{n}\int_{-\pi}^{\pi} e^{imx}dx \\
&= 1
\end{aligned}$$

Now since $F_n(x) = \frac{1}{n}(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}})^2$ which is positive. Therefore from 1 we get 2.

To prove (3) i.e. $\int_{\delta\leq|x|\leq\pi}|F_n(x)|dx \to 0$ as $n\to\infty\ \forall\delta>0$

$$\begin{aligned}\int_{\delta\leq|x|\leq\pi}|F_N(x)|dx &\leq \int\limits_{\delta\leq|x|\leq\pi}\frac{1}{n}(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}})^2dx \\ &\leq \int\limits_{\delta\leq|x|\leq\pi}\frac{1}{n}(\frac{1}{\sin\frac{x}{2}})^2dx \qquad \ldots(a)\end{aligned}$$

Now to prove that $(\sin\frac{\delta}{2})^2 \leq (\sin\frac{x}{2})^2$

$x\in[-\pi,\pi]$ this implies that $\frac{x}{2}\in[\frac{-\pi}{2},\frac{\pi}{2}]$, Since $\delta\leq|x|\leq\pi$

$\sin(\frac{-\pi}{2})\leq\sin(\frac{-\delta}{2})\leq\sin(\frac{\delta}{2})\leq\sin(\frac{\pi}{2})$ after squaring we get the result and substituting in (a) we get

$\int\limits_{\delta\leq|x|\leq\pi}\frac{1}{n}(\frac{1}{\sin\frac{x}{2}})^2dx \leq \frac{1}{n}\frac{1}{(\sin\frac{\delta}{2})^2}2(\delta-\pi)\to 0\ as\ n\to\infty$ □

Corollary 4.3.0.4. *Trigonometric polynomials are dense in L^p.*

Proof. let f $\in L^p(\mathbb{T})$ and h is a trigonometric function

where $h = F_N * f = \sigma_N f$.

Since $\{F_N\}_{N\in\mathbb{N}}$ is an approximation to the identity the for any f $\in L^p(\mathbb{T})$ with $1\leq p<\infty$

$$||F_N * f - f||_p \to 0\ as\ n\to\infty$$

□

Corollary 4.3.0.5. *(Riemann- Lebesgue Lemma)*

For any integrable function i.e. ($f\in L^1$) we have $\hat{f}(n)\to 0\ as\ n\to\infty$

Proof. Let P be trigonometric polynomial such that $\|f-P\|_1 < \varepsilon\ for\ \varepsilon>0$

P has a degree N then $\hat{P}(n)=0$ for any $|n|>N$

Hence for any $|n|>N$ we have

$|\hat{f}(n)| = |\hat{f}(n)-\hat{P}(n)| = |(f\hat{-}P)(n)| \leq \|f-P\_1 < \varepsilon$ □

Theorem 4.3.0.6. *(Uniqueness theorem)*
If $\hat{f}(n) = 0\ \forall n \in \mathbb{Z}$ *then* $f = 0$ *a.e*

Proof. Suppose $\hat{f}(n) = 0, \forall n \in \mathbb{Z}$

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = 0\ \forall N \in \mathbb{N}$$

$\sigma_N f(x) = F_N * f(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n f(x) = 0$

$$\|f\|_1 = \|\sigma_N f - f\|_1 \to 0 \ as\ n \to \infty$$

hence f is 0 almost everywhere. □

We also have if $\hat{f}(n) = \hat{g}(n) \forall n \in \mathbb{Z}$ then $f = g$ almost everywhere.
Next is a theorem for sufficient condition for almost everywhere pointwise convergence

Theorem 4.3.0.7. *Suppose* $\sum_{n\in\mathbb{N}} |\hat{f}(n)| < \infty$ *the* $S_N f$ *converges to f almost everywhere. In particular f is equal to a continuous function almost everywhere.*

Proof. $|S_N f(x)| \leq \sum_{n\in\mathbb{N}} |\hat{f}(n)|$ for every x , $S_N f(x)$ converges point wise to some function g .Further the convergence is uniform as $\sum_{n\in\mathbb{N}} |\hat{f}(n)| < \infty$,then there exists some $p \in \mathbb{N}$ such that for all $n > p \sum_{k=n}^{\infty} |\hat{f}(n)| < \varepsilon$.
Hence for $N, M > p$

$$|S_N f(x) - S_M f(x)| \leq \sum_{n=M+1}^{N} |\hat{f}(n)|$$

and this does not depend on x.

since $S_N f$ is continuous implies g is continuous.

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} S_N f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \hat{f}(n)e^{inx}\right) e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \hat{f}(n)\ for\ every\ n \in \mathbb{N}
\end{aligned}$$

Since $\hat{f}(n) = \hat{g}(n)$ which implies that f=g almost everywhere and f is continuous almost everywhere. $\square$

Chapter 5

L^P CONVERGENCE

5.1 L^2 CONVERGENCE

We start with

Definition 5.1.0.1. A function f is convex on interval [a,b] if for any two points x_1 *and* x_2 *in* $[a,b]$ and any $0 < \lambda < 1$,

$$f(\lambda(x_1) + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

Using the definition of convex function we prove the Hölders inequality.

Proposition 5.1.0.2. Hölders inequality gives for $f \in L^p$ *and* $g \in L^q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof. Since $f \in L^p$ *and* $g \in L^q$ we have $0 < \|f\|_p, \|g\|_q < \infty$
Wlg define $F(x) = \frac{|f(x)|}{\|f\|_p}$ and $G(x) = \frac{|g(x)|}{\|g\|_q}$

$$\int F^p d\mu = \int \frac{|f(x)|^p}{\|f\|_p^p} d\mu = \frac{1}{\|f\|_p^p} \int |f(x)|^p d\mu = 1$$

similarly $\int G^q = 1$

Now we define $s(x) = \log\left(\frac{|f(x)|}{\|f\|_p}\right)^p$ and $t(x) = \log\left(\frac{|g(x)|}{\|g\|_q}\right)^q$

$$e^{s(x)} = \left(\frac{|f(x)|}{\|f\|_p}\right)^p$$

which implies $F(x) = e^{\frac{s(x)}{p}}$ and similarly $G(x) = e^{\frac{t(x)}{q}}$

e^x is a convex function we put $\lambda = \frac{1}{q}$

$$e^{\frac{1}{q}t(x)+\frac{1}{p}s(x)} \leq \frac{1}{q}e^{\frac{t(x)}{q}} + \frac{1}{p}e^{\frac{s(x)}{p}}$$

which gives

$$F(x)G(x) \leq \frac{G(x)^q}{q} + \frac{f(x)^p}{p} \tag{5.1}$$

Integrating left side of (i)we get

$$\|FG\|_1 = \int |FG| d\mu = \int \frac{|fg|_1}{\|f\|_p \|g\|_q} d\mu = \frac{\|fg\|}{\|f\|_p \|g\|_q} d\mu \tag{5.2}$$

integrating right hand side of (i)

$$\int \frac{F(x)^p}{p} + \frac{G(x)^q}{q} d\mu = \frac{1}{p}\int F(x)^p d\mu + \frac{1}{q}\int G(x)^q d\mu = \frac{1}{p} + \frac{1}{q} = 1 \tag{5.3}$$

from (5.1), (5.2) and (5.3) we get the result. □

Proposition 5.1.0.3. $L^2(\mathbb{T})$ is a Hilbert space with inner product $\langle f, g \rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}dx$

Proof. $\langle f,g\rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}dx \le \|f\|_2\|\bar{g}\|_2 < \infty$ hence the inner product is well defined. To check the conditions for inner product space

Let f,g and h $\in L^2$

1) $\langle af+bg,h\rangle = \int (af+bg)(x)\overline{h(x)}dx = a\langle f,h\rangle + b\langle g,h\rangle$

2) $\overline{\langle g,f\rangle} = \int \overline{g(x)\overline{f(x)}} = \int \overline{g(x)}f(x)dx = \langle f,g\rangle$

3) $\langle f,f\rangle = \int |f(x)|^2 dx \ge 0$

If $\langle f,f\rangle = 0 \Leftrightarrow \int f(x)\overline{g(x)}dx = 0 \Leftrightarrow \int |f(x)|^2 dx = 0 \; a.e.$ □

Proposition 5.1.0.4. Let $e_n(t) = e^{int}$ the $\{e_n\}_{n\in\mathbb{Z}}$ is orthogonal set

$$\langle e_m, e_n\rangle = \int_{\mathbb{T}} e^{i(m-n)t}dt = \begin{cases} 1 \; if \; m=n \\ 0 \; if \; m\neq n \end{cases}$$

Proposition 5.1.0.5. Let $f \in L^2$

1) $S_N f$ is the best degree N trigonometric L^2 approximation to f. I.e. for any given trigonometric polynomial P of degree atmost N , we have $\|f-S_N f\|_2 \le \|f-P\|_2$

2)$\|f-S_N f\|_2 \to 0 \; as \; n\to\infty$

3)(Parsevals identity)

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

Proof. Let P be a trigonometric polynomial of degree atmost N. Let $Q = S_N f - P$ then Q is a trigonometric polynomial as well having degree atmost N

Let $Q = \sum_{n=-N}^{N} a_n e_n$ then we have

$$\begin{aligned}
\langle f - S_N f, Q\rangle &= \langle f, \sum_{n=-N}^{N} a_n e_n\rangle - \langle \sum_{n=-N}^{N} \hat{f(n)} e_n, \sum_{n=-N}^{N} a_n e_n\rangle \\
&= \int f(x) \sum_{n=-N}^{N} \overline{a_n} e_{-n} - \int \sum_{n=-N}^{N} \hat{f}(n) e_n \overline{a_n} e_{-n} \\
&= \sum_{n=-N}^{N} \hat{f}(n)\overline{a_n} - \sum_{n=-N}^{N} \hat{f}(n)\overline{a_n} \\
&= 0 \qquad\qquad\qquad\qquad \ldots(i)
\end{aligned}$$

Similarly $\langle Q, f - S_N f\rangle = 0$ (ii)

$$\begin{aligned}
\|f - P\|_2^2 &= \langle f - P, f - P\rangle \\
&= \langle f - (S_N f - Q), f - (S_N f - Q)\rangle \\
&= \langle (f - S_N f) - Q, (f - S_N f) - Q\rangle \\
&= \langle (f - S_N f), (f - S_N f)\rangle - \langle Q, (f - S_N f)\rangle \\
&\quad - \langle (f - S_N f), Q\rangle + \langle Q, Q\rangle \\
&= \|f - S_N f\|_2^2 + \|Q\|_2^2 \\
&\geq \|f - S_N f\|_2^2
\end{aligned}$$

Since $\sigma_N f = F_N * f$ is a trigonometric polynomial of degree less than N and

$$\|f - \sigma_N f\|_2 \to 0 \; as \; N \to \infty$$

therefore

$$\|f - S_N f\|_2 \leq \|f - \sigma_N f\|_2 \to 0 \; as \; N \to \infty$$

Since $S_N f$ is a trigonometric function by (i) and (ii) we have
$\langle (f - S_N f), S_N f\rangle = \langle S_N f, (f - S_N f)\rangle = 0$

$$\begin{aligned}
\|f\|_2^2 &= \langle f, f\rangle \\
&= \langle (f - S_N f) + S_N f, (f - S_N f) + S_N f\rangle \\
&= \|f - S_N f\|_2^2 + \langle (S_N f), (S_N f)\rangle \\
&= \|f - S_N f\|_2^2 + \int_{\mathbb{T}} (\sum_{n=-N}^{N} \hat{f}(n) e_n)(\sum_{n=-N}^{N} \overline{\hat{f}(n) e_n} d\mu \\
&= \|f - S_N f\|_2^2 + \int_{\mathbb{T}} \sum_{n=-N}^{N} |\hat{f}(n)|^2 d\mu \\
&= \|f - S_N f\|_2^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2 x
\end{aligned}$$

as $n \to \infty$

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

□

5.2 L^P CONVERGENCE

Definition 5.2.0.1. For a pair p,q with $1 \leq p < q < \infty$ and a continuous linear operator $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$ the operator norm is defined by $\|T\|_{p,q} = \sup\{\|Tf\|_q : \|f\|_p = 1\}$

To prove that S_n is a continuous linear operator.Let f,g be functions inL^p and a be a real value

$$\begin{aligned}
S_n(af + g) &= \sum_{k=-n}^{n} (a\hat{f+}g)(k) e^{ikx} \\
&= \sum_{k=-n}^{n} a(\hat{f})(k) e^{ikx} + \hat{g}(k) e^{ikx} \\
&= aS_n f + S_n g
\end{aligned}$$

hence linear.

To prove that S_n is bounded

$$\begin{aligned}
|S_n f| &= \left|\sum_{k=-n}^{n} \hat{f}(k)e^{ikx}\right| \leq \sum_{k=-n}^{n} \left|\hat{f}(k)\right| \\
&= \sum_{k=-n}^{n} \left|\int_{\mathbb{T}} f(y)e^{-iky}dy\right| \\
&\leq \sum_{k=-n}^{n} \int_{\mathbb{T}} |f(y)|dy = \sum_{k=-n}^{n} \|f\|_1 \\
&\leq \sum_{k=-n}^{n} \|f\|_p \|1\|_q \qquad \ldots (Hölders\ inequality) \\
&= \sum_{k=-n}^{n} \|f\|_p = (2N+1)\|f\|_p
\end{aligned}$$

hence S_n is a continuous linear operator.

Theorem 5.2.0.2. *Let $1 \leq p < \infty$ then following are equivalent.*

1) $\|S_n f - f\|_p \to 0$ as $n \to \infty$ $\forall f \in L^p(\mathbb{T})$

2) $\sup\limits_{n\in\mathbb{N}} \|S_n\|_p < \infty$
where $\|S_n\|_p = \|S_n\|_{L^p \to L_p}$ is an operator norm when viewing S_n as an operator from L^p to itself.

Proof. Suppose (2) is true. Then $\forall n \in \mathbb{N}$ we have

$\sup\limits_{n\in\mathbb{N}} \|S_n\|_p < \infty$. Now $\sigma_n f$ is a trigonometric polynomial and and since trigonometric polynomials are dense in L^p we have N such that

$$\|\sigma_n f - f\| < \varepsilon \ \forall n > N.$$

Now $\sigma_n f$is a trigonometric polynomial therefore by proposition 4.1.0.3 $S_n(\sigma_n f) = \sigma_n f$

$$\begin{aligned}
\|S_n f - f\|_p &= \|S_n f - S_n(\sigma_n f) + \sigma_n f - f\|_p \\
&\leq \|S_n(f - \sigma_n f)\|_p + \|\sigma_n f - f\|_p \\
&\leq \|S_n\|_p \|f - \sigma_n f\|_p + \|\sigma_n f - f\|_p \\
&\leq (\|S_n\| + 1)\varepsilon \\
&\leq \sup_{n\in\mathbb{N}}(\|S_n\| + 1)\varepsilon \to 0 \ldots as\ n \to \infty
\end{aligned}$$

Conversely, $\|S_n f - f\| \to 0\ as\ n \to \infty\ \forall f \in L^p(\mathbb{T})$

$\|S_n f - f\| < M_f$ for some $M_f\ and\ \forall n \in \mathbb{N}$

So

$$\|S_n f\|_p \leq \|f\|_p + \|S_N f - f\|_p = \|f\|_p + M_f\ \forall n \in \mathbb{N}$$

We use the Uniform boundedness theorem and get $\|S_n\| \leq M$ for some real M and $\forall n$.

Hence

$$\sup \|S_n\| \leq M < \infty$$

□

This theorem also holds for continuous function by density of continuous functions in L^p functions.

Theorem 5.2.0.3. $\sup\limits_{N\in\mathbb{N}} \|S_N\|_1 = \infty$ *and* $\sup\limits_{N\in\mathbb{N}} \|S_N\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} = \infty$
Consequently $\exists f \in L^1(\mathbb{T})$ *and* $\exists g \in \mathscr{C}(\mathbb{T})$ *such that* $S_N f$ *and* $S_N g$ *do not converge to* f *and* g *in* L^1 *and* L^∞ *norm respectively.*

Proof. $\|F_N\|_1 = \int\limits_{\mathbb{T}} F_N(x)dx = 1$

and $\|S_N\|_1 = \sup\{\|S_N f\|_1 : \|f\|_1 = 1\}$

Therefore

$$\|S_N\|_1 \geq \|S_N(F_M)\|_1 = \|D_N * F_M\|_1 \to \|D_N\|_1\ as\ M \to \infty$$

as $\|D_N\| > c\log N$ we have $\sup\limits_{N\in\mathbb{N}} \|S_N\|_1 = \infty$
Let $f = sgn(D_N)$ then $\|f\|_\infty = 1$

$$\begin{aligned} \|S_N\|_\infty &\geq \|S_N f\|_\infty \geq |S_N f(0)| \\ &= |(D_N * f)(0)| = |\int_{\mathbb{T}} D_N(y) f(0-y)dy| \\ &= |\int_{\mathbb{T}} D_N(y) sgnD_N(-y)dy| \\ &= \|D_N\|_1 > c\log N \end{aligned}$$

Hence we have $\sup\limits_{N\in\mathbb{N}} \|S_N\|_\infty = \infty$
Now if $f \notin \mathscr{C}(\mathbb{T})$ and for $\varepsilon > 0$, $\exists g \in \mathscr{C}(\mathbb{T})$ such that $\|g-f\|_1 < \varepsilon$ and $\|g\|_\infty = 1$

$$\begin{aligned} \|S_N(g-f)\|_\infty &= \sup_{x\in\mathbb{T}} |S_N(g-f)(x)| a.e \\ &= \sup_{x\in\mathbb{T}} |\sum_{n=N}^{N} (g \hat{-} f)(n) e^{inx}| \\ &\leq \sup_{x\in\mathbb{T}} \sum_{n=N}^{N} |(g \hat{-} f)(n)| \\ &< \varepsilon(2N+1) \end{aligned}$$

For continuous function g we have

$$\begin{aligned} \|S_N\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} &\geq \|S_N f\|_\infty \\ &\geq \|S_N g\|_\infty - \|S_N(g-f)\|_\infty \\ &\geq C\log N - (2N+1)\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$

$$\|S_N\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} \geq C\log N$$

Hence

$$\sup_{N\in\mathbb{N}} \|S_N\|_{\mathscr{C}(\mathbb{T})\to\mathscr{C}(\mathbb{T})} \to \infty$$

$$\|S_N f - f\|_1 \geq \int_{\mathbb{T}} |S_N f(x)|dx - \int_{\mathbb{T}} |f(x)|dx \to \infty \; as \; N \to \infty$$

$$\|S_N g - g\|_\infty = \sup|S_N g - g| \; a.e \geq \|S_N g\|_\infty - \|g\|_\infty \to \infty \; as \; N \to \infty$$

$\square$

5.3 HILBERT TRANSFORM

For $1 < p < \infty$ we will investigate the issues of convergence in L^p by considering the Hilbert transform.

Definition 5.3.0.1. For trigonometric polynomial f we define the Hilbert transform of f by

$$Hf(t) = \sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)e^{int}$$

The Riesz projection P_+ *and* P_- are defined by

$$P_+f(t) = \sum_{n=1}^{\infty} \hat{f}(n)e^{int} \; and \; P_-f(t) = \sum_{n=-\infty}^{-1} \hat{f}(n)e^{int}$$

$$Af(t) = P_+f(t) + \hat{f}(0) = \sum_{n=0}^{\infty} \hat{f}(n)e^{int}$$

$$S_N^+ f = \sum_{n=0}^{2N} \hat{f}(n)e^{int}$$

All the above operators are well defined for trigonometric polynomial. This is because for trigonometric polynomial $f(x) = \sum\limits_{n\in\mathbb{Z}} a_n e^{inx}$, $\hat{f}(n) = f(n)$ and $a_n = 0 \forall n > N$ where N is the degree of the polynomial.

$S_N^+ f$ is well defined for all L^p functions.

for $g \in L^p$

$|S_N^+ f(t)| = |\sum\limits_{n=0}^{2N} \int\limits_{\mathbb{T}} f(x)e^{in(x-t)}dx| = \|f\|_1 \leq \|f\|_p \; for \; 1 \leq p < \infty$

Theorem 5.3.0.2. *Let $f \in \mathscr{C}(\mathbb{T})$ then $\hat{f}(n) = O(\frac{1}{n^m}) \forall m > 0$*
In particular $\hat{f}(n) = O(n^{-2})$ and $\sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$

Proof. $(\hat{f'})(n) = in\hat{f}(n)$. By continuously differentiating m times we get

$$(\hat{f^m})(n) = in^m \hat{f}(n) \; \forall m \in \mathbb{N}$$

$(\hat{f^m})(n) \to 0 \; as \; n \to \infty$ by Riemann Lebesgue Lemma. So $\{\hat{f^m}(n)\}_{n\in\mathbb{Z}}$ s a bounded sequence.

$$(\hat{f^m})(n) = (in)^m \hat{f}(n)$$

$$|(\hat{f^m})(n)| = |(in)^m \hat{f}(n)|$$

$$|\hat{f}(n)| = \frac{|(\hat{f^m})(n)|}{n^m} \leq \frac{C}{n^m}$$

for some real C.

Hence operators are well defined for function in $\mathscr{C}^\infty$ □

Proposition 5.3.0.3. Let $1 < p < \infty$ the following are equivalent :

1)$\|H\|_p < \infty$

2)$\|P_+\|_p < \infty$

3) $\sup\limits_{N\in\mathbb{N}} \|S_N\|_p < \infty$

4) $\sup\limits_{N\in\mathbb{N}} \|S_N^+\|_p < \infty$

5)$\|A\|_p < \infty$

Proof. f can be written as

$$f(x) = P_+ f(x) + P_- f(x) + \hat{f}(0) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

and

$$Hf = -iP_+ f + iP_- f \text{ and } iHF = P_+ f - P_- f$$

$$f + iHf = 2P_+ F + \hat{f}(0)$$

$$\Rightarrow P_+ f = \frac{1}{2}(f + iHf - \hat{f}(0)$$

and also $|\hat{f}(0)| \leq \|f\|_1 \leq \|f\|_p$

Hence by the above equation the boundedness of P_+ and H is equivalent

This shows (1) $\Leftrightarrow$ (2)

If $g(t) = f(t)e^{iNt}$

then

$$\hat{f}(n-N) = \int_{\mathbb{T}} f(x) e^{i(n-N)x} dx = \int_{\mathbb{T}} \hat{f}(n) e^{iNx} dx = \hat{g}(n) dx$$

$$\sum_{n=N}^{N} \hat{f}(n) e^{int} = e^{-iNt} \sum_{n=0}^{2N} \hat{f}(n-N) e^{int} = \sum_{n=0}^{2N} \hat{g}(n) e^{int}$$

Hence $\|S_N f\|p = \|S_N^+ g\|p$ we also have $\|f\| = \|g\|$

$$\|S_N\|p = \|S_N^+\|p$$

Hence we have (3)$\Leftrightarrow$ (4)

Suppose (4) holds therefore $\sup_{N\in\mathbb{N}} \|S_N^+\|_p < \infty$

$$\|Af\|_p = \|\liminf_{N\to\infty} S_N^+ f\|_p \leq \liminf_{N\to\infty} \|S_N^+ f\|_p \leq (\sup_{N\in\mathbb{N}} \|S_N^+\|_p)\|f\|_p$$

$$\Rightarrow \|Af\|_p < \infty\ \forall f \Rightarrow \|A\|_p < \infty$$

Hence (4)$\Rightarrow$ (5)

$$\begin{aligned} S_N^+ f(t) &= \sum_{n=0}^{\infty} \hat{f}(n)e^{int} - \sum_{n=2N+1}^{\infty} \hat{f}(n)e^{int} \\ &= Af(t) - e^{i(2N+1)t} \sum_{n=0}^{\infty} \hat{f}(n+2n+1)e^{int} \\ &= Af(t) - e^{i(2n+1)t} Ag(t) \end{aligned}$$

where $g = f(n+2N+1)$ *and* $\|f\|_p = \|g\|_p$

$$\sup_{N\in\mathbb{N}} \|S_N^+ f\| \leq \|A\|_p\|f\|_p + \|A\|_p\|g\|_p = 2\|A\|_p\|f\|_p$$

for smooth f

Hence by density of f in L^p we have (5) $\Rightarrow$ (4)

Since $Af = P_+ f + \hat{f}(0) \Rightarrow (5) \Leftrightarrow (1)$

(We have (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2)) $\square$

Corollary 5.3.0.4. *Let* $1 < p < \infty$ *then* $S_N f$ *converges to f in* L^p *norm*

$\forall f \in L^p(\mathbb{T}) \Leftrightarrow \|H\|_p < \infty$

Proof. For $1 < p < \infty$ we have

$\|S_N f - f\|_p \to 0$ *as* $n \to \infty \Leftrightarrow \sup_{n\ in\mathbb{N}} \|S_N\|_p < \infty \Leftrightarrow \|H\|_p < \infty$ $\square$

Theorem 5.3.0.5. *Riesz Thorin interpolation theorem :*

Let 1 $\leq p_1, p_2, q-1, q_2 \leq \infty$. *Suppose T is an operator which is bounded mapping from* $L^{p_1}(\mathbb{T})$ *to* $L^{q_1}(\mathbb{T})$ *and* $L^{p_2}(\mathbb{T})$ *to* $L^{q_2}(\mathbb{T})$

Let $A = \|T\|_{L^{p_1}\to L^{q_1}}$ *and* $B = \|T\|_{L^{p_2}\to L^{q_2}}$ *then for any* $0 \leq \alpha \leq 1$ *we have*

$$\|Tf\|_{q_\alpha} \leq A^{1-\alpha}B^{\alpha}\|f\|_{p_\alpha}$$

where $\frac{1}{p_\alpha} = \frac{1-\alpha}{p-1} + \frac{\alpha}{p_2}$ *and* $\frac{1}{q_\alpha} = \frac{1-\alpha}{q-1} + \frac{\alpha}{q_2}$

Theorem 5.3.0.6. *Let $1 < p < \infty$ then there exist constant A_p such that*

$$\|Hf\|_p \leq A_p \|f\|_p$$

for all trigonometric polynomials f and H extends to a bounded operator from $L^p(\mathbb{T}) \to L^p(\mathbb{T})$
Hence, $S_N f$ converges to f in the L^p norm for all $f \in L^p(\mathbb{T})$ for $1 < p < \infty$.

Proof. Let f be a non zero real valued trigonometric polynomial with $\hat{f}(0) = 0$ then

$$\hat{f}(-n) = \int_{\mathbb{T}} f(x)e^{inx}dx = \overline{\int_{\mathbb{T}} \overline{f(x)}e^{inx}} = \overline{\hat{f}(n)}$$

as f(x) is a real valued function.

$$\begin{aligned} Hf(x) &= -i\sum_{n>0}\hat{f}(n)e^{inx} + i\sum_{n<0}\hat{f}(n)e^{inx} \\ &= \sum_{n>0} -i\hat{f}(n)e^{inx} + i\sum_{n>0}\hat{f}(-n)e^{-inx} \\ &= \sum_{n>0}\hat{f}(n)e^{inx} + i\overline{\hat{f}(n)e^{inx}} \\ &= 2Re\left(\sum_{n>0} -i\hat{f}(n)e^{inx}\right) \end{aligned}$$

Hence H is also a real valued function.

$$\begin{aligned} (f + iHf)(x) &= f(x) + i(\sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)e^{inx}) \\ &= \sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx} + i(\sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)e^{inx}) \\ &= \sum_{n\in\mathbb{Z}}(1 + i(-isgn(n))\hat{f}(n)e^{inx} = \sum_{n>0}\hat{f}(n)e^{inx} \end{aligned}$$

If we take $k \in \mathbb{N}$ then $(f+iHf)^{2k}$ will have positive frequency and $\int_{\mathbb{T}}(f+iHf)^{2k} = 0$ this is because f is a trigonometric polynomial.

$$0 = \int_{\mathbb{T}} (f+iHf)^{2k} = \int_{\mathbb{T}} \sum_{j=0}^{2k} i^j \binom{2k}{j} (Hf)^j f^{2k-j}$$

Combining the real parts

$$\sum_{j=0}^{k} \int_{\mathbb{T}} (-1)^k \binom{2k}{2j} (Hf)^{2j} f^{2k-2j} = 0$$

$$\int_{\mathbb{T}} (-1)^j Hf^{2k} = -\sum_{j=0}^{k-1} \int_{\mathbb{T}} (-1)^j \binom{2k}{2j} (Hf)^{2j} f^{2k-2j}$$

Hence

$$\begin{aligned}
\|Hf\|_{2k}^{2k} &= |\int_{\mathbb{T}} Hf|^{2k} \\
&= |-\sum_{j=0}^{k-1} \int_{\mathbb{T}} (-1)^j \binom{2k}{2j} (Hf)^{2j} f^{2k-2j}| \\
&\leq \sum_{j=0}^{k-1} \binom{2k}{2j} \|(Hf)^{2j} f^{2k-2j}\|_1
\end{aligned}$$

For each j we have

$$\begin{aligned}
\|(Hf)^{2j} f^{2k-2j}\|_1 &\leq \|(Hf)^{2j}\|_{\frac{k}{j}} \|f^{2k-2j}\|_{\frac{k}{k-j}} \quad \textit{by holders inequality} \\
&= (\int_{\mathbb{T}} |Hf^{2j}|^{\frac{k}{j}})^{\frac{j}{k}} (\int_{\mathbb{T}} |f^{2k-2j}|^{\frac{k}{k-j}})^{\frac{k-j}{k}} \\
&= \|Hf\|_{2k}^{2j} \|f\|_{2k}^{2k-2j}
\end{aligned}$$

Let $R = \frac{\|Hf\|_{2k}}{\|f\|_{2k}}$ hence we have $R^{2k} \leq \sum_{j=0}^{k-1} \binom{2k}{2j} R^{2j}$

Polynomial $x^{2k} - \sum_{j=0}^{k-1} \binom{2k}{2j} x^{2j}$ has an even degree and the polynomial tends to infinity as x tends to infinity. If $x^{2k} - \sum_{j=0}^{k-1} \binom{2k}{2j} x^{2j} \leq 0$ then there exists a constant C_{2k} such that $|x| \leq C_{2k}$.

Hence we have $|R| \leq C_{2k}$

Therefore we have $\|Hf\|_{2k} \leq C_{2k}\|f\|_{2k}$ for all trigonometric polynomial with $\hat{f}(0) = 0$

now we remove the restriction that $\hat{f}(0) = 0$

Hilbert transform of an constant function is 0

$$\begin{aligned} \|Hf\|_{2k} = \|Hf - H\hat{f}(0)\|_{2k} &= \|H(f-\hat{f})(0)\|_{2k} \\ &\leq C_{2k}\|(f-\hat{f})(0)\|_{2k} \leq 2C_{2k}(\|f\|_{2k}) \end{aligned}$$

Now suppose f is not necessarily a real valued trigonometric polynomial

let $f(x) = \sum_{-N}^{n} a_n e^{inx}$ the we have

$$\begin{aligned} f(x) &= \sum_{-N}^{N} a_n e^{inx} = \sum_{-N}^{N} \frac{a_n + \overline{a_{-n}}}{2} e^{inx} + \sum_{-N}^{N} \frac{a_n - \overline{a_{-n}}}{2} e^{inx} \\ &= \left(\left(\frac{a_0 + \overline{a_0}}{2} \right) + \sum_{n=1}^{N} a_n + \overline{a_{-n}} + a_{-n} + \overline{a_n} \frac{e^{inx} + E^{-inx}}{2} \right) \\ &+ i \left(\left(\frac{a_0 - \overline{a_0}}{2} \right) + \sum_{n=1}^{N} a_n - \overline{a_{-n}} - a_{-n} + \overline{a_n} \frac{e^{inx} + e^{-inx}}{2} \right) \\ &= (\Re a_0 + \sum_{n=1}^{N} \Re(a_n + a_{-n}) \cos(nx)) + i(\Im a_0 + \sum_{n=1}^{N} \Re(a_n - a_{-n}) \sin(nx)) \end{aligned}$$

If we take P as the real part of f(x) and Q to be the imaginary part of f(x) then P and Q are the real valued trigonometric polynomials and $f = P + iQ$

$$\|Hf\|_{2k} \leq \|HP\|_{2k} + \|iHQ\|_{2k} \leq 2C_{2k}(\|P\|_{2k} + \|Q\|_{2k}) \leq 4C_{2k}\|f\|_{2k}$$

hence there is a A_p such that $\|Hf\|_p \leq A_p\|f\|_p$ $\forall$ trigonometric polynomial when p= 2k with $k \in \mathbb{N}$

Since any p$\geq$ 2 is in the interval of the form [2k,2k+2] the Riesz Thorin interpolation theorem extends this result for all $p \geq 2$. Since trigonometric polynomials are dense in $L^p(\mathbb{T})$ H extends to a bounded operator on $L^p(\mathbb{T})$ as well.

For $1 < p < 2$ we use the fact that H is skew adjoint i.e. for f,g $\in L^2(\mathbb{T})$ we have

$$\begin{aligned}
\langle Hf, g\rangle &= \int_{\mathbb{T}} Hf(x)\overline{g(x)}dx = \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)e^{inx}\overline{g(x)}dx \\
&= \sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)\int_{\mathbb{T}} e^{inx}\overline{g(x)}dx = \sum_{n\in\mathbb{Z}} -isgn(n)\hat{f}(n)\overline{\hat{g}(n)} \\
&= \sum_{n\in\mathbb{Z}} -isgn(n)\int_{\mathbb{T}} f(x)e^{-inx}dx\overline{\hat{g}(n)} \\
&= \int_{\mathbb{T}} f(x)\sum_{n\in\mathbb{Z}} -isgn(n)\overline{\hat{g}(n)e^{inx}} = \int_{\mathbb{T}} -f(x)\overline{Hg(x)}dx = \langle -f, Hg\rangle
\end{aligned}$$

If $\frac{1}{p}+\frac{1}{q} = 1$ with $q \geq 2$ then

$$|\langle Hf, g\rangle| = |\langle f, Hg\rangle| \leq \|f\|_p\|Hg\|_q \leq C_q\|f\|_p\|g\|_q$$

If f is a trigonometric polynomial and $g = \overline{\left(\frac{|Hf|^p}{Hf}\right)}$ then $g \in L^q(\mathbb{T})$

$$\|g\|_q = \left(\int_{\mathbb{T}} \left|\overline{\frac{|Hf|^p}{Hf}}\right|^q\right)^{\frac{1}{q}} = \left(\int_{\mathbb{T}} \left|\overline{\frac{|Hf|^p}{Hf}}\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = \|Hf\|_p^{p-1}$$

we get

$$\|Hf\|_p^p = \left|\int_{\mathbb{T}} Hf\frac{|Hf|^p}{Hf}\right| = |\langle Hf, g\rangle| \leq C_q\|f\|_p\|Hf\|_p^{p-1}$$

hence $\|Hf\|_p \leq C_q\|f\|_p$ and H is a bounded for trigonometric polynomials for $1 < p < 2$. By the density of trigonometric polynomials in $L^p(\mathbb{T})$ we can extend H to bounded operator on $L^p(\mathbb{T})$. $\square$

5.4 UNIFORM CONVERGENCE

Definition 5.4.0.1. Let $0 < \alpha < 1$ the $f : \mathbb{T} \to \mathbb{C}$ satisfies a Lip-α condition if there is a constant C such that

$$|f(t) - f(x)| \leq C|t - x|^{\alpha} \ \forall t, x \in \mathbb{T}$$

The minimum such C is called the Lip-α constant of f and is denoted by $[f]_{\alpha}$ and $\mathscr{C}^{\alpha}(\mathbb{T})$ is a set of all functions satisfying the Lip-α condition.
The minimum value of C exists by the compactness of $\mathbb{T}$
if $\beta \geq \alpha$ and $f \in \mathscr{C}^{\beta}(\mathbb{T})$ then for all x, t $\in \mathbb{T}$

$$|f(t) - f(x)| \leq C|t - x|^{\beta} \leq C|t - x|^{\alpha}|t - x|^{\beta - \alpha} \leq C'|t - x|^{\alpha}$$

Hence $f \in \mathscr{C}^{\alpha}(\mathbb{T})$
hence $\mathscr{C}^{\beta} \subseteq \mathscr{C}^{\alpha}$

Theorem 5.4.0.2. *Suppose* $0 < \alpha < 1$ *the there exist a constant* k_{α} *such that* $\forall f \in \mathscr{C}^{\alpha}$ *we have*

$$\|S_N f - f\| < K_{\alpha}[f]_{\alpha} N^{-\alpha} \log N$$

In particular $\|S_N f - f\| \to 0$ *as* $n \to \infty$

Proof. $F_N(x) = \frac{1}{N}(\frac{\sin\frac{Nx}{2}}{\sin\frac{x}{2}})^2$ we use the property that if $|\frac{x}{2}| \leq \frac{\pi}{2}$ then $\sin\frac{x}{2} \geq \frac{2x}{2\pi}$ and $|sin(nx)| \leq n|\sin x|$.

we get estimated value of $F_N(x) \leq \min\{N, \frac{\pi^2}{Nx^2}\}$

$$
\begin{aligned}
|\sigma_N f(0) - f(0)| &= |\int_{\mathbb{T}} f(x)F_N(-x)dx - \int_{\mathbb{T}} f(0)F_N(-x)dx| \\
&\leq \int_{\mathbb{T}} |f(x) - f(0)||F_N(-x)|dx \\
&\leq \frac{1}{2\pi} \int_{|x|<\frac{\pi}{N}} [f]_\alpha |x|^\alpha N dx + \int_{\frac{\pi}{N}<|x|<\pi} |x|^\alpha (\frac{\pi^2}{N|x|^2}) dx \\
&= [f]_\alpha \left(\frac{2N(\frac{\pi}{N})^{\alpha+1}}{\alpha+1} + \frac{2\pi^2}{N(\alpha-1)}(\pi^{\alpha-1} - (\frac{\pi}{N})^{\alpha-1}) \right) \\
&\leq C_\alpha [f]_\alpha N^{-\alpha}
\end{aligned}
$$

for some constant C_α

If we take f = $\tau_y(x)f = f(x-y)$ in the previous result we get

$$|\sigma_N f(x) - f(x)| \leq C_\alpha [f]_\alpha N^{-\alpha}$$

Now since $S_N(\sigma_N f) = \sigma_N f$ as $\sigma_N f$ is trigonometric polynomial

$$
\begin{aligned}
\|S_N f - f\|_\infty &= \|S_N f - S_N(\sigma_N f) + \sigma_N f - f\|_\infty \\
&\leq \|D_N * (f - \sigma_N f)\|_\infty + \|\sigma_N f - f\|_\infty \\
&\leq (\|D_N\|_1 + 1)\|\sigma_N f - f\|_\infty \\
&= (C \log N + 1) C_\alpha [f]_\alpha N^{-\alpha} \\
&< K_\alpha [f]_\alpha N^{-\alpha} \log N
\end{aligned}
$$

$\mathscr{C}^1(\mathbb{T}) \subseteq \mathscr{C}^\alpha(\mathbb{T})$ as $0 < \alpha < 1$

hence the uniform convergence of Fourier series follows for $\mathscr{C}^1$ But the rate of convergence is not same. $\square$

5.5 CONVERGENCE IN SOBOLEV SPACES

One family of subspace of $L^2(\mathbb{T})$ which does provide rate of L^2 convergence is the family of Sobolev spaces.

Definition 5.5.0.1. Let $s > 0$ the Sobolev space $H^s(\mathbb{T})$ is defined to be the set of functions f such that

$$\|f\|_{H^s} = (|\hat{f}(0)| + \sum_{n\in\mathbb{Z}} |n|^{2s}|\hat{f}(n)|^2)^{\frac{1}{2}} < \infty$$

Proposition 5.5.0.2. $\|S_N f - f\|_2 \le \frac{1}{N^{2s}}\|f\|_{H^s}$

Proof.

$$\begin{aligned}
\|S_N f - f\|_2 &= \|\sum_{n=-N}^{N} \hat{f}(n)e^{inx} - f(x)\|_2 \\
&= (\int_{\mathbb{T}} |\sum_{n=-N}^{N} \hat{f}(n)e^{inx} - f(x)|^2 dx)^{\frac{1}{2}} \\
&= (\int_{\mathbb{T}} |\sum_{n=-N}^{N} \hat{f}(n)e^{inx} - \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}|^2 dx)^{\frac{1}{2}} \\
&= (\int_{\mathbb{T}} |\sum_{|n|>N} \hat{f}(n)e^{inx}|^2 dx)^{\frac{1}{2}} \le \sum_{|n|>N} |\hat{f}(n)|^2 \\
&\le \frac{1}{N^{2s}} \sum_{|n|>N} ||n|^{2s}\hat{f}(n)|^2 \\
&\le \frac{1}{N^{2s}}\|f\|_{H^s}
\end{aligned}$$

□

5.6 DIVERGENCE OF FOURIER SERIES

Example due to Fejer of a continuous function whose Fourier series is Divergent at a point.

Theorem 5.6.0.1. *$\exists f \in \mathscr{C}(\mathbb{T})$ such that $S_N f(x)$ diverges for some x.*

Proof. Let p and n be positive integers and let

$$Q_{p,n} = \frac{cospx}{n} + \frac{cos(p+1)x}{n-1} + \cdots + \frac{cos(p+n-1)x}{1} - \frac{cos(p+n+1)x}{1} - \cdots - \frac{cos(p+2n)x}{n}$$

$$\hat{Q_{p,n}}(m) = \begin{cases} \frac{1}{2(n-k)} \; if |m| = p+k \; for \; 0 \le k \le n-1 \\ \frac{-1}{2k} \; if \; |m| = p+n+k \; for \; 1 \le k \le n \\ 0 \; otherwise \end{cases}$$

We get this by using the identity $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ Now we use the identity that $\cos(a-b) = 2\sin a \sin b$ and we get

$$Q_{p,n}(x) = \sum_{k=1}^{n} \frac{\cos(p+n_k)x + \cos(p+n+k)x}{k} = 2\sin((p+n)x) \sum_{k=1}^{n} \frac{\sin kx}{k}$$

Partial sums of $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ are uniformly bounded .Hence there is a C such that
$|Q_{p,n}| < C \; \forall p,n,x$

$|Q_{p,n}|$ is bounded and has a convergent subsequence which is also continuous say $|Q_{p_k,n_k}|$
If $\exists \sum_{k=1}^{\infty} |a_k| < \infty$ then $\sum_{k=1}^{\infty} a_k Q_{p_k,n_k}$ converges uniformly to a continuous function f_{a_k,p_k,n_k} for any sequence $\{p_k\}\{n_k\}$

In particular if $p_{k+1} > p_k + 2n_k \forall k$ then $Q_{p_k,n_k} and Q_{p_{k+1},n_{k+1}}$ have the disjoint frequencies for all k so,

$$\hat{f}(m) = \begin{cases} \frac{a_k}{2(n_k - j_0)} & \text{if } |m| = p_k + j \text{ for } 0 \leq j < n_k - 1 \\ \frac{-a_k}{2j} & \text{if } |m| = p_k + n_k + j \text{ for } 1 < j < n_k \\ 0 & otherwise \end{cases}$$

Let $a_k = \frac{1}{k^2}$ and $p-k = n-k = 2^{k^2}$

Let $f = f_{a_k, p_k, n_k}$ be defined as above then

$$\begin{aligned} |S_{p_k+n_k-1} f(0) - S_{p_k-1} f(0)| &= |\sum_{p_k \leq |j| \leq p_k+n_k-1} \hat{f}(m)| \\ &= |2 \sum_{n=0}^{n_k-1} \frac{1}{2k^2(n_k - j)}| = \frac{1}{k^2} \sum_{n=0}^{n_k-1} \frac{1}{(n_k - j)} \\ &= \frac{1}{k^2} \sum_{n=1}^{n_k} \frac{1}{(j)} \geq \frac{1}{k^2} \log(n_k) = \log 2 \end{aligned}$$

Hence Fourier series at point 0 does not exist. □

Theorem 5.6.0.2. *There is a continuous function g whose Fouries series diverges at everywhere dense set of points.*

Proof. let f be a function like in the above theorem

let $\{x_i\}_{i \in \mathbb{N}}$ be an everywhere dense subset of $\mathbb{T}$ having rational order.

Then if $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be such that for $\varepsilon_i > 0$, $\sum_{i \in \mathbb{N}} \varepsilon_i < \infty$ for all i, then the sum $\sum_{i=1}^{\infty} \varepsilon_1 f(x - x_i)$ converges uniformly to a continuous function g.

This sum of the coefficient diverges at point x_i hence we get the result. □

5.7 ALMOST EVERYWHERE CONVERGENCE IN L^P SPACE

Definition 5.7.0.1. Let f $\in L^1(\mathbb{T})$ the maximal operator M is defined by

$$Mf(x) = \sup_{N\in\mathbb{N}} |S_N f(x)|$$

Then we have the theorem:

Let $1 < p < \infty$ then there is a constant C for which

$$\|Mf\|_p \le C\|f\|_p \ \forall f \in L^p(\mathbb{T})$$

Theorem 5.7.0.2. *(Markov's inequality)*
Let $f \in L^p(\mathbb{T})$ *and* $\lambda > 0$ *let* $E_\lambda = \{x : |f(x)| \ge \lambda\}$ *then* $m(E_\lambda) \le \frac{\|f\|_p^p}{\lambda^p}$

Proof. We estimate
$\|f\|_p^p = \int_{\mathbb{T}} |f|^p \ge \int_{E_\lambda} |f|^p \ge \int_{E_\lambda} \lambda^p \ge \lambda^p.m(E_\lambda)$
Hence we get the result. □

Theorem 5.7.0.3. *(Carleson's Theorem)*
Let $f \in L^p(\mathbb{T})$ *, where* $1 < p < \infty$ *the* $S_N f$ *converges to f almost everywhere.*

Proof. Let g be a trigonometric polynomial then $S_N g = g$ for large enough N . then for any x we have

$$\begin{aligned}\limsup_N |S_N f(x) - f(x)| &= \limsup_N |S_N f(x) - S_N g(x) + g(x) - f(x)| \\ &\leq \limsup_N |S_N (g-f)(x)| + |g(x) - f(x)| \\ &\leq M(f-g)(x) + |g(x) - f(x)|\end{aligned}$$

For any $\varepsilon > 0$

Let $D_\varepsilon = \{x : \limsup_N |S_N f(x) - f(x)| \geq \varepsilon\}$

$E_\varepsilon = \{x : |M(f-g)(x) \geq \frac{\varepsilon}{2}\}$

$F_\varepsilon = \{x : |g(x) - f(x)| \geq \frac{\varepsilon}{2}\}$

If $\limsup_N |S_N f(x) - f(x)| \geq \varepsilon$ then $M(f-g)(x) \geq \frac{\varepsilon}{2}$ or $|g(x) - f(x)| \geq \frac{\varepsilon}{2}$

Hence $D_\varepsilon \subseteq E_\varepsilon \cup F_\varepsilon$ and $m(D_\varepsilon) \leq m(E_\varepsilon) + m(F_\varepsilon)$

By Markov's inequality

$m(E_\varepsilon) \leq \frac{2^p}{\varepsilon^p} \|M(f-g)\|_p^p \leq \frac{2^p}{\varepsilon^p} \|(f-g)\|_p^p$

$m(F_\varepsilon) \leq \frac{2^p}{\varepsilon^p} \|(f-g)\|_p^p$

Hence $m(D_\varepsilon) \leq \frac{2^p}{\varepsilon^p} (C_p + 1)^p \|(f-g)\|_p^p$

since trigonometric polynomials are dense in L^p it follows that $m(D_\varepsilon) = 0 \ \forall \varepsilon$

thus

$m(x : \limsup_N |S_N f(x) - f(x)| \geq 0)$

$= m(\bigcup_{k=1}^{\infty} \{x : \limsup_N |S_N f(x) - f(x)| \geq \frac{1}{k}\}) = m(\bigcup_{k=1}^{\infty} D_{\frac{1}{k}}) = 0$

Hence the proof. $\square$

Chapter 6

ANALYSIS AND CONCLUSIONS

In the introductory chapter the formulas derived for Fourier coefficient is useful to find the Fourier series of periodic integrable functions. Properties of Fourier coefficients simplify theorems in later chapters.

In the second chapter we prove the Fubinis theorem for two variables which helps in computation of double integral using iterated integral of a Lebesgue integral function . We can interchange the order of iterated integration using Fubinis theorem.

In the third chapter we define convolution of periodic integrable functions. Convolution of periodic integral functions have distributive, commutative, associative properties. Convolution of two periodic integrable function is continuous and Fourier coefficient of convolution of the two periodic integrable functions is product of the Fourier coefficient of first function and the Fourier coefficient of the second function.

We also prove that if we have an integrable function then the function can be approximated to a continuous function

In the fourth chapter we prove that Fourier series of trigonometric polynomials converge back to itself, continuous functions are dense in L^p spaces, Dirichlet kernel is not a approximation to the identity but Fejer kernel is approximation to the identity which is very important result as using it we prove trigonometric functions are dense in L^p spaces, Riemann-Lebesgue lemma and uniqueness theorem.

Also, If the absolute sum of Fourier coefficient is finite then Fourier series of a function converges to the function, which is a sufficient condition for almost everywhere pointwise convergence.

In the fifth chapter we prove Holders inequality and use it to prove that inner product of L^2 space is defined and then prove L^2 is a Hilbert space. If f is a function in L^2 then Fourier series of the function converges to the function.

We conclude that if the supremum of the operator norm of partial sum where the operator of partial sums is from L^p to L^p is finite then function in L^p approximates to the a Fourier series and vice versa.

Next we defined Hilbert transforms using it we get convergence of Fourier series of a function in L^p to the function in L^p space.If a function satisfies the Lip-α condition then partial sums of Fourier series converges uniformly to the function. Taking an example we show that there is a function which is continuous that cannot converge to Fourier series of the function. Lastly we have if a function in L^p then partial sums of the Fourier series converges to the function almost everywhere.

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