A Non-linear Extension of Generalised Fibonacci Sequence

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by

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "A Non-linear Extension of Generalised Fibonacci Sequence" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Manvendra Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "A Non-linear Extension of Generalised Fibonacci Sequence" is a bonafide work carried out by Mr. Saish Savalaram Kanolkar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " A Non-linear Extension of Generalised Fibonacci Sequence." This survey is divided into seven chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of the Fibonacci sequence and the history of Number theory.

SECOND CHAPTER:

This chapter deals with extension of Fibonacci sequence P_n. In this topic we have studied identities, divisibility properties. We have also determined congruence properties of extended sequence P_n.

THIRD CHAPTER:

This chapter deals with extension of Lucas sequence T_n. In this topic we have obtained various divisibility and gcd properties. We have also determined cycles of T_n mod m and a pisano period table of L_n mod m. Also obtained some properties related to pisano period and worked on some generalised formula. Besides this obtained some relations between P_n and T_n.

FOURTH CHAPTER:

This chapter deals with extension of pell sequence D_n. In this topic we have obtained a non-linear second order recurrence relation. Also worked on recurrence relation and gcd properties of some other generalisation of Fibonacci.

FIFTH CHAPTER.

This chapter deals with difference relation to the sequences of k-Fibonacci numbers. In this chapter we learn various properties related to k-Fibonacci difference sequence. We also find formulas for the sum of the elements of these new sequences as well as their generating functions. Finally, we study the k–Fibonacci Newton polynomial interpolation.

SIXTH CHAPTER.

In this chapter we learn Binet formula and identities of k-Fibonacci numbers.

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First and foremost, I would like to express my gratitude to my Mentor, Dr. Manvendra Tamba, who was a continual source of inspiration. He pushed me to think imaginatively and urged me to do this homework without hesitation. His vast knowledge, extensive experience, and professional competence in Number Theory enabled me to successfully accomplish this project. This endeavour would not have been possible without his help and supervision.

ABSTRACT

Fibonacci numbers or Fibonacci sequence is among the most popular numbers or sequence in mathematics. One of the important features arising from the Fibonacci sequence is the Golden Ratio. It is the ratio of the consecutive numbers in the Fibonacci sequence which converges to 1.61803398875. In this paper, a new extension of Fibonacci sequence which yields a non-linear second order recurrence relation is defined. Some identities, divisibility and congruence properties for the new sequence is obtained. We obtain general formulas to find any term of the i^{th} k–Fibonacci difference sequence from the initial k–Fibonacci numbers.

The main aim of this article is, in general to prove some basic results related to generalised Fibonacci sequence and to generalise some identities and properties of the same. Besides this we do find out properties related to pisano periods of sequences discussed in the paper.

Keywords: Fibonacci sequence; lucas sequence; pell sequence; generalised fibonacci sequence; non-linear recurrence relation; Congruence and gcd properties; Pisano period

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Notations and Abbreviations

ϕ	Eulers phi function
Δ	Difference operator
$\pi(m)$	Pisano period of F_n mod m
$\eta(m)$	Pisano period of L_n mod m
$\nu(m)$	Pisano period of C_n mod m
p(m, G_n)	Pisano period of Generalised fibonacci number
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$b(m)$	Base length of extended sequences
$t(m)$	Tail period of extended sequences
$M_n = 2^n - 1$	Mersenne number
(x, y)	Gcd of x and y

Chapter 1

INTRODUCTION

Number theory (or arithmetic or higher arithmetic in older usage) is a branch of pure mathematics devoted primarily to the study of the integers and arithmetic functions. German mathematician Carl Friedrich Gauss (1777–1855) said, "Mathematics is the queen of the sciences—and number theory is the queen of mathematics." Number theorists study prime numbers as well as the properties of mathematical objects constructed from integers (for example, rational numbers), or defined as generalizations of the integers (for example, algebraic integers).

The main goal of number theory is to discover interesting and unexpected relationships between different sorts of numbers and to prove that these relationships are true.

In mathematics, the Fibonacci sequence is a sequence in which each number is the sum of the two preceding ones. The sequence commonly starts from 0 and 1. Applications of Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings, such as branching in trees, the arrangement of leaves on a stem, the fruit sprouts of a pineapple, the flowering of an artichoke, and the arrangement of a pine cone's bracts,

though they do not occur in all species.
Fibonacci numbers are also strongly related to the golden ratio: Binet's formula expresses the n-th Fibonacci number in terms of n and the golden ratio, and implies that the ratio of two consecutive Fibonacci numbers tends to the golden ratio as n increases.
The different generalisation of Fibonacci sequence is a Lucas sequence which is an integer sequence named after the mathematician François Édouard Anatole Lucas (1842–1891), Lucas numbers and Fibonacci numbers form complementary instances of Lucas sequences.This sequence also has a variety of relationships with the Fibonacci numbers, like the fact that adding any two Fibonacci numbers two terms apart in the Fibonacci sequence results in the Lucas number in between.
Similarly, we have Pell numbers which may be calculated by means of a recurrence relation similar to that for the Fibonacci numbers, and sequence of numbers grow exponentially, proportionally to powers of the silver ratio $1+\sqrt{2}$. As well as being used to approximate the square root of two, Pell numbers can be used to find square triangular numbers, to construct integer approximations to the right isosceles triangle, and to solve certain combinatorial enumeration problems.

Chapter 2

FIBOSENNE SEQUENCE P_n

2.1 Basic definitions

The well known Fibonacci sequence F_n is defined by

$$F_0 = 0, F_1 = 1, \ \ F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0$$

In this note, we present yet another extension of F_n and call it Fibosenne sequence $\{P_n\}$ (see [5]) defined by

$$P_n = 2^{F_n} - 1, \ \ n \geq 0$$

where F_n is n^{th} Fibonacci number. We call P_n the n^{th} Fibosenne number in view of its form like Mersenne number M_n. It is clear that $P_n = M_{F_n}$.We shall establish various relations for P_n in line with those of F_n. We shall also study some congruence properties of P_n.

2.2 Identities, divisibility and gcd properties of $\{P_n\}$

Proposition 2.2.0.1. *(see [5])*

1. For $n \geq 2$, $\log_2(1+P_n) = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$,

Where α, β are the roots of the equation $x^2 = x+1$

Proof. First we prove the Binet's formula for F_n :

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

To derive a general formula for the Fibonacci numbers we can look at the interesting quadratic $x^2 - x - 1 = 0$

Roots of the above equation are given by $x = \frac{1 \pm \sqrt{5}}{2}$

The quadratic equation is also written as $x^2 = x+1$

$x^1 = x+0 = F_1x + F_0$

$x^2 = x+1 = F_2x + F_1$

$x^3 = xx^2 = x(x+1) = x^2 + x = x+1+x$

$\therefore\ x^3 = 2x+1 = F_3x + F_2$

$x^4 = xx^3 = x(2x+1) = 2x^2 + x = 2(x+1) + x$

$\therefore\ x^4 = 3x+2 = F_4x + F_3$

$x^5 = xx^4 = x(3x+2) = 3x^2 + 2x = 3x+3+2x$

$\therefore\ x^5 = 5x+3 = F_5x + F_4$

$x^6 = xx^5 = x(5x+3) = 5x^2 + 3x = 5(x+1) + 3x$

$\therefore\ x^6 = 8x+5 = F_6x + F_5$

$\vdots$

Continuing in the same manner, we get

$x^n = F_nx + F_{n-1}$

Let $\sigma = \frac{1+\sqrt{5}}{2}$ & $\tau = \frac{1-\sqrt{5}}{2}$

since σ & τ both satisfy roots of quadratic equation $x^2 = x+1$,

they both must satisfy $x^n = F_nx + F_{n-1}$

i.e. $\sigma^n = F_n\sigma + F_{n-1}$ and $\tau^n = F_n\tau + F_{n-1}$

by using above equations we get,

$$\begin{aligned}
\sigma^n - \tau^n &= F_n(\sigma - \tau) \\
&= F_n\left\{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right\} \\
\Rightarrow F_n &= \frac{1}{\sqrt{5}}(\sigma^n - \tau^n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}
\end{aligned}$$

Next,

$$\begin{aligned}
\log_2(1+P_n) &= \log_2(2^{F_n}) = F_n \\
&= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \\
&= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)
\end{aligned}$$

where α & β are the roots of the equation $x^2 = x + 1$

□

2. For $n \geq 0$, P_n satisfies the non linear second order recurrence relation $P_{n+2} = P_{n+1} + P_n + P_{n+1}P_n$ with initial conditions $P_0 = 0$ & $P_1 = 1$

Proof.

$$\begin{aligned}
P_{n+2} &= 2^{F_{n+2}} - 1 = 2^{F_{n+1}+F_n} - 1 \\
&= 2^{F_{n+1}+F_n} - 2^{F_{n+1}} + 2^{F_{n+1}} - 1 \\
&= 2^{F_{n+1}}(2^{F_n} - 1) - (2^{F_n} - 1) + (2^{F_n} - 1) + (2^{F_{n+1}} - 1) \\
&= (2^{F_{n+1}} - 1)(2^{F_n} - 1) + (2^{F_n} - 1) + (2^{F_{n+1}} - 1) \\
&= P_{n+1} + P_n + P_{n+1}P_n
\end{aligned}$$

□

3. For $n \geq 1$,

$$P_{-n} = \begin{cases} P_n & \text{if} \quad \text{n is odd} \\ \frac{-P_n}{1+P_n} & \text{if} \quad \text{n is even} \end{cases}$$

Proof. We have $F_{-n} = (-1)^{n+1}F_n$

Now, $P_{-n} = 2^{F_{-n}} - 1 = 2^{(-1)^{n+1}F_n} - 1$

If n is odd, $P_{-n} = 2^{F_n} - 1 = P_n$

If n is even, $P_{-n} = 2^{-F_n} - 1 = \frac{1}{2^{F_n}} - 1 = \frac{1-2^{F_n}}{2^{F_n}} = \frac{-P_n}{1+P_n}$ □

Proposition 2.2.0.2.

1. for $n \geq 2$,

$$1 + P_{n+2} = \frac{(1+P_n)^3}{(1+P_{n-2})} \qquad \text{(see [5])}$$

Proof. We have, $F_{n+2} + F_{n-2} = 3F_n$

Now,

$$\begin{aligned} (1+P_{n+2})(1+P_{n-2})(1+P_n) &= 2^{F_{n+2}}2^{F_{n-2}}2^{F_n} \\ &= 2^{F_{n+2}+F_{n-2}+F_n} = 2^{4F_n} \\ &= (1+P_n)^4 \\ \Rightarrow \quad 1 + P_{n+2} &= \frac{(1+P_n)^3}{(1+P_{n-2})} \end{aligned}$$

□

2. $\forall\, n \geq 2, \quad 1 + P_{n+1} = \frac{(1+P_n)^2}{(1+P_{n-2})}$ (see [5])

Proof. We have $1 + P_{n+2} = (1+P_n)(1+P_{n+1})$ (i)

Also by Proposition 2.2.0.2 (1) , $1 + P_{n+2} = \frac{(1+P_n)^3}{(1+P_{n-2})}$ (ii)

Hence using (i) & (ii) we get, $(1+P_n)(1+P_{n+1}) = \frac{(1+P_n)^3}{1+P_{n-2}}$

$$\Rightarrow (1+P_{n+1}) = \frac{(1+P_n)^2}{1+P_{n-2}} \qquad \square$$

3. For $m,n \geq 1$, $\quad P_{m+n} = \left\{(1+P_m)^{F_{n+1}}(1+P_n)^{F_{m-1}}\right\} - 1$ (see [5])

Proof. First we prove that $F_{m+n} = F_mF_{n+1} + F_nF_{m-1}$

For $n = 0$, $F_{m+0} = F_m = F_mF_{0+1} + F_0F_{m-1}$

Assume that the result holds for all $n \ni 1 \leq n \leq k$

Next for $n = k+1$,

$$\begin{aligned} F_{m+n} = F_{m+(k+1)} &= F_{m+k} + F_{m+k-1} \\ &= F_mF_{k+1} + F_{m-1}F_k + F_mF_k + F_{m-1}F_{k-1} \text{ (by induction hypothesis)} \\ &= F_m(F_{k+1}+F_k) + F_{m-1}(F_k+F_{k-1}) \\ &= F_mF_{k+2} + F_{m-1}F_{k+1} \end{aligned}$$

Hence the result true $\forall\, n \geq 0$

So next, $1+P_{m+n} = 2^{F_{m+n}} = 2^{F_mF_{n+1}+F_{m-1}F_n} = \left(2^{F_m}\right)^{F_{n+1}}\left(2^{F_n}\right)^{F_{m-1}}$

$$\Rightarrow \quad P_{m+n} = \left\{(1+P_m)^{F_{n+1}}(1+P_n)^{F_{m-1}}\right\} - 1 \qquad \square$$

4. For $m,n \geq 1$, (see [5])

$$P_{mn} = \begin{cases} \left\{\prod_{k=1}^{\frac{mn}{2}}(1+P_{2k-1})\right\} - 1 & \text{if } mn \text{ is even, } mn \geq 2 \\ \left\{(1+P_1)\prod_{k=0}^{\frac{mn-1}{2}}(1+P_{2k})\right\} - 1 & \text{if } mn \text{ is odd} \end{cases}$$

Proof. We have identity for F_n,

$$F_{mn} = \begin{cases} \sum_{k=1}^{\frac{mn}{2}} F_{2k-1} & \text{if } mn \text{ is even, } mn \geq 2 \\ 1 + \sum_{k=0}^{\frac{mn-1}{2}} F_{2k} & \text{if } mn \text{ is odd} \end{cases}$$

Using above identity we get the desired result. □

5. For $n \geq 0$, $\quad (1+P_{n+1})^{F_n} = \prod_{k=1}^{n}(1+P_k)^{F_k}$

Proof. First we prove that $F_nF_{n+1} = \sum_{k=1}^{n} F_k^2 \quad$ (see [6])

$$\begin{aligned} F_nF_{n+1} &= F_n(F_n+F_{n-1}) = F_n^2 + F_nF_{n-1} \\ &= F_n^2 + (F_{n-1}+F_{n-2})F_{n-1} = F_n^2 + F_{n-1}^2 + F_{n-2}F_{n-1} \\ &\vdots \\ F_nF_{n+1} &= \sum_{k=1}^{n} F_k^2 \end{aligned}$$

$$\begin{aligned} (1+P_{n+1})^{F_n} &= (2^{F_{n+1}})^{F_n} = 2^{F_{n+1}F_n} = 2^{\sum_{k=1}^{n} F_k^2} = \prod_{k=1}^{n}(2^{F_k})^{F_k} \\ &= \prod_{k=1}^{n}(1+P_k)^{F_k} \end{aligned}$$

□

6. For $n \geq 1$, $\quad \prod_{k=1}^{n}(1+P_k)^k = \frac{(1+P_{n+2})^n(1+P_3)}{(1+P_{n+3})}$

Proof. We have $\sum_{k=1}^{n} kF_k = nF_{n+2} - F_{n+3} + F_3 \quad$ (see [6])

$$\begin{aligned} \prod_{k=1}^{n}(1+P_k)^k = \prod_{k=1}^{n}(2^{F_k})^k &= \prod_{k=1}^{n} 2^{kF_k} \\ &= 2^{\sum_{k=1}^{n} kF_k} = 2^{nF_{n+2}-F_{n+3}+F_3} = \frac{(2^{F_{n+2}})^n 2^{F_3}}{2^{F_{n+3}}} \\ &= \frac{(1+P_{n+2})^n(1+P_3)}{(1+P_{n+3})} \end{aligned}$$

□

Proposition 2.2.0.3.

1. For $n \geq 0$, $(P_n, P_{n+1}) = 1 \quad$ (see [5])

Proof. Let $d = (P_n, P_{n+1})$ then $d|P_n$ and $d|P_{n+1}$ (i)

but $P_{n+1} = P_n + P_{n-1} + P_n P_{n-1}$ (ii)

$\Rightarrow\ d|P_{n-1}$ by (i) & (ii), Next $d|P_n$ and $d|P_{n-1} \Rightarrow d|P_{n-2}$

Continuing in the same manner we get

$d|P_1 \quad \Rightarrow\ d|1, \quad \therefore d = 1$

hence $(P_n, P_{n+1}) = 1$ □

2. For $m,n \in \mathbb{N}$, If $n|F_n$ then $M_n|P_m$ (see [5])

Proof. First we prove that $n|r \Rightarrow M_n|M_r$

Since $n|r \Rightarrow \exists\, k \in \mathbb{Z} \ni r = kn$

$$\begin{aligned} M_r &= 2^r - 1 = 2^{kn} - 1 = (2^n)^k - 1 \\ &= (2^n - 1)(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 1) \\ &= M_n(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 1) \end{aligned}$$

$\Rightarrow M_n|M_r$

Now $n|F_m \Rightarrow M_n|M_{F_m} \Rightarrow M_n|P_m$ □

3. For $m,n \in \mathbb{N}$, If $n|m$ then $P_n|P_m$ (see [5])

Proof. To prove that $F_n|F_{rn}\ \forall\, r \in \mathbb{N}$

for $r = 1$, $F_n|F_n$ and for $r = 2$, $F_{2n} = F_{n+n} = F_{n-1}F_n + F_nF_{n+1} \Rightarrow F_n|F_{2n}$

Assume that $F_n|F_{(s-1)n}$ where $s > 2$

$F_{sn} = F_{sn-n+n} = F_{(s-1)n+n} = F_{(s-1)n-1}F_n + F_{(s-1)n}F_{n+1}$

Since $F_n|F_n$ and $F_n|F_{(s-1)n}$ by induction hypothesis

hence $F_n|F_{sn} \quad \therefore F_n|F_{rn}\ \forall\, n \in \mathbb{N}$

Since $n|m\ \exists\, k \in \mathbb{Z} \ni m = nk$ then $F_n|F_{nk} = F_m$

Now, $n|m \Rightarrow F_n|F_m \Rightarrow M_{F_n}|M_{F_m} \Rightarrow P_n|P_m$ □

4. For $m \in \mathbb{N}$, If $p > 2$ is a prime $\ni$ $(p-1)|F_m$ then $p|P_m$ (see [5])

Proof. Now Since $p > 2, \Rightarrow (p,2) = 1$

$\Rightarrow 2^{p-1} \equiv 1(\text{mod } p)$ by Euler's identity (i)

Since $(p-1)|F_m \; \exists \; r \in \mathbb{Z} \ni F_m = (p-1)r$

$P_m = 2^{F_m} - 1 = (2^{p-1})^r - 1 \equiv 0(\text{mod } p)$ by (i) $\Rightarrow p|P_m$ □

5. For $m,n \geq 0$, $P_{(m,n)} = (P_m, P_n)$

Proof. First we prove that $F_{(m,n)} = (F_m, F_n) \; for \; m,n \geq 0$ (see [6])

Let $d = (m,n) \Rightarrow d|m$ and $d|n \Rightarrow m = m_1 d$ and $n = n_1 d$ where $m_1, n_1 \in \mathbb{Z}$

$\because d = (m,n) \Rightarrow d = mx + ny$ for some $x, y \in \mathbb{Z}$

Now, $F_d | F_{m_1 d} = F_m \; \& \; F_d | F_{n_1 d} = F_n$

Suppose $t|F_m \; \& \; t|F_n$

$F_d = F_{mx+ny} = F_{mx-1} F_{ny} + F_{mx} F_{ny+1}$

since $t|F_n$ and $F_n|F_{ny} \Rightarrow t|F_{mx-1}F_{ny}$ also $t|F_m$ and $F_m|F_{mx} \Rightarrow t|F_{mx}F_{ny+1}$

$\therefore t|F_d$ i.e. $t|F_{(m,n)}$

hence $(F_m, F_n) = F_{(m,n)}$

$$\begin{aligned} P_{(m,n)} &= 2^{F_{(m,n)}} - 1 = 2^{(F_m, F_n)} - 1 \\ &= (2^{F_m} - 1, 2^{F_n} - 1) = (P_m, P_n) \end{aligned}$$

□

2.3 Some Congruence properties of $\{P_n\}$

In this section, we present some congruence properties of Fibosenne sequence. The following table gives Fibosenne numbers $0 \leq n \leq 12$. (see [5])

A look at the table 1 reveals that the last digit of P_n follow the pattern $0113715, 113715, \ldots$

Given $m \geq 2$, we have $P_0 \equiv 0(\text{mod } m)$ and $P_1 \equiv 1(\text{mod } m)$. For $n \geq 2$, the congruence properties of P_n can be found by using the recurrence relation obtained in Proposition 2.2.0.1 (2). For $2 \leq m \leq 10$ we show the cycles of $P_n(\text{mod } m)$ in Table 2. (see [5])

Table 1: $\mathbf{P_n}$, $\mathbf{0 \leq n \leq 12}$ (see [5])

n	P_n
0	0
1	1
2	1
3	3
4	7
5	31
6	255
7	8191
8	2097151
9	17179869183
10	36028797018963967
11	618970019642690137449562111
12	22300745198530623141535718272648361505980415

Table 2: Cycles of $\mathbf{P_n(mod\ m), b(m), t(m), 2 \leq m \leq 10}$ (see [5])

m	$P_n(\text{mod } m)$	Base length $b(m)$	Tail period $t(m)$
2	**0,** 1,1,1,1,1,...	1	1
3	0,1,1, 0,1,1, 0,1,1,...	0	3
4	**0,1,1,** 3,3,3,3,3,...	3	1
5	0,1,1,3,2,1, 0,1,1,3,2,1,...	0	6
6	**0,** 1,1,3, 1,1,3, 1,1,3,...	1	3
7	0,1,1,3,0,3,3,1, 0,1,1,3,0,3,3,1,...	0	8
8	**0,1,1,3,** 7,7,7,7,7,7,...	4	1
9	0,1,1,3,7,4,3,1,7,6,1,4,0,4,4,6,7,1,6,4,7,3,4,1,...	0	24
10	**0,** 1,1,3,7,1,5, 1,1,3,7,1,5,...	1	6

Here we observe that P_n modulo m show some pattern. After initial terms which we call "Base" there is periodic repetition of terms. We call it "Tail". This suggests that there is a pattern for congruence residues, which is different from that of Fibonacci sequence. For all sequences, which show such a pattern for congruence modulo m, we define the Base length as the number of terms in the base and denote it by $b(m)$, the base length of the sequence modulo m. Similarly, we define the Tail Period as the minimum number of terms repeating in the tail of the sequence modulo m and denote it by $t(m)$. For example when $m = 4$, the sequence is $0,1,1,3,3,\ldots$ Here $0,1,1$ is the base and $3,3,3,\ldots$ the tail. There are three terms in the base and hence $b(4) = 3$. In the tail number 3 repeats and so $t(4) = 1$. In the second column of Table 2, base is shown bold text. If there are no initial terms forming base, then $b(m) = 0$. It is noteworthy that for $m = 11$, $b(11) = 46$ and $t(11) = 12$.

From Table 2, for $m = 10$ the following result follows immediately.

Proposition 2.3.0.1. *(see [5])*

1. For $u \geq 0$,

$$P_{6u+r} \equiv \left\{ \begin{array}{lll} 1 (\mathrm{mod}\ 10) & if & r = 1,2,5 \\ 3 (\mathrm{mod}\ 10) & if & r = 3 \\ 5 (\mathrm{mod}\ 10) & if & r = 6 \\ 7 (\mathrm{mod}\ 10) & if & r = 4 \end{array} \right\}$$

2. For $1 \leq r \leq 4$ and $u \geq 0$, $\quad P_{6u+r} \equiv P_r (\mathrm{mod}\ 5)$

3. For $1 \leq r \leq 3$ and $u \geq 0$, $\quad P_{3u+r} \equiv P_r (\mathrm{mod}\ 6)$

Definition 2.3.0.2. For $n \geq 1$, let $[n]_F$ be the largest in $\mathbb{N} \ni F_{[n]_F} \leq n$. (see [5])

For example, $[1]_F = 2, [2]_F = 3, [5]_F = 5$

Proposition 2.3.0.3. *For $n \geq 1$, $P_k \equiv 2^n - 1 (mod\ \ 2^n)$, for $k \geq [n]_F$. In this case $b(2^n) = [n]_F$ and $t(2^n) = 1$. (see [5])*

Proof. For $k \geq [n]_F$.

$$P_k = 2^{F_k} - 1 = 2^n(2^{F_k - n}) - 1, \text{ as } F_k - n \geq 0$$
$$\equiv -1(\text{mod } 2^n)$$
$$\equiv 2^n - 1(\text{mod } 2^n)$$

□

Given $m \geq 2$, considering the Fibonacci numbers F_n and the smallest residues $R_n(m)$ of the terms modulo m. It was observed that the sequence $R_n(m)$ repeats after $\pi(m)$ terms as shown in below table:

Table 3: Values of $\pi(\mathbf{m})$ for $\mathbf{2 \leq m \leq 30}$ (see [5])

m		2	3	4	5	6	7	8	9	10
π(m)		3	8	6	20	24	16	12	24	60
m	11	12	13	14	15	16	17	18	19	20
π(m)	10	24	28	48	40	24	36	24	18	60
m	21	22	23	24	25	26	27	28	29	30
π(m)	16	30	48	24	100	84	72	48	14	120

$\pi(m)$ is called pisano period of F_n mod m. From the definition of $\pi(m)$ we obtained a lemma given below:

Lemma 2.3.0.4. *For $m \geq 2$ and $u \geq 0$, $F_{\pi(m)u+r} \equiv F_r(mod\ m)$ where $0 \leq r < \pi(m)$ (see [5])*

Using this lemma we prove the following:

Proposition 2.3.0.5. *For $m \geq 2$ and $u \geq 0$, we have $F_{\pi(m)u+r} \equiv F_r(mod\ m)$, where $0 \leq r < \pi(m)$ and $2^m \equiv 1(mod\ k)$ for some $k > 1$, then $P_{\pi(m)u+r} \equiv P_r(mod\ k)$ (see [5])*

Proof. We have

$$\begin{aligned} P_{\pi(m)u+r} &= 2^{F_{\pi(m)u+r}} - 1 \\ &= 2^{mj+F_r} - 1, \text{ for some } j \geq 0 \\ &= (2^m)^j 2^{F_r} - 1 \\ &\equiv 2^{F_r} - 1 (\text{mod } k) \equiv P_r (\text{mod } k) \end{aligned}$$

□

Notice that in Proposition 2.3.0.5, $2^m \equiv 1(\text{mod } k)$ suggests that k is odd. The case when k is a power of 2 is dealt in Proposition 2.3.0.3, when k is a even but not a power of 2, we have the following:

Proposition 2.3.0.6. *For $m \geq 2$ and $u \geq 0$, if $F_{\pi(m)u+r} \equiv F(r)(mod\, m)$, where $0 \leq r < \pi(m)$ and $2^m \equiv 1(mod\, k)$ for some $k > 1$, then for $s \geq 1$, $P_{\pi(m)u+r} \equiv N_r(mod\ \ 2^s k)$, where N_r is independent of u. (see [5])*

Proof. We see for $F_{\pi(m)u+r} - s \geq 0$ that

$$\begin{aligned} P_{\pi(m)u+r} &= 2^{F_{\pi(m)u+r}} - 1 \\ &= 2^{mj+F_r} - 1, \text{ for some } j \geq 0 \\ &= 2^s(2^{mj+F_r-s}) - 1 \\ &= 2^s(kq + r_1) - 1, \text{where } r_1 \text{ is the remainder when } 2^{mj+F_r-s} \text{ is divisible by } k \\ &= 2^s kq + 2^s r_1 - 1 \\ &\equiv 2^s r_1 - 1 (\text{mod } 2^s k) \end{aligned}$$

□

We have the following corollaries:

Corollary 2.3.0.7. *For $n \geq 0$, we have (see [5])*

$$P_{3n+r} \equiv \left\{ \begin{array}{lll} 0(mod\,3) & if & r=0 \\ 1(mod\,3) & if & r=1,2 \end{array} \right\}$$

Proof. Take $\pi(2)=3$ and $k=3$ in Proposition 2.3.0.5 □

Corollary 2.3.0.8. *For $n \geq 0$, (see [5])*

$$P_{6n+r} \equiv \left\{ \begin{array}{lll} 0(mod\,15) & if & r=0 \\ 1(mod\,15) & if & r=1,2,5 \\ 3(mod\,15) & if & r=3 \\ 7(mod\,15) & if & r=4 \end{array} \right\}$$

Proof. As $\pi(4)=6$, $F_{6u+r} \equiv (\mathrm{mod}\ 4)$ so taking $k=15$ and making use of Proposition 2.3.0.5 we get desired result. □

Corollary 2.3.0.9. *For $n \geq 0$, (see [5])*

$$P_{20n+r} \equiv \left\{ \begin{array}{lll} 0(mod\,31) & if & r=0,5,10,15 \\ 1(mod\,31) & if & r=1,2,8,19 \\ 3(mod\,31) & if & r=3,14,16,17 \\ 7(mod\,31) & if & r=4,6,7,13 \\ 15(mod\,31) & if & r=9,11,12,18 \end{array} \right\}$$

Proof. As $\pi(5)=20$, taking $k=31$ and making use of Proposition 2.3.0.5 we get desired result. □

We have the following results on $t(m)$, the tail period.

Proposition 2.3.0.10. *(see [5])*

1. If $m > 2$ is an odd integer then $b(m) = 0$ and $t(m)|\pi(\phi(m))$ where ϕ is Eulers function. In particular, if $p > 2$ is a prime then $t(p)|\pi(p-1)$.

Proof. Now, $m > 2$ and m is odd then $\phi(m) \geq 2$
by using Lemma 2.3.0.4, $F_{\pi(\phi(m))u+r} \equiv F_r (\text{mod } \phi(m))$ where $0 \leq r < \pi(\phi(m))$
$\Rightarrow \phi(m) | \left(F_{\pi(\phi(m))u+r} - F_r\right) \Rightarrow F_{\pi(\phi(m))u+r} - F_r = \phi(m)k$, where $k \in \mathbb{Z}$

$$\begin{aligned} P_{\pi(\phi(m))u+r} &= 2^{F_{\pi(\phi(m))u+r}} - 1 = 2^{\phi(m)k+F_r} - 1 \\ &\equiv 2^{F_r} - 1 (\text{mod } m) \equiv P_r (\text{mod } m) \end{aligned}$$

Thus the pattern in $P_n(\text{mod } m)$ repeats after $\pi(\phi(m))$ terms. So $\pi(\phi(m))$ must be a multiple of $t(m)$. hence $t(m)|\pi(\phi(m))$
since the cycle repeats right from the begining, $b(m) = 0$ □

2. For $m \geq 1$, $t(2^m) = 1$

Proof. This result follows from Proposition 2.3.0.3 □

3. Let $m \geq 2$ be an integer $\ni 2^m \equiv 1 (\text{mod } k)$ then $t(k)|\pi(m)$

Proof. We use Lemma 2.3.0.4 so that
$F_{\pi(m)u+r} - F_r = mz$, where $z \in \mathbb{Z}$

$$\begin{aligned} P_{\pi(m)u+r} &= 2^{F_{\pi(m)u+r}} - 1 \\ &= 2^{mz+F_r} - 1 \\ &\equiv 2^{F_r} - 1 (\text{mod } k) \equiv P_r (\text{mod } k) \end{aligned}$$

Thus the pattern in $P_n(\text{mod } k)$ repeats after $\pi(m)$ terms. So $\pi(m)$ must be a multiple of $t(k)$. hence $t(k)|\pi(m)$ □

Using Proposition 2.3.0.10 (3) we have the following properties:

Proposition 2.3.0.11. *(see [5])*

1. For $u \geq 1$, if $m|M_{2^u}$ then $t(m)|3(2^{u-1})$

Proof. $\because m|M_{2^u} \Rightarrow 2^{2^u} \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(2^u)$
but by result in [6], $F_{3(2^{u-1})} \equiv 0(\text{mod } 2^u)$ and $F_{3(2^{u-1})+1} \equiv 1(\text{mod } 2^u)$
$\therefore\ \pi(2^u)|3(2^{u-1})$
hence $t(m)|3(2^{u-1})$ □

2. For $u \geq 1$, if $m|M_{5^u}$ then $t(m)|4(5^u)$

Proof. $\because m|M_{5^u} \Rightarrow 2^{5^u} \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(5^u)$
but by result in [6], $F_{4(5^u)} \equiv 0(\text{mod } 5^u)$ and $F_{4(5^u)+1} \equiv 1(\text{mod } 5^u)$
$\therefore\ \pi(5^u)|4(5^u)$
hence $t(m)|4(5^u)$ □

3. For $u \geq 1$, if r is the largest integer $\ni\ \pi(p^r) = \pi(p)$ and $m|M_{p^u}$ for some prime p then $t(m)|p^{u-r}\pi(p)\ \forall\ u > r$

Proof. $\because m|M_{p^u} \Rightarrow 2^{p^u} \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(p^u)$
but by result in [6], If q is the largest $\ni \pi(p^q) = \pi(p)$ then $\pi(p^s) = p^{s-q}\pi(p)\ \forall\ s > q$
hence $\pi(p^u) = p^{u-r}\pi(p)\ \forall\ u > r$
$\therefore\ t(m)|p^{u-r}\pi(p)\ \forall\ u > r$ □

4. If $p \neq 5$ is a prime and $m|M_p$ then $t(m)|p^2 - 1$

Proof. $\because\ m|M_p \Rightarrow\ 2^p \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(p)$
but by result in [6], if $p \neq 5$ is a prime then $\pi(p)|p^2-1$
$\therefore\ t(m)|p^2-1$ □

5. If p is a prime of the form $10k \pm 1$ and $m|M_p$ then $t(m)|p-1$

Proof. $\because\ m|M_p \Rightarrow\ 2^p \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(p)$
but by result in [6], if p is a prime of the form $10k \pm 1$ then
$F_{p-1} \equiv 0(\text{mod } p)$ and $F_p \equiv 1(\text{mod } p)$
$\Rightarrow\ \pi(p)|p-1$
$\therefore\ t(m)|p-1$ □

6. If p is a prime of the form $10k \pm 3$ and $m|M_p$ then $t(m)|2p+2$

Proof. $\because\ m|M_p \Rightarrow\ 2^p \equiv 1(\text{mod } m)$ then by Proposition 2.3.0.10 (3), $t(m)|\pi(p)$
but by result in [6], if p is a prime of the form $10k \pm 3$ then
$F_{p+1} \equiv 0(\text{mod } p)$ and $F_p \equiv -1(\text{mod } p)$
$F_{2p+2} = F_{2(p+1)} = F_{p+1}L_{p+1}$ and $F_{p+1} \equiv 0(\text{mod } p)\ \Rightarrow F_{2p+2} \equiv 0(\text{mod } p)$

$$\begin{aligned} F_{2p+3} &= F_{2p+2} + F_{2p+1} \equiv F_{2p+1}(\text{mod } p) = F_{p+(p+1)}(\text{mod } p) \\ &= (F_{p-1}F_{p+1} + F_pF_{p+2})(\text{mod } p) \equiv F_pF_{p+2}(\text{mod } p) \\ &= F_p(F_{p+1} + F_p)(\text{mod } p) = (F_pF_{p+1} + F_p^2)(\text{mod } p) \\ &\equiv (-1)^2(\text{mod } p) \equiv 1(\text{mod } p) \end{aligned}$$

hence $\pi(p)|2p+2, \quad \therefore\ t(m)|2p+2$ □

Chapter 3

LUCASENNE SEQUENCE T_n

3.1 Basic definitions

The well known Lucas sequence L_n is defined by

$$L_0 = 2, L_1 = 1, \;\; L_{n+2} = L_{n+1} + L_n \text{ for } n \geq 0$$

In this note, we present yet another extension of L_n and call it Lucasenne sequence $\{T_n\}$ defined by

$$T_n = 2^{L_n} - 1, \;\; n \geq 0$$

where L_n is n^{th} Lucas number. We call T_n the n^{th} Lucasenne number in view of its form like Mersenne number M_n. It is clear that $T_n = M_{L_n}$.We shall establish various relations for T_n in line with those of L_n. We shall also study some congruence properties of T_n

3.2 Identities, divisibility and gcd properties of T_n

Proposition 3.2.0.1.

1. For $n \geq 0$, $\log_2(1+T_n) = \alpha^n + \beta^n$ where α, β are roots of equation $x^2 = x+1$

Proof. First we prove the binet's formula for Lucas numbers (see [6])

$$L_n = \alpha^n + \beta^n \text{ where } \alpha, \beta \text{ are roots of equation } x^2 = x+1$$

For $n = 0$, $L_0 = \alpha^0 + \beta^0 = 1+1 = 2$
For $n = 1$, $L_1 = \alpha^1 + \beta^1 = \left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right) = 1$

Assume that the result holds for all $k < n+1$ where $k > 1$
i.e.$L_k = \alpha^k + \beta^k = \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^k$
Next, T.P.T. $L_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ &= \alpha^n + \beta^n + \alpha^{n-1} + \beta^{n-1}, \text{ by induction hypothesis} \\ &= \alpha^{n-1}(\alpha+1) + \beta^{n-1}(\beta+1) \\ &= \alpha^{n-1}\alpha^2 + \beta^{n-1}\beta^2, \quad \text{since } \alpha^2 = \alpha+1 \text{ and } \beta^2 = \beta+1 \\ &= \alpha^{n+1} + \beta^{n+1} \end{aligned}$$

hence $L_n = \alpha^n + \beta^n \ \forall\, n \geq 0$
Finally, $\log_2(1+T_n) = \log_2(2^{L_n}) = L_n = \alpha^n + \beta^n$ □

2. For $n \geq 0$, T_n satisfies the non linear second order recurrence relation $T_{n+2} = T_{n+1} + T_n + T_{n+1}T_n$ with initial conditions $T_0 = 3$ & $T_1 = 1$

Proof.

$$\begin{aligned} T_{n+2} &= 2^{L_{n+2}} - 1 = 2^{L_{n+1}+L_n} - 1 = 2^{L_{n+1}+L_n} - 2^{L_{n+1}} + 2^{L_{n+1}} - 1 \\ &= 2^{L_{n+1}}(2^{L_n} - 1) - (2^{L_n} - 1) + (2^{L_n} - 1) + (2^{L_{n+1}} - 1) \\ &= (2^{L_{n+1}} - 1)(2^{L_n} - 1) + (2^{L_n} - 1) + (2^{L_{n+1}} - 1) \\ &= T_{n+1} + T_n + T_{n+1}T_n \end{aligned}$$

□

3. For $n \geq 1$,

$$T_{-n} = \begin{cases} T_n & \text{if} \quad \text{n is even} \\ \frac{-T_n}{1+T_n} & \text{if} \quad \text{n is odd} \end{cases}$$

Proof. We prove that $L_{-n} = (-1)^n L_n$ (see [6])

$$L_{-n} = \alpha^{-n} + \beta^{-n} = \frac{\alpha^n + \beta^n}{(\alpha\beta)^n} = (-1)^n L_n \text{ where } \alpha, \beta \text{ are roots of eqn } x^2 = x + 1$$

Now, $T_{-n} = 2^{L_{-n}} - 1 = 2^{(-1)^n L_n} - 1$

If n is even, $T_{-n} = 2^{L_n} - 1 = T_n$

If n is odd, $T_{-n} = 2^{-L_n} - 1 = \frac{1}{2^{L_n}} - 1 = \frac{1-2^{L_n}}{2^{L_n}} = \frac{-T_n}{1+T_n}$ □

4. For $n \geq 1$, $\displaystyle\prod_{k=1}^{n}(1+T_k)^{L_k} = \frac{(1+T_{n+1})^{L_n}}{(1+T_0)}$

Proof. We have $\displaystyle\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - L_0$ (see [6])

$$\prod_{k=1}^{n}(1+T_k)^{L_k} = \prod_{k=1}^{n}(2^{L_k})^{L_k} = \prod_{k=1}^{n} 2^{L_k^2}$$

$$= 2^{\sum_{k=1}^{n} L_k^2} = 2^{L_n L_{n+1} - L_0}$$
$$= \frac{(2^{L_{n+1}})^{L_n}}{2^{L_0}} = \frac{(1+T_{n+1})^{L_n}}{1+T_0}$$

□

5. For $n \geq m \geq 1$,

$$(1+T_m)^{L_n} = \begin{cases} (1+T_{n+m})(1+T_{n-m}) & \text{if} \quad \text{n is even} \\ \frac{(1+T_{n+m})}{(1+T_{n-m})} & \text{if} \quad \text{n is odd} \end{cases}$$

Proof. The above result follows by identity, $L_m L_n = L_{n+m} + (-1)^n L_{n-m}$ (see [6]) □

Proposition 3.2.0.2.

1. $\forall\, n \geq 0, \quad (T_n, T_{n+1}) = 1$

Proof. Suppose $(T_n, T_{n+1}) > 1$

Then by Fundamental theorem of arithmetic, $\exists$ a prime $p \ni \ p|(T_n, T_{n+1})$

$\Rightarrow p|T_n$ and $p|T_{n+1}$

$\Rightarrow p|T_{n-1}$ since $T_{n+1} = T_{n-1} + T_n + T_{n-1}T_n$

$\because p|T_{n-1}$ and $p|T_n \Rightarrow p|T_{n-2}$

Continuing in the same manner we get,

$p|T_1$ i.e. $p|1$ which is $\Rightarrow\Leftarrow$

hence $(T_n, T_{n+1}) = 1$ □

2. Let n be an odd positive integer. If $\phi(n)|L_m \Rightarrow n|T_m$

Proof. Since n is an odd positive integer $\Rightarrow (n,2) = 1$

By Euler's identity, $2^{\phi(n)} \equiv 1 (\text{mod } n)$

Since $\phi(n)|L_m \; \exists\, z \in \mathbb{Z} \;\ni\; L_m = \phi(n)z$

$$\begin{aligned} T_m &= 2^{L_m} - 1 = 2^{\phi(n)z} - 1 \\ &= (2^{\phi(n)})^z - 1 \equiv 0 (\text{mod } n) \end{aligned}$$

□

3. Let $m,n \in \mathbb{N}$. If $n|m$ and $\frac{m}{n}$ is odd then $T_n|T_m$

Proof. We have $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = Q^n$ for $n \in \mathbb{N}$ where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
and $\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = AQ^n$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ $(\star)$

Also $A = 2Q - I$ ($\therefore$ A & Q commutes) and $A^2 = 5I$

Using eqn $(\star)$, we get Cassinis identity: $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$ (see [6])

To prove that $5 \nmid L_n \; \forall\, n \geq 0$

Clearly $5 \nmid L_0 = 2$.

Suppose $5|L_n$ for some $n \in \mathbb{N}$

$\Rightarrow 5|L_n^2$ and $5|5(-1)^{n+1} \Rightarrow 5|L_n^2 + 5(-1)^{n+1} = L_{n+1}L_{n-1}$ by Cassinis identity

$\Rightarrow 5|L_{n+1}L_{n-1} \Rightarrow 5|L_{n+1}$ or $5|L_{n-1}$

If $5|L_{n+1}$ and $5|L_n \Rightarrow 5|(L_{n+1}, L_n) = 1$ which is $\Rightarrow\Leftarrow$

If $5|L_{n-1}$ and $5|L_n \Rightarrow 5|(L_{n-1}, L_n) = 1$ which is $\Rightarrow\Leftarrow$

hence $5 \not| L_n \; \forall\, n \geq 0, \Rightarrow \; (5, L_n) = 1$

Let $\frac{m}{n} = k$ and k is odd so $k = 2r+1, r \geq 0$

Notice that $\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = AQ^n$ is diagonal matrix mod L_n

$$\begin{aligned} \begin{pmatrix} 5^r L_{nk+1} & 5^r L_{nk} \\ 5^r L_{nk} & 5^r L_{nk-1} \end{pmatrix} &= 5^r AQ^{nk} = (5I)^r AQ^{nk} \\ &= (A^2)^r AQ^{nk} = A^{2r+1}Q^{nk} \\ &= A^k Q^{nk} = (AQ^n)^k \end{aligned}$$

$\because AQ^n$ is diagonal matrix mod L_n $\Rightarrow (AQ^n)^k$ is diagonal matrix mod L_n

$\Rightarrow\ 5^r L_{nk} \equiv 0 (\text{mod } L_n)$ $\Rightarrow\ L_n | 5^r L_{nk}$

but $(5, L_n) = 1$ $\Rightarrow\ (5^r, L_n) = 1$ and $L_n | 5^r L_{nk}$

hence $L_n | L_{nk} = L_m$ by Euclid's lemma

Now, $n|m \ \Rightarrow\ L_n | L_m \ \Rightarrow\ M_{L_n} | M_{L_m} \ \Rightarrow\ T_n | T_m$ □

4. Let $m, n \in \mathbb{N}$. If $n|m$ and $\frac{m}{n}$ is even then $T_n | P_m$

Proof. We have $F_{2n} = L_n F_n$ (see [6]) and $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = Q^n$ where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
$Q^{2n} = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix} = \begin{pmatrix} F_{2n+1} & F_n L_n \\ F_n L_n & F_{2n-1} \end{pmatrix}$ $\Rightarrow\ Q^{2n}$ is diagonal matrix mod L_n
Let $k = \frac{m}{n}$, since k is even, $k = 2r$
$\begin{pmatrix} F_{nk+1} & F_{nk} \\ F_{nk} & F_{nk-1} \end{pmatrix} = Q^{nk} = Q^{2nr} = (Q^{2n})^r$ which is diagonal matrix mod L_n
$\Rightarrow F_m = F_{nk} \equiv 0 \text{ mod } L_n,$ hence $L_n | F_m$

Now, $n|m \ \Rightarrow\ L_n | F_m \ \Rightarrow\ M_{L_n} | M_{F_m} \ \Rightarrow\ T_n | P_m$ □

5. Let $m, n \in \mathbb{N}$ and $d = (m, n)$. If $\frac{m}{d}$ and $\frac{n}{d}$ both are odd

then $T_{(m,n)} = (T_m, T_n)$

Proof. By using the proof of Proposition 3.2.0.2 (3), we get

$L_d | L_{(\frac{m}{d})d} = L_m$ as $\frac{m}{d}$ is odd and $L_d | L_{(\frac{n}{d})d} = L_n$ as $\frac{n}{d}$ is odd

$\Rightarrow\ L_d | (L_m, L_n) \Rightarrow\ L_{(m,n)} | (L_m, L_n)$

Next, To show that $(L_m, L_n) | L_{(m,n)}$

If $(L_m, L_n) = 1$ then we are done

If $(L_m, L_n) > 1$

since $5 \nmid L_n,\ n \geq 0\ \Rightarrow (5, L_n) = 1 \Rightarrow (5^k, L_n) = 1$ for $k \geq 1$

since $d = (m, n) \Rightarrow (\frac{m}{d}, \frac{n}{d}) = 1$ $\Rightarrow \exists\, a, b \in \mathbb{Z} \ni (\frac{m}{d})a + (\frac{n}{d})b = 1$

$\Rightarrow\ am + bn = d$, here one of a, b must be odd and other must be even

$\therefore\ a + b = 2k + 1, k \geq 0$

$\begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix} = AQ^m$ is diagonal matrix mod (L_m, L_n)
$\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = AQ^n$ is diagonal matrix mod (L_m, L_n)
$\Rightarrow (AQ^m)^a(AQ^n)^b$ is diagonal matrix mod (L_m, L_n)
Note that

$$\begin{aligned}(AQ^m)^a(AQ^n)^b &= A^{a+b}Q^{am+bn} = A^{2k+1}Q^d \\ &= (A^2)^k AQ^d = (5I)^k AQ^d \\ &= 5^k AQ^d \text{ which is diagonal matrix mod } (L_m, L_n)\end{aligned}$$

$\Rightarrow 5^k L_d \equiv 0 \bmod (L_m, L_n) \quad \Rightarrow L_d \equiv 0 \bmod \frac{(L_m,L_n)}{(5^k,(L_m,L_n))}$
but $(5^k, (L_m, L_n)) = ((5^k, L_m), L_n) = (1, L_n) = 1$
$\Rightarrow L_d \equiv 0 \bmod (L_m, L_n) \quad \Rightarrow L_{(m,n)} \equiv 0 \bmod (L_m, L_n)$
$\Rightarrow (L_m, L_n) | L_{(m,n)}$, hence $(L_m, L_n) = L_{(m,n)}$

Now, $T_{(m,n)} = 2^{L_{(m,n)}} - 1 = 2^{(L_m,L_n)} - 1 = (2^{L_m} - 1, 2^{L_n} - 1) = (T_m, T_n)$ □

3.3 Some Congruence properties of Lucasenne sequence $\{T_n\}$

In this section, we present some congruence properties of Lucasenne sequence. The following table gives Lucasenne numbers $0 \leq n \leq 11$.
A look at the table 1 reveals that the last digit of T_n follow the pattern $317577, 317577, \ldots$
Given $m \geq 4$, we have $T_0 \equiv 3(\bmod\ m)$ and $T_1 \equiv 1(\bmod\ m)$. For $n \geq 2$, the congruence properties of T_n can be found by using the recurrence relation obtained in Proposition 3.2.0.1 (2). For $2 \leq m \leq 10$ we show the cycles of $T_n(\bmod\ m)$ in Table 2.

Table 1: $\mathbf{T_n},\ \mathbf{0 \leq n \leq 11}$

n	T_n
0	3
1	1
2	7
3	15
4	127
5	2047
6	262143
7	536870911
8	140737488355327
9	75557863725914323419135
10	10633823966279326983230456482242756607
11	80346902212949513777098104617058130126110149689139641765068 7

Table 2: Cycles of $\mathbf{T_n(mod\ m)},\ \mathbf{b(m)},\ \mathbf{t(m)}$

m	$T_n(\text{mod } m)$	Base length b(m)	Tail period t(m)
2	1,1,1,1,1,...	0	1
3	0,1,1, 0,1,1, 0,1,1,...	0	3
4	**3,1,** 3,3,3,3,...	2	1
5	3,1,2,0,2,2, 3,1,2,0,2,2,...	0	6
6	3,1,1, 3,1,1, 3,1,1,...	0	3
7	3,1,0,1,1,3,0,3, 3,1,0,1,1,3,0,3,...	0	8
8	**3,1,** 7,7,7,7,...	2	1
9	3,1,7,6,1,4,0,4,4,6,7,1,6,4,7,3,4,1,0,1,1,3,7,4, 3,1,...	0	24
10	3,1,7,5,7,7, 3,1,7,5,7,7,...	0	6

From Table 2, for $m = 10$ the following result follows immediately.

Proposition 3.3.0.1.

1. For $u \geq 0$,

$$T_{6u+r} \equiv \left\{ \begin{array}{lll} 1(\text{mod } 10) & if & r=1 \\ 3(\text{mod } 10) & if & r=0 \\ 5(\text{mod } 10) & if & r=3 \\ 7(\text{mod } 10) & if & r=2,4,5 \end{array} \right\}$$

2. For $0 \leq r \leq 4$ and $u \geq 0$, $\quad T_{6u+r} \equiv T_r(\text{mod } 5)$
3. For $0 \leq r \leq 2$ and $u \geq 0$, $\quad T_{3u+r} \equiv T_r(\text{mod } 6)$

Definition 3.3.0.2. For $n \geq 1$, let $[n]_L$ be the largest in $\mathbb{N} \ni L_{[n]_L} \leq n$.
For example, $[1]_L = 1, [2]_L = 1, [4]_L = 3$

Proposition 3.3.0.3. *For $n \geq 1$, $T_k \equiv 2^n - 1(mod\ \ 2^n)$, for all $k \geq b(2^n)$. In this case $t(2^n) = 1$.*

Proof. For $k \geq b(2^n)$.

$$\begin{aligned} T_k &= 2^{L_k} - 1 = 2^n(2^{L_k - n}) - 1, \text{ since } L_k - n \geq 0 \text{ as } k \geq b(2^n) \geq [n]_L \\ &\equiv -1(\text{mod } 2^n) \\ &\equiv 2^n - 1(\text{mod } 2^n) \end{aligned}$$

□

Given $m \geq 2$, considering the Lucas numbers L_n and the smallest residues $R_n^{'}(m)$ of the terms modulo m. It was observed that the sequence $R_n^{'}(m)$ repeats after $\eta(m)$ terms as shown in below table:

Table 3: Values of $\eta(\mathrm{m})$ for $2 \leq \mathrm{m} \leq 30$ (see [6])

m		2	3	4	5	6	7	8	9	10
η(m)		3	8	6	4	24	16	12	24	12
m	11	12	13	14	15	16	17	18	19	20
η(m)	10	24	28	48	8	24	36	24	18	12
m	21	22	23	24	25	26	27	28	29	30
η(m)	16	30	48	24	20	84	72	48	14	24

$\eta(m)$ is called pisano period of L_n mod m. From the definition of $\eta(m)$ we obtained a lemma given below:

Lemma 3.3.0.4. *For $m \geq 2$ and $u \geq 0$, $L_{\eta(m)u+r} \equiv L_r (mod\ m)$ where $0 \leq r < \eta(m)$*

Using this lemma we prove the following:

Proposition 3.3.0.5. *For $m \geq 2$ and $u \geq 0$, we have $L_{\eta(m)u+r} \equiv L_r(mod\ m)$, where $0 \leq r < \eta(m)$ and $2^m \equiv 1(mod\ k)$ for some $k > 1$, then $T_{\eta(m)u+r} \equiv T_r(mod\ k)$*

Proof. We have

$$
\begin{aligned}
T_{\eta(m)u+r} &= 2^{L_{\eta(m)u+r}} - 1 \\
&= 2^{mj+L_r} - 1, \text{ for some } j \geq 0 \\
&= (2^m)^j 2^{L_r} - 1 \\
&\equiv 2^{L_r} - 1 (\text{mod } k) \equiv T_r (\text{mod } k)
\end{aligned}
$$

□

Notice that in Proposition 3.3.0.5, $2^m \equiv 1(\text{mod } k)$ suggests that k is odd. The case when k is a power of 2 is dealt in Proposition 3.3.0.3, when k is a even but not a power of 2, we have the following:

Proposition 3.3.0.6. *For $m \geq 2$ and $u \geq 0$, if $L_{\eta(m)u+r} \equiv L_r(mod\, m)$, where $0 \leq r < \eta(m)$ and $2^m \equiv 1(mod\, k)$ for some $k > 1$, then for $s \geq 1$, $T_{\eta(m)u+r} \equiv N_r(mod\ \ 2^s k)$, where N_r is independent of u.*

Proof. We see for $L_{\eta(m)u+r} - s \geq 0$ that

$$\begin{aligned}
T_{\eta(m)u+r} &= 2^{L_{\eta(m)u+r}} - 1 \\
&= 2^{mj+L_r} - 1, \text{ for some } j \geq 0 \\
&= 2^s(2^{mj+L_r-s}) - 1 \\
&= 2^s(kq+r_1) - 1, \text{where } r_1 \text{ is the remainder when } 2^{mj+L_r-s} \text{ is divisible by } k \\
&= 2^s kq + 2^s r_1 - 1 \\
&\equiv 2^s r_1 - 1(\text{mod } 2^s k)
\end{aligned}$$

□

We have the following corollaries:

Corollary 3.3.0.7. *For $u \geq 0$,*

$$T_{8u+r} \equiv \left\{ \begin{array}{lll} 0(mod\, 7) & if & r = 2,6 \\ 1(mod\, 7) & if & r = 1,3,4 \\ 3(mod\, 7) & if & r = 0,5,7 \end{array} \right\}$$

Proof. Note that $\eta(3) = 8$ and $2^3 \equiv 1(\text{mod } 7)$
Also $L_{8u+r} \equiv L_r(\text{mod } 3)$ where $0 \leq r < \eta(3)$ and use Proposition 3.3.0.5 □

Corollary 3.3.0.8. *For* $n \geq 0$,

$$T_{6n+r} \equiv \left\{ \begin{array}{lll} 0(mod\,15) & if & r=3 \\ 1(mod\,15) & if & r=1 \\ 3(mod\,15) & if & r=0 \\ 7(mod\,15) & if & r=2,4,5 \end{array} \right\}$$

Proof. Note that $\eta(4) = 6$ and $2^4 \equiv 1(\text{mod } 15)$

Also $L_{6u+r} \equiv L_r(\text{mod } 4)$ where $0 \leq r < \eta(4)$ and use Proposition 3.3.0.5 □

Proposition 3.3.0.9.

1. If $m > 2$ is an odd integer then $b(m) = 0$ and $t(m)|\eta(\phi(m))$ where ϕ is Eulers function. In particular, if $p > 2$ is a prime then $t(p)|\eta(p-1)$.

Proof. Now, $m > 2$ and m is odd then $\phi(m) \geq 2$

by using Lemma 3.3.0.4, $L_{\eta(\phi(m))u+r} \equiv L_r(\text{mod } \phi(m))$ where $0 \leq r < \eta(\phi(m))$

$\Rightarrow \phi(m)|\left(L_{\eta(\phi(m))u+r} - L_r\right) \Rightarrow L_{\eta(\phi(m))u+r} - L_r = \phi(m)k$, where $k \in \mathbb{Z}$

$$\begin{aligned} T_{\eta(\phi(m))u+r} &= 2^{L_{\eta(\phi(m))u+r}} - 1 = 2^{\phi(m)k+L_r} - 1 \\ &\equiv 2^{L_r} - 1(\text{mod } m) \equiv T_r(\text{mod } m) \end{aligned}$$

Thus the pattern in $T_n(\text{mod } m)$ repeats after $\eta(\phi(m))$ terms. So $\eta(\phi(m))$ must be a multiple of $t(m)$. hence $t(m)|\eta(\phi(m))$

since the cycle repeats right from the begining, $b(m) = 0$ □

2. For $m \geq 1$, $t(2^m) = 1$

Proof. This result follows from Proposition 3.3.0.3 □

3. Let $m \geq 2$ be an integer $\ni 2^m \equiv 1 (\bmod k)$ then $t(k) | \eta(m)$

Proof. We use Lemma 3.3.0.4, so that
$L_{\eta(m)u+r} - L_r = mz$, where $z \in \mathbb{Z}$

$$\begin{aligned} T_{\eta(m)u+r} &= 2^{L_{\eta(m)u+r}} - 1 \\ &= 2^{mz+L_r} - 1 \\ &\equiv 2^{L_r} - 1 (\bmod k) \equiv T_r (\bmod k) \end{aligned}$$

Thus the pattern in $T_n (\bmod k)$ repeats after $\eta(m)$ terms. So $\eta(m)$ must be a multiple of $t(k)$. hence $t(k) | \eta(m)$ □

Using Proposition 3.3.0.9 (3) we have the following properties:
Properties:

1. If $u \geq 1$ and $m | M_{2^u}$ then $t(m) | 3(2^{u-1})$

Proof. $\because\ m | M_{2^u} \Rightarrow\ 2^{2^u} \equiv 1 (\bmod m)$ then by Proposition 3.3.0.9 (3), $t(m) | \eta(2^u)$
but by result in [6], $F_{3(2^{u-1})} \equiv 0 (\bmod 2^u)$ and $F_{3(2^{u-1})+1} \equiv 1 (\bmod 2^u)$
$\Rightarrow F_{3(2^{u-1})-1} = F_{3(2^{u-1})+1} - F_{3(2^{u-1})} \equiv 1 (\bmod 2^u)$
Now, $L_{3(2^{u-1})} = F_{3(2^{u-1})+1} + F_{3(2^{u-1})-1} \equiv 2 (\bmod 2^u)$
and $L_{3(2^{u-1})+1} = F_{3(2^{u-1})+2} + F_{3(2^{u-1})} = F_{3(2^{u-1})+1} + 2F_{3(2^{u-1})} \equiv 1 (\bmod 2^u)$
$\therefore\ \eta(2^u) | 3(2^{u-1})$ hence $t(m) | 3(2^{u-1})$ □

2. If $u \geq 1$ and $m | M_{5^u}$ then $t(m) | 4(5^{u-1})$

Proof. $\because m|M_{5^u} \Rightarrow\ 2^{5^u} \equiv 1(\text{mod } m)$ then by Proposition 3.3.0.9 (3), $t(m)|\eta(5^u)$ but by result in [6], Let G_n be any generalised Fibonacci sequence, then

$$\pi(5^u) = \begin{cases} \mathrm{p}(5^u, G_n) & \text{if} \quad G_1^2 - G_0G_2 \text{ is not divisible by } 5 \\ 5 \times \mathrm{p}(5^u, G_n) & \text{if} \quad G_1^2 - G_0G_2 \text{ is divisible by } 5 \end{cases}$$

where $\mathrm{p}(m, G_n)$ is pisano period of G_n mod m

$\Rightarrow\ \pi(5^u) = 5 \times \eta(5^u)$ since $L_1^2 - L_0L_2 = -5$ which is divisible by 5

since $\pi(5^u)|4(5^u) \quad \Rightarrow 5 \times \eta(5^u)|4(5^u)$

$\Rightarrow \eta(5^u)|4(5^{u-1}) \quad$ hence $t(m)|4(5^{u-1})$ □

3. If p is a prime of the form $10k \pm 1$ and $m|M_p$ then $t(m)|p-1$

Proof. $\because\ m|M_p \Rightarrow\ 2^p \equiv 1(\text{mod } m)$ then by Proposition 3.3.0.9 (3), $t(m)|\eta(p)$ but by result in [6], if p is a prime of the form $10k \pm 1$ then

$L_{p-1} \equiv 2(\text{mod } p)$ and $L_p \equiv 1(\text{mod } p)$

$\Rightarrow\ \eta(p)|p-1 \qquad \therefore\ t(m)|p-1$ □

3.4 Relations between P_n and T_n

Proposition 3.4.0.1. *For* $n \geq 1, \quad T_n = P_{n+1}P_{n-1} + P_{n+1} + P_{n-1}$

Proof. We have $L_n = F_{n+1} + F_{n-1} \; \forall \; n \geq 1 \quad$ (see [6])

$$\begin{aligned} T_n &= 2^{L_n} - 1 = 2^{F_{n+1}+F_{n-1}} - 1 = 2^{F_{n+1}+F_{n-1}} - 2^{F_{n+1}} + 2^{F_{n+1}} - 1 \\ &= 2^{F_{n+1}}(2^{F_n} - 1) - (2^{F_{n-1}} - 1) + (2^{F_{n-1}} - 1) + (2^{F_{n+1}} - 1) \\ &= (2^{F_{n+1}} - 1)(2^{F_{n-1}} - 1) + (2^{F_{n-1}} - 1) + (2^{F_{n+1}} - 1) \\ &= P_{n+1}P_{n-1} + P_{n+1} + P_{n-1} \end{aligned}$$

□

Proposition 3.4.0.2. *For* $n \geq 2, \quad T_n = \frac{P_{n+2}-P_{n-2}}{1+P_{n-2}}$

Proof. We have $L_n = F_{n+2} - F_{n-2} \; \forall \; n \geq 2 \quad$ (see [6])

$$\begin{aligned} T_n &= 2^{L_n} - 1 = 2^{F_{n+2}-F_{n-2}} - 1 \\ &= \frac{2^{F_{n+2}} - 2^{F_{n-2}}}{2^{F_{n-2}}} = \frac{(2^{F_{n+2}} - 1) - (2^{F_{n-2}} - 1)}{2^{F_{n-2}}} \\ &= \frac{P_{n+2} - P_{n-2}}{1 + P_{n-2}} \end{aligned}$$

□

Proposition 3.4.0.3. *For* $n \geq 1, \quad (1+P_n)^5 = T_{n+1}T_{n-1} + T_{n+1} + T_{n-1} + 1$

Proof. We have $5F_n = L_{n+1} + L_{n-1} \; \forall \; n \geq 1 \quad$ (see [6])

$$\begin{aligned} (1+P_n)^5 &= 2^{5F_n} = 2^{L_{n+1}+L_{n-1}} = 2^{L_{n+1}+L_{n-1}} + 2^{L_{n+1}} - 2^{L_{n-1}} \\ &= 2^{L_{n+1}}(2^{L_{n-1}} - 1) - (2^{L_{n-1}} - 1) + (2^{L_{n-1}} - 1) + 2^{L_{n+1}} \end{aligned}$$

$$= (2^{L_{n+1}} - 1)(2^{L_{n-1}} - 1) + (2^{L_{n-1}} - 1) + 2^{L_{n+1}}$$
$$= T_{n+1}T_{n-1} + T_{n-1} + T_{n+1} + 1$$

□

Proposition 3.4.0.4. *For $n \geq 0$,*

$$(1+T_n) = \frac{(1+P_{n+1})^2}{(1+P_n)}$$

Proof. We have $L_n + F_n = 2F_{n+1} \; \forall \, n \geq 0$ (see [6])
$(1+P_n)(1+T_n) = 2^{F_n+L_n} = 2^{2F_{n+1}} = (2^{F_{n+1}})^2 = (1+P_{n+1})^2$ □

Proposition 3.4.0.5. *For $n \geq 1$,* $(1+T_n)^{L_n} = 2^{(-1)^n F_3}(1+P_{2n+1})(1+P_{2n-1})$

Proof. We have $L_n^2 = L_{2n} + 2(-1)^n \; \forall \, n \geq 0$ (see [6])

$$(1+T_n)^{L_n} = 2^{L_n^2} = 2^{2(-1)^n + L_{2n}} = 2^{(-1)^n F_3 + F_{2n+1} + F_{2n-1}}$$
$$= 2^{(-1)^n F_3}(1+P_{2n+1})(1+P_{2n-1})$$

□

Proposition 3.4.0.6. *For $n \geq 0$,*

$$(1+P_n)^{5F_n} = \frac{(1+T_{2n})^2}{(1+T_n)^{L_n}}$$

Proof. We have $2L_{2n} = L_n^2 + 5F_n^2 \;\; \forall \, n \geq 0$ (see [6])

$$(1+T_{2n})^2 = 2^{2L_{2n}} = 2^{L_n^2+5F_n^2} = (2^{L_n})^{L_n}(2^{F_n})^{5F_n}$$
$$= (1+T_n)^{L_n}(1+P_n)^{5F_n}$$

□

3.5 Generalised Fibonacci

Definition 3.5.0.1. For $a, b \in \mathbb{N} \cup \{0\}$,
Define $X_0 = a$, $X_1 = b$ and $X_n = X_{n-1} + X_{n-2} \; \forall \; n \geq 2$

Definition 3.5.0.2. We present another extension of generalised Fibonacci sequence $\{Y_n\}$ defined by the relation

$$Y_n = 2^{X_n} - 1 \; \forall \; n \geq 0$$

Proposition 3.5.0.3. *For $n \geq 0$, Y_n satisfies the non linear second order recurrence relation $Y_{n+2} = Y_{n+1} + Y_n + Y_{n+1}Y_n$ with initial conditions $Y_0 = 2^a - 1$ & $Y_1 = 2^b - 1$*

Proof.

$$\begin{aligned} Y_{n+2} &= 2^{X_{n+2}} - 1 = 2^{X_{n+1}+X_n} - 1 \\ &= 2^{X_{n+1}+X_n} - 2^{X_{n+1}} + 2^{X_{n+1}} - 1 \\ &= 2^{X_{n+1}}(2^{X_n} - 1) - (2^{X_n} - 1) + (2^{X_n} - 1) + (2^{X_{n+1}} - 1) \\ &= (2^{X_{n+1}} - 1)(2^{X_n} - 1) + (2^{X_n} - 1) + (2^{X_{n+1}} - 1) \\ &= Y_{n+1} + Y_n + Y_{n+1}Y_n \end{aligned}$$

□

Proposition 3.5.0.4. *For $a, b, n \geq 0$, the following are equivalent:*
(1) $(a, b) = 1$
(2) $(X_n, X_{n+1}) = 1$
(3) $(Y_n, Y_{n+1}) = 1$

Proof. Suppose $(a, b) = 1$
$\because (X_0, X_1) = 1 \Rightarrow (X_0 + X_1, X_1) = 1 \Rightarrow (X_2, X_1) = 1$
$\because (X_2, X_1) = 1 \Rightarrow (X_2, X_2 + X_1) = 1 \Rightarrow (X_2, X_3) = 1$

$$\vdots$$

$\because (X_n, X_{n-1}) = 1 \Rightarrow (X_n, X_{n-1} + X_n) = 1 \Rightarrow (X_n, X_{n+1}) = 1$

hence (1) $\Rightarrow$ (2)

Suppose $(X_n, X_{n+1}) = 1$

then $(Y_n, Y_{n+1}) = (2^{X_n} - 1, 2^{X_{n+1}} - 1) = 2^{(X_n, X_{n+1})} - 1 = 2 - 1 = 1$

hence (2) $\Rightarrow$ (3)

Suppose $(Y_n, Y_{n+1}) = 1$

In particular $(Y_0, Y_1) = 1 \Rightarrow (2^{X_0} - 1, 2^{X_1} - 1) = 1 \Rightarrow (2^a - 1, 2^b - 1) = 1$

$\Rightarrow 2^{(a,b)} - 1 = 1 \Rightarrow 2^{(a,b)} = 2 \Rightarrow (a, b) = 1$

hence (3) $\Rightarrow$ (1)

$\square$

Chapter 4

PELLENE SEQUENCE D_n

4.1 Basic definitions

The well known Pell sequence C_n is defined by

$$C_0 = 0, C_1 = 1, \;\; C_{n+2} = 2C_{n+1} + C_n \text{ for } n \geq 0$$

In this note, we present yet another extension of C_n and call it pellenne sequence $\{D_n\}$ defined by

$$D_n = 2^{C_n} - 1, \;\; n \geq 0$$

where C_n is n^{th} pell number. We call D_n the n^{th} pellenne number in view of its form like Mersenne number M_n. It is clear that $D_n = M_{C_n}$.We shall establish various relations for D_n in line with those of C_n. We shall also study some congruence properties of D_n

4.2 Identities, divisibility and gcd properties of D_n

Proposition 4.2.0.1.

1. For $n \geq 0$,

$$\log_2(1+D_n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ where } \alpha, \beta \text{ are roots of equation } x^2 = 2x+1$$

Proof. First we prove the binet's formula for Pell numbers (refer [2])

$$C_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n \geq 0 \text{ where } \alpha, \beta \text{ are roots of equation } x^2 = 2x+1$$

For $n = 0$, $C_0 = \frac{\alpha^0 - \beta^0}{\alpha - \beta} = 0$
For $n = 1$, $C_1 = \frac{\alpha^1 - \beta^1}{\alpha - \beta} = 1$
Assume that the result holds for all $k < n+1$ where $k > 1$
i.e. $C_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$
Next, T.P.T. $C_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$

$$\begin{aligned}
C_{n+1} &= 2C_n + C_{n-1} \\
&= \frac{2\alpha^n - 2\beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}, \text{ by induction hypothesis} \\
&= \frac{\alpha^{n-1}(2\alpha+1) - \beta^{n-1}(2\beta+1)}{\alpha - \beta} \\
&= \frac{\alpha^{n-1}\alpha^2 - \beta^{n-1}\beta^2}{\alpha - \beta}, \quad \text{since } \alpha^2 = 2\alpha + 1 \text{ and } \beta^2 = 2\beta + 1 \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}
\end{aligned}$$

Finally, $\log_2(1+D_n) = \log_2(2^{C_n}) = C_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ □

2. For $n \geq 0$, D_n satisfies the non linear second order recurrence relation
$D_{n+2} = D_{n+1}(D_n+1)(D_{n+1}+2)+D_n$ with initial conditions $D_0 = 0$ & $D_1 = 1$

Proof.

$$\begin{aligned} D_{n+2} &= 2^{C_{n+2}} - 1 = 2^{2C_{n+1}+C_n} - 1 = 2^{2C_{n+1}+C_n} - 2^{C_n} + 2^{C_n} - 1 \\ &= (2^{2C_{n+1}} - 1)2^{C_n} + 2^{C_n} - 1 \\ &= (2^{C_{n+1}} - 1)(2^{C_{n+1}} + 1)2^{C_n} + 2^{C_n} - 1 \\ &= D_{n+1}(D_n+1)(D_{n+1}+2) + D_n \end{aligned}$$

□

3. For $n \geq 1$,

$$D_{-n} = \begin{cases} D_n & \text{if} \quad \text{n is odd} \\ \frac{-D_n}{1+D_n} & \text{if} \quad \text{n is even} \end{cases}$$

Proof. We prove that $C_{-n} = (-1)^{n+1}C_n$

$$C_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\beta^n - \alpha^n}{(\alpha\beta)^n(\alpha - \beta)} = (-1)^{n+1}C_n \text{ where } \alpha, \beta \text{ are roots of eqn } x^2 = 2x+1$$

Now, $D_{-n} = 2^{C_{-n}} - 1 = 2^{(-1)^{n+1}C_n} - 1$

If n is odd, $D_{-n} = 2^{C_n} - 1 = D_n$

If n is even, $D_{-n} = 2^{-C_n} - 1 = \frac{1}{2^{C_n}} - 1 = \frac{1-2^{C_n}}{2^{C_n}} = \frac{-D_n}{1+D_n}$ □

Proposition 4.2.0.2.

1. $\forall\, n \geq 0, \quad (D_n, D_{n+1}) = 1$

Proof. Let $d = (D_n, D_{n+1})$

$\Rightarrow d|D_n$ and $d|D_{n+1}$ but $D_{n+1} = D_n(D_n+2)(D_{n-1}+1) + D_{n-1} \Rightarrow d|D_{n-1}$

Also $d|D_n$ and $d|D_{n-1} \Rightarrow d|D_{n-2}$

$\vdots$

Continuing in the same manner we get, $d|D_1 \Rightarrow d|1 \Rightarrow d = 1$

Hence $(D_n, D_{n+1}) = 1$ □

2. Let n be an odd positive integer. If $\phi(n)|C_m \Rightarrow n|D_m$

Proof. Since n is an odd positive integer $\Rightarrow (n,2) = 1$

By Euler's identity, $2^{\phi(n)} \equiv 1 (\text{mod } n)$

Since $\phi(n)|C_m \; \exists\, z \in \mathbb{Z} \ni C_m = \phi(n)z$

$$D_m = 2^{C_m} - 1 = 2^{\phi(n)z} - 1$$
$$= (2^{\phi(n)})^z - 1 \equiv 0 (\text{mod } n)$$

□

3. For $m, n \in \mathbb{N}$, If $n|m$ then $D_n|D_m$

Proof. We have $\begin{pmatrix} C_{n+1} & C_n \\ C_n & C_{n-1} \end{pmatrix} = J^n$ for $n \in \mathbb{N}$ where $J = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

Clearly, J^n is diagonal matrix mod C_n

Since $n|m \; \exists\, r \in \mathbb{Z} \ni m = rn$ so that $\begin{pmatrix} C_{rn+1} & C_{rn} \\ C_{rn} & C_{rn-1} \end{pmatrix} = J^{rn}$

$\Rightarrow J^{rn} = (J^n)^r$ is diagonal matrix mod $C_n \Rightarrow C_{rn} \equiv 0 (\text{mod } C_n)$

Hence $C_n|C_{rn}$ i.e. $C_n|C_m$

Finally, $n|m \Rightarrow C_n|C_m \Rightarrow M_{C_n}|M_{C_m} \Rightarrow D_n|D_m$ □

4. For $m, n \in \mathbb{N}$, $(D_m, D_n) = D_{(m,n)}$

Proof. Since $(m,n)|m$ and $(m,n)|n \Rightarrow C_{(m,n)}|C_m$ and $C_{(m,n)}|C_n \Rightarrow C_{(m,n)}|(C_m, C_n)$

Next, To prove that $(C_m, C_n)|C_{(m,n)}$

Let $d = (m,n)$ then $\exists\, x,y \in \mathbb{Z} \ni d = mx+ny$
$\begin{pmatrix} C_{d+1} & C_d \\ C_d & C_{d-1} \end{pmatrix} = J^d = J^{mx+ny} = (J^m)^x (J^n)^y$
$\begin{pmatrix} C_{m+1} & C_m \\ C_m & C_{m-1} \end{pmatrix} = J^m$ is diagonal matrix mod (C_m, C_n)
and $\begin{pmatrix} C_{n+1} & C_n \\ C_n & C_{n-1} \end{pmatrix} = J^n$ is diagonal matrix mod (C_m, C_n)
$\Rightarrow J^d = (J^m)^x (J^n)^y$ is diagonal matrix mod (C_m, C_n)
$\Rightarrow C_{(m,n)} = C_d \equiv 0 (\text{mod } (C_m, C_n)) \ \Rightarrow (C_m, C_n) | C_{(m,n)} \ \Rightarrow C_{(m,n)} = (C_m, C_n)$
Finally, $(D_m, D_n) = (2^{C_m} - 1, 2^{C_n} - 1) = 2^{(C_m, C_n)} - 1 = 2^{C_{(m,n)}} - 1 = D_{(m,n)}$ □

4.3 Some Congruence properties of Pellene sequence $\{D_n\}$

In this section, we present some congruence properties of Pellene sequence. The following table gives pellene numbers $0 \le n \le 7$. A look at the table 1 reveals that the last digit of D_n follow the pattern $01315, 1315, \ldots$ Given $m \ge 2$, we have $D_0 \equiv 0 (\text{mod } m)$ and $D_1 \equiv 1 (\text{mod } m)$. For $n \ge 2$, the congruence properties of D_n can be found by using the recurrence relation obtained in Proposition 4.2.0.1(2). For $2 \le m \le 10$ we show the cycles of $D_n (\text{mod } m)$ in Table 2.

Table 1: $\mathbf{D_n, 0 \le n \le 7}$

n	D_n
0	0
1	1
2	3
3	31
4	4095
5	536870911
6	1180591620717411303423
7	748288838313422294120286634350736906063837462003711

Table 2: Cycles of $\mathbf{D_n}(\mathbf{mod\ m})$, $\mathbf{b}(\mathbf{m})$, $\mathbf{t}(\mathbf{m})$

m	$D_n(\text{mod } m)$	Base length b(m)	Tail period t(m)
2	**0,** 1,1,1,1,1,...	1	1
3	0,1, 0,1, 0,1,...	0	2
4	**0,1,** 3,3,3,3,...	2	1
5	0,1,3,1, 0,1,3,1,...	0	4
6	**0,** 1,3 1,3, 1,3,...	1	2
7	0,1,3,3,0,3,1,1, 0,1,3,3,0,3,1,1,...	0	8
8	**0,1,3,** 7,7,7,7,...	3	1
9	0,1,3,4,0,4,6,1, 0,1,3,4,0,4,6,1,...	0	8
10	**0,** 1,3,1,5, 1,3,1,5,...	1	4

From Table 2, for $m = 10$ the following result follows immediately.

Proposition 4.3.0.1.

1. For $u \geq 0$,

$$D_{4u+r} \equiv \left\{ \begin{array}{ll} 1(\text{mod } 10) & if \quad r = 1,3 \\ 3(\text{mod } 10) & if \quad r = 2 \\ 5(\text{mod } 10) & if \quad r = 4 \end{array} \right\}$$

2. For $0 \leq r \leq 3$ and $u \geq 0$, $\quad D_{4u+r} \equiv D_r(\text{mod } 5)$

Definition 4.3.0.2. For $n \geq 1$, let $[n]_C$ be the largest in $\mathbb{N} \ni C_{[n]_C} \leq n$.
For example, $[1]_C = 1, [2]_C = 2, [5]_C = 3$

Proposition 4.3.0.3. *For $n \geq 1$, $D_k \equiv 2^n - 1(mod\ 2^n)$, for all $k \geq b(2^n)$. In this case $t(2^n) = 1$.*

Proof. For $k \geq b(2^n)$.

$$D_k = 2^{C_k} - 1 = 2^n(2^{C_k - n}) - 1, \text{ since } C_k - n \geq 0 \text{ as } k \geq b(2^n) \geq [n]_C$$
$$\equiv -1 (\text{mod } 2^n)$$
$$\equiv 2^n - 1 (\text{mod } 2^n)$$

□

Given $m \geq 2$, considering the pell numbers C_n and the smallest residues $R_n^{''}(m)$ of the terms modulo m. It was observed that the sequence $R_n^{''}(m)$ repeats after $\nu(m)$ terms as shown in below table:

Table 3: Values of ν(m) for $2 \leq$ m ≤ 30

m		2	3	4	5	6	7	8	9	10
ν(m)		2	8	4	12	8	6	8	24	12
m	11	12	13	14	15	16	17	18	19	20
ν(m)	24	8	28	6	24	16	16	24	40	12
m	21	22	23	24	25	26	27	28	29	30
ν(m)	24	24	22	8	60	28	72	12	20	24

$\nu(m)$ is called pisano period of C_n mod m. From the definition of $\nu(m)$ we obtained a lemma given below:

Lemma 4.3.0.4. *For $m \geq 2$ and $u \geq 0$, $C_{\nu(m)u+r} \equiv C_r(mod\ m)$*
where $0 \leq r < \nu(m)$

Using this lemma we prove the following:

Proposition 4.3.0.5. *For $m \geq 2$ and $u \geq 0$, we have $C_{\nu(m)u+r} \equiv C_r(mod\ m)$, where $0 \leq r < \nu(m)$ and $2^m \equiv 1(mod\ k)$ for some $k > 1$, then*
$D_{\nu(m)u+r} \equiv D_r(mod\ k)$

Proof. We have

$$\begin{aligned} D_{\nu(m)u+r} &= 2^{C_{\nu(m)u+r}} - 1 = 2^{mj+C_r} - 1, \text{ for some } j \geq 0 \\ &= (2^m)^j 2^{C_r} - 1 \\ &\equiv 2^{C_r} - 1 (\text{mod } k) \equiv D_r (\text{mod } k) \end{aligned}$$

□

Notice that in Proposition 4.3.0.5, $2^m \equiv 1(\text{mod } k)$ suggests that k is odd. The case when k is a power of 2 is dealt in Proposition 4.3.0.3, when k is a even but not a power of 2, we have the following:

Proposition 4.3.0.6. *For $m \geq 2$ and $u \geq 0$, if $C_{\nu(m)u+r} \equiv C_r(\text{mod } m)$, where $0 \leq r < \nu(m)$ and $2^m \equiv 1(\text{mod } k)$ for some $k > 1$, then for $s \geq 1$, $D_{\nu(m)u+r} \equiv N_r(\text{mod } \; 2^s k)$, where N_r is independent of u.*

Proof. We see for $C_{\nu(m)u+r} - s \geq 0$ that

$$\begin{aligned} D_{\nu(m)u+r} &= 2^{C_{\nu(m)u+r}} - 1 = 2^{mj+C_r} - 1, \text{ for some } j \geq 0 \\ &= 2^s(2^{mj+C_r-s}) - 1 \\ &= 2^s(kq + r_1) - 1, \text{where } r_1 \text{ is the remainder when } 2^{mj+C_r-s} \text{ is divisible by } k \\ &= 2^s kq + 2^s r_1 - 1 \\ &\equiv 2^s r_1 - 1 (\text{mod } 2^s k) \end{aligned}$$

□

We have the following corollaries:

Corollary 4.3.0.7. *For $u \geq 0$,*

$$D_{8u+r} \equiv \left\{ \begin{array}{lll} 0(mod\, 7) & if & r=0,4 \\ 1(mod\, 7) & if & r=1,6,7 \\ 3(mod\, 7) & if & r=2,3,5 \end{array} \right\}$$

Proof. Note that $\nu(3)=8$ and $2^3 \equiv 1(\text{mod } 7)$

Also $C_{\nu(3)u+r} \equiv C_r(\text{mod } 3)$ where $0 \leq r < \nu(3)$ and use Proposition 4.3.0.5 □

Corollary 4.3.0.8. *For $n \geq 0$,*

$$D_{12n+r} \equiv \left\{ \begin{array}{lll} 0(mod\, 31) & if & r=0,6,9 \\ 1(mod\, 31) & if & r=1,3,11 \\ 3(mod\, 31) & if & r=2,4 \\ 7(mod\, 31) & if & r=8,10 \\ 15(mod\, 31) & if & r=5,7 \end{array} \right\}$$

Proof. Note that $\nu(5)=12$ and $2^5 \equiv 1(\text{mod } 31)$

Also $C_{\nu(5)u+r} \equiv C_r(\text{mod } 5)$ where $0 \leq r < \nu(5)$ and use Proposition 4.3.0.5 □

Proposition 4.3.0.9.

1. If $m > 2$ is an odd integer then $b(m)=0$ and $t(m)|\nu(\phi(m))$ where ϕ is Eulers function. In particular, if $p > 2$ is a prime then $t(p)|\nu(p-1)$.

Proof. Now, $m > 2$ and m is odd then $\phi(m) \geq 2$

by using Lemma 4.3.0.4, $C_{\nu(\phi(m))u+r} \equiv C_r(\text{mod } \phi(m))$ where $0 \leq r < \nu(\phi(m))$

$\Rightarrow \phi(m) | \left(C_{\nu(\phi(m))u+r} - C_r\right) \Rightarrow\ C_{\nu(\phi(m))u+r} - C_r = \phi(m)k$, where $k \in \mathbb{Z}$

$$\begin{aligned} D_{\nu(\phi(m))u+r} &= 2^{C_{\nu(\phi(m))u+r}} - 1 = 2^{\phi(m)k+C_r} - 1 \\ &\equiv 2^{C_r} - 1(\text{mod } m) \equiv D_r(\text{mod } m) \end{aligned}$$

Thus the pattern in $D_n(\bmod m)$ repeats after $\nu(\phi(m))$ terms. So $\nu(\phi(m))$ must be a multiple of $t(m)$. hence $t(m)|\nu(\phi(m))$
since the cycle repeats right from the begining, $b(m) = 0$ □

2. For $m \geq 1$, $t(2^m) = 1$

Proof. This result follows from Proposition 4.3.0.3 □

3. Let $m \geq 2$ be an integer $\ni 2^m \equiv 1(\bmod k)$ then $t(k)|\nu(m)$

Proof. We have lemma: For $m \geq 2$ and $u \geq 0$, $C_{\nu(m)u+r} \equiv C_r(\bmod m)$
where $0 \leq r < \nu(m)$
$C_{\nu(m)u+r} - C_r = mz$, where $z \in \mathbb{Z}$

$$\begin{aligned} D_{\nu(m)u+r} &= 2^{C_{\nu(m)u+r}} - 1 \\ &= 2^{mz+C_r} - 1 \\ &\equiv 2^{C_r} - 1(\bmod k) \equiv D_r(\bmod k) \end{aligned}$$

Thus the pattern in $D_n(\bmod k)$ repeats after $\nu(m)$ terms. So $\nu(m)$ must be a multiple of $t(k)$. hence $t(k)|\nu(m)$ □

Using Proposition 4.3.0.9 (3) we have the following properties:

Properties:

1. For $u \geq 1$, If $m|M_{2^u}$ then $t(m)|2^u$

Proof. $\because$ $m|M_{2^u} \Rightarrow 2^{2^u} \equiv 1(\bmod m)$ then by Proposition 4.3.0.9 (3), $t(m)|\nu(2^u)$
but $\nu(2^u)|2^u$ hence $t(m)|2^u$ □

2. For $u \geq 1$, If $m|M_{3^u}$ then $t(m)|8(3^{u-1})$

Proof. $\because$ $m|M_{3^u} \Rightarrow 2^{3^u} \equiv 1 (\bmod m)$ then by Proposition 4.3.0.9 (3), $t(m)|\nu(3^u)$ but $\nu(3^u)|8(3^{u-1})$ hence $t(m)|8(3^{u-1})$ □

3. If p is a prime of the form $10k \pm 1$ and $m|M_p$ then $t(m)|2p+2$

Proof. $\because$ $m|M_p \Rightarrow 2^p \equiv 1 (\bmod m)$ then by Proposition 4.3.0.9 (3), $t(m)|\nu(p)$
but if p is a prime of the form $10k \pm 1$ then $\nu(p)|2p+2$
$\therefore$ $t(m)|2p+2$ □

Conjucture: The above properties are not proved mathematically but it can be observed from pisano period table of $C_n (\bmod m)$

Definition 4.3.0.10. For a positive integer c, the sequence, say G_n is defined recurrently by $G_n = cG_{n-1} + G_{n-2}$ for $n \geq 2$ with initial conditions $G_0 = 0$ and $G_1 = 1$.
$\begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} = Z^n$ for $n \in \mathbb{N}$ where $Z = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$,
Define $W_n = 2^{G_n} - 1 \; \forall \, n \geq 0$

Proposition 4.3.0.11. *For $n, m \in \mathbb{N} \cup \{0\}$, If $n|m \Rightarrow W_n|W_m$*

Proposition 4.3.0.12. *For $n, m \in \mathbb{N} \cup \{0\}$, $W_{(n,m)} = (W_n, W_m)$*

Proof. A proof is similar to that of Proposition 4.2.0.2 (3),(4), so it is left to the reader. □

4.4 Some other Generalisation of Fibonacci

Definition 4.4.0.1. Define $A_0 = 0, A_1 = 1$ and $A_n = 2A_{n-1} + 3A_{n-2} \; \forall \; n \geq 2$

Some initial few terms of sequence are: $0, 1, 2, 7, 20, 61, ...$

Definition 4.4.0.2. We present another extension of generalised Fibonacci sequence $\{B_n\}$ defined by the relation

$$B_n = 2^{A_n} - 1 \; \forall \; n \geq 0$$

Some initial few terms of sequence are: $0, 1, 3, 127, ...$

Proposition 4.4.0.3. *For $n \geq 0$, B_n satisfies the non linear second order recurrence relation*

$$B_{n+2} = B_{n+1}B_n(B_{n+1}+2)\left[(B_n+2)(B_n+1)+1\right] + B_{n+1}(B_{n+1}+2) + B_n\left[(B_n+2)(B_n+1)+1\right]$$

with initial conditions $B_0 = 0$ & $B_1 = 1$

Proof.

$$\begin{aligned}
B_{n+2} &= 2^{A_{n+2}} - 1 = 2^{2A_{n+1}+3A_n} - 1 = 2^{2A_{n+1}+3A_n} - 2^{2A_{n+1}} + 2^{2A_{n+1}} - 1 \\
&= 2^{2A_{n+1}}(2^{3A_n} - 1) - (2^{3A_n} - 1) + (2^{3A_n} - 1) + 2^{2A_{n+1}} - 1 \\
&= (2^{2A_{n+1}} - 1)(2^{3A_n} - 1) + (2^{3A_n} - 1) + 2^{2A_{n+1}} - 1 \\
&= (2^{A_{n+1}} - 1)(2^{A_{n+1}} + 1)(2^{A_n} - 1)(2^{2A_n} + 2^{A_n} + 1) \\
&\quad + (2^{A_n} - 1)(2^{2A_n} + 2^{A_n} + 1) + (2^{A_{n+1}} - 1)(2^{A_{n+1}} + 1) \\
&= B_{n+1}B_n(B_{n+1}+2)[B_n(B_n+2) + 3 + B_n] \\
&\quad + B_{n+1}(B_{n+1}+2) + B_n[B_n(B_n+2) + B_n + 3] \\
&= B_{n+1}B_n(B_{n+1}+2)\left[(B_n+2)(B_n+1)+1\right] + B_{n+1}(B_{n+1}+2) \\
&\quad + B_n\left[(B_n+2)(B_n+1)+1\right]
\end{aligned}$$

□

Proposition 4.4.0.4. *For $n \geq 0$, we have $(A_n, A_{n+1}) = 1$ or 3*

Proof. For $n = 0$, $(A_0, A_1) = (0, 1) = 1$

For $n = 1$, $(A_1, A_2) = (1, 2) = 1$

Assume $(A_k, A_{k+1}) = 1$ where $k \geq 2$

Next, let $d = (A_{k+1}, A_{k+2})$

then $d|A_{k+1}$ and $d|A_{k+2} \Rightarrow d|A_{k+1}$ and $d|(2A_{k+1} + 3A_k)$

$\Rightarrow d|2A_{k+1}$ and $d|(2A_{k+1} + 3A_k) \Rightarrow d|3A_k$

Now, $d|3A_k$ and $d|3A_{k+1} \Rightarrow d|(3A_k, 3A_{k+1})$

$\Rightarrow d|3(A_k, A_{k+1}) \Rightarrow d|3 \Rightarrow d = 1$ or 3

hence $(A_n, A_{n+1}) = 1$ or 3 □

Conjucture: It is observed that $(A_n, A_{n+1}) = 1$

Proposition 4.4.0.5. *For $n \geq 0$, we have $(B_n, B_{n+1}) = 1$ or 7*

Proof. Now,

$$(B_n, B_{n+1}) = (2^{A_n} - 1, 2^{A_{n+1}} - 1) = 2^{(A_n, A_{n+1})} - 1$$

$\Rightarrow (B_n, B_{n+1}) = 1$ or 7 by Proposition 4.4.0.4 □

Conjucture: It is observed that $(B_n, B_{n+1}) = 1$

Chapter 5

k-FIBONACCI DIFFERENCE SEQUENCE

5.1 Introduction

The content of this topic is taken from the reference **[1]**. In this section we study k–Fibonacci numbers with some of their properties and the difference sequences.

Definition 5.1.0.1. For a positive integer k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1 \tag{5.1}$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

For $k = 1$, the clasical Fibonacci sequence $F = \{0, 1, 1, 2, 3, 5, 8, ..\}$ is obtained and for $k = 2$ the pell sequence $P = \{0, 1, 2, 5, 12, 29, ...\}$ appears.
Hence, the k-Fibonacci sequence is $F_k = \{0, 1, k, k^2 + 1, k^3 + 2k, k^4 + 3k^2 + 1, ...\}$
Moreover, we define $F_{k,-n} = (-1)^{n+1}F_{k,n}$
The well known Binet formula fin the Fibonacci numbers theory allows us to express the

k-Fibonacci numbers in function of the roots σ_1 and σ_2 of the characteristic equation, associated to the recurrence relation $r^2 = kr + 1$:

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \quad \text{with } \sigma_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2} \tag{5.2}$$

It is verified the limit of the quotient of two terms is

$$\lim_{n\to\infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma_{k,1}^r \tag{5.3}$$

In particular, $\sigma_{1,1}$ is the Golden Ratio, $\phi = \frac{1+\sqrt{5}}{2}$; $\sigma_{2,1}$ is the Silver Ratio, $\sigma_{3,1}$ the Bronze ratio and $\sigma_{4,1}$ the copper ratio.
Moreover, from the characteristic equation we deduce the following formula:

$$\sigma^2 = k\sigma + 1 \tag{5.4}$$

Finally, we define $F_{k,-n} = (-1)^{n-1} F_{k,n}$
Difference sequences: Given the numerical ordered sequence $\{a_0, a_1, a_2, \dots\}$, the first difference $\Delta(a_n)$ is defined as $\Delta(a_n) = a_{n+1} - a_n,\ n \geq 0$
The ith difference of the sequence a_n is written as $\Delta^i(a_n)$ and is defined recursively as

$$\Delta^i(a_n) = \Delta^{i-1}(a_{n+1}) - \Delta^{i-1}(a_n) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} a_{i+n-j}$$

5.2 Finite difference on the k-Fibonacci numbers

In this section we apply the concept of the finite difference to the k–Fibonacci numbers and we will call the k–Fibonacci difference sequence to the sequence obtained.

5.2.1 The k-Fibonacci difference sequences

If we apply the relation of the finite difference to the k-Fibonacci sequence $F_k = \{F_{k,n}\}$, then $\Delta(F_{k,n}) = F_{k,n+1} - F_{k,n}$, $\Delta^2(F_{k,n}) = \Delta(\Delta(F_{k,n})) = \Delta(F_{k,n+1} - F_{k,n}) = F_{k,n+2} - F_{k,n+1} - F_{k,n+1} + F_{k,n} = F_{k,n+2} - 2F_{k,n+1} + F_{k,n}$, etc. We will write

$$\Delta^{(i)}(F_k) = F_k^{(i)} = \{\Delta^{(i)}(F_{k,n})\} = \{F_{k,n}^{(i)}\} = \{F_{k,n+1}^{(i-1)} - F_{k,n}^{(i-1)}\} \text{ indistinctly.} \tag{5.5}$$

If $i = 0$, then $F_{k,n}^{(0)} = F_{k,n}$ and if $i = 1$, it is $\Delta^{(1)} = \Delta$.
From the definition of difference relation, it is
$\Delta(F_k) = \{1, k-1, k^2-k+1, k^3-k^2+2k-1, k^4-k^3+3k^2-2k+1, \dots\}$
$\Delta^2(F_k) = \{k-2, k^2-2k+2, k^3-2k^2+3k-2, k^4-2k^3+4k^2-4k+2, \dots\}$ etc. and as general form,

$$F_{k,n}^{(i)} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} F_{k,i+n-j} \tag{5.6}$$

For instance, The 3-Fibonacci sequence is $\{0, 1, 3, 10, 33, 109, \dots\}$. The first five 3-Fibonacci difference sequences are:

1. $F_3^{(1)} = \{1, 2, 7, 23, 76, 251, 829, 2738, 9043, \dots\}$
2. $F_3^{(2)} = \{1, 5, 16, 53, 175, 578, 1909, 6305, 20824, \dots\}$
3. $F_3^{(3)} = \{4, 11, 37, 122, 403, 1331, 4396, 14519, 47953, \dots\}$
4. $F_3^{(4)} = \{7, 26, 85, 281, 928, 3065, 10123, 33434, 110425, \dots\}$
5. $F_3^{(5)} = \{19, 59, 196, 647, 2137, 7058, 23311, 76991, 254284, \dots\}$

For $i \geq 3$, the 3-Fibonacci difference sequences $\Delta^i(F_3)$, are not cited in OEIS.
Next we will prove the k-Fibonacci difference sequences verify also the initial relation (5.1).

Lemma 5.2.1.1. *The k-Fibonacci difference numbers verify the recurrence relation of the k-Fibonacci numbers:*

$$F_{k,n+1}^{(i)} = kF_{k,n}^{(i)} + F_{k,n-1}^{(i)} \tag{5.7}$$

Proof. By induction. For $i = 1$,

$$F_{k,n+1}^{(1)} = F_{k,n+2} - F_{k,n+1} = (kF_{k,n+1} + F_{k,n}) - (kF_{k,n} + F_{k,n-1})$$
$$= k(F_{k,n+1} - F_{k,n}) + (F_{k,n} - F_{k,n-1}) = kF_{k,n}^{(1)} + F_{k,n-1}^{(1)}$$

Let us suppose this formula is true for i : $F_{k,n+1}^{(i)} = kF_{k,n}^{(i)} + F_{k,n-1}^{(i)}$.
then:

$$F_{k,n+1}^{(i+1)} = F_{k,n+2}^{(i)} - F_{k,n+1}^{(i)} = kF_{k,n+1}^{(i)} + F_{k,n}^{(i)} - kF_{k,n}^{(i)} - F_{k,n-1}^{(i)}$$
$$= k(F_{k,n+1}^{(i)} - F_{k,n}^{(i)}) + (F_{k,n}^{(i)} - F_{k,n-1}^{(i)}) = kF_{k,n}^{(i+1)} + F_{k,n-1}^{(i+1)}$$

□

From this relation we will be able to find the general term of the difference sequence $F_k^{(i)} = \{F_{k,n}^{(i)}\}$ in relation to the first initial terms $F_{k,0}^{(i)}$ and $F_{k,1}^{(i)}$. However $\{F_{k,n}^{(i)}\}$ is not a k-Fibonacci sequence because the initial values are not necessarily 0 and 1, respectively.

5.2.2 The Binet identity for $F_{k,n}^{(i)}$

From the relation (5.7) we can deduce $F_{k,n}^{(i)} = c_1\sigma_1^n + c_2\sigma_2^n$, where $\sigma_{1,2} = \frac{k\pm\sqrt{k^2+4}}{2}$, respectively. Then, for $n = 0 \Rightarrow c_1 + c_2 = F_{k,0}^{(i)}$ and for $n = 1 \Rightarrow c_1\sigma_1 + c_2\sigma_2 = F_{k,1}^{(i)}$. The solution of this system is the Binet identity $F_{k,n}^{(i)} = c_1\sigma_1^n + c_2\sigma_2^n$ where

$$c_1 = \frac{F_{k,1}^{(i)} - \sigma_2 F_{k,0}^{(i)}}{\sigma_1 - \sigma_2}, \quad c_2 = \frac{-F_{k,1}^{(i)} - \sigma_1 F_{k,0}^{(i)}}{\sigma_1 - \sigma_2}$$

We must take into account $F_{k,0}^{(i)}$ and $F_{k,1}^{(i)}$ change in every k-Fibonacci difference sequence. The Binet identity can be reduced taking into account the formula (5.2) and $\sigma_1\sigma_2 = -1$, in the following form:

$$
\begin{aligned}
F_{k,n}^{(i)} &= c_1\sigma_1^n + c_2\sigma_2^n = \frac{F_{k,1}^{(i)} - \sigma_2 F_{k,0}^{(i)}}{\sigma_1 - \sigma_2}\sigma_1^n + \frac{-F_{k,1}^{(i)} - \sigma_1 F_{k,0}^{(i)}}{\sigma_1 - \sigma_2}\sigma_2^n \\
&= F_{k,1}^{(i)}\frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} + F_{k,0}^{(i)}\frac{\sigma_1^{n-1} - \sigma_2^{n-1}}{\sigma_1 - \sigma_2} = F_{k,n}F_{k,1}^{(i)} + F_{k,n-1}F_{k,0}^{(i)}
\end{aligned}
$$

From this formula and taking into account the formula (5.3), we deduce

$$
\lim_{n\to\infty}\frac{F_{k,n+r}^{(i)}}{F_{k,n}^{(i)}} = \sigma_1^r
$$

In the next theorem, we give a new relationship between the terms of the sequence $\Delta^i(F_k)$ and the terms of the preceding sequence $\Delta^{i-1}(F_k)$.

Theorem 5.2.2.1. *(Second relationship). For $i,n \geq 1$,*

$$
F_{k,n}^{(i)} = (k-1)F_{k,n}^{(i-1)} + F_{k,n-1}^{(i-1)}, \text{ being } F_{k,0}^{(i)} = F_{k,1}^{(i-1)} - F_{k,0}^{(i-1)} \tag{5.8}
$$

Proof. From the definition of the k-Fibonacci difference numbers,

$$
F_{k,n-1}^{(i-1)} = \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}F_{k,i-2+n-j} \text{ and } F_{k,n}^{(i-1)} = \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}F_{k,i-1+n-j}
$$

then

$$
\begin{aligned}
(k-1)F_{k,n}^{(i-1)} + F_{k,n-1}^{(i-1)} &= k\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}F_{k,i-1+n-j} - \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}F_{k,i-1+n-j} \\
&\quad + \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}F_{k,i-2+n-j} \\
&= \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(kF_{k,i-1+n-j} + F_{k,i-2+n-j})
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} F_{k,i-1+n-j} \\
&= \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} F_{k,i+n-j} - \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} F_{k,i-1+n-j} \\
&= F_{k,i+n} + \sum_{j=1}^{i-1} (-1)^j \binom{i-1}{j} F_{k,i+n-j} - \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} F_{k,i-1+n-j} \\
&= F_{k,i+n} + \sum_{j=1}^{i-1} (-1)^j \left[\binom{i-1}{j} + \binom{i-1}{j-1} \right] F_{k,i+n-j} - (-1)^{i-1} F_{k,n} \\
&= F_{k,i+n} + \sum_{j=1}^{i-1} (-1)^j \binom{i}{j} F_{k,i+n-j} + (-1)^i F_{k,n} \\
&= \sum_{j=0}^{i} (-1)^j \binom{i}{j} F_{k,i+n-j} = F_{k,n}^{(i)}
\end{aligned}
$$

□

Curiously, for the classical Fibonacci sequence $(k=1)$, it is $F_{k,n}^{(i)} = F_{k,n-1}^{(i-1)}$
If we apply iterately the formula (5.8), then we will get the following formula. For $r \leq i$ and $k > 1$,

$$F_{k,n}^{(i)} = \sum_{j=0}^{r} \binom{r}{j} (k-1)^{r-j} F_{k,n-j}^{(i-r)} \tag{5.9}$$

If we represent the sum $\sum_{j=0}^{n} \binom{n}{j} p^{n-j} F_{k,q-j}$ in symbolic form as $[p + F_{k,q}]^{(n)}$, this last formula can be expressed like

$$F_{k,n}^{(i)} = \left[(k-1) + F_{k,n}^{(i-r)} \right]^{(r)} = F_{k,n}^{(i)} = \sum_{j=0}^{r} \binom{r}{j} (k-1)^{r-j} F_{k,n-j}^{(i-r)}$$

For $r = i$ we can indicate the k-Fibonacci difference numbers depending on the k-Fibonacci numbers in the manner indicated in the following corollary.

Corollary 5.2.2.2. *If* $r = i$,

$$F_{k,n}^{(i)} = \sum_{j=0}^{i} \binom{i}{j} (k-1)^{i-j} F_{k,n-j} \tag{5.10}$$

For the classical Fibonacci sequence $(k = 1)$, we cannot apply this formula and must apply the formula (5.6) of the definition.
In the next theorem we give the formula for calculating the sum of the k-Fibonacci difference numbers.

Theorem 5.2.2.3.

$$\sum_{j=0}^{n} F_{k,j}^{(i)} = F_{k,n+1}^{(i-1)} - F_{k,0}^{(i-1)} \tag{5.11}$$

Proof. Apllying the formula (5.7) and later the (5.5), we obtain

$$\begin{aligned}
\sum_{j=0}^{n} F_{k,j}^{(i)} &= F_{k,0}^{(i)} + \sum_{j=1}^{n} F_{k,j}^{(i)} \\
&= F_{k,0}^{(i)} + \frac{1}{k} \sum_{j=0}^{n} \left(F_{k,j+1}^{(i)} F_{k,j-1}^{(i)} \right) \\
&= F_{k,0}^{(i)} + \frac{1}{k} \left(F_{k,n+1}^{(i)} + F_{k,n}^{(i)} - F_{k,1}^{(i)} - F_{k,0}^{(i)} \right) \\
&= F_{k,0}^{(i)} + \frac{1}{k} \left(F_{k,n+2}^{(i-1)} - F_{k,n+1}^{(i-1)} + F_{k,n+1}^{(i-1)} - F_{k,n}^{(i-1)} - F_{k,2}^{(i-1)} + F_{k,1}^{(i-1)} - F_{k,1}^{(i-1)} + F_{k,0}^{(i-1)} \right) \\
&= F_{k,1}^{(i-1)} - F_{k,0}^{(i-1)} + \frac{1}{k} \left(F_{k,n+2}^{(i-1)} - F_{k,n}^{(i-1)} - (F_{k,2}^{(i-1)} - F_{k,0}^{(i-1)}) \right) \\
&= F_{k,1}^{(i-1)} - F_{k,0}^{(i-1)} + F_{k,n+1}^{(i-1)} - F_{k,1}^{(i-1)} \\
&= F_{k,n+1}^{(i-1)} - F_{k,0}^{(i-1)}
\end{aligned}$$

□

If we substitute the formula (5.10) in (5.11), we can find out a formula for this last sum in function of the k-Fibonacci numbers

Theorem 5.2.2.4. *The sum of the $n+1$ first terms of the k-Fibonacci difference sequence is given by*

$$\sum_{j=0}^{n} F_{k,j}^{(i)} = \sum_{j=0}^{i-1} \binom{i-1}{j} (k-1)^{i-1-j} (F_{k,n+1-j} + (-1)^j F_{k,j})$$

Proof.

$$\begin{aligned}\sum_{j=0}^{n} F_{k,j}^{(i)} &= F_{k,n+1}^{(i-1)} - F_{k,0}^{(i-1)} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} (k-1)^{i-1-j} F_{k,n+1-j} \\ &\quad - \sum_{j=0}^{i-1} \binom{i-1}{j} (k-1)^{i-1-j} F_{k,-j} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} (k-1)^{i-1-j} (F_{k,n+1-j} + (-1)^j F_{k,j})\end{aligned}$$

□

taking into account $F_{k,-j} = (-1)^{j+1} F_{k,j}$.

We can give a formula for the sum about the terms of this same sequence. Let $S_n^{(i)}$ be this sum. Then

$$\begin{aligned} S_{k,n}^{(i)} &= \qquad F_{k,0}^{(i)} + F_{k,1}^{(i)} + F_{k,2}^{(i)} + \cdots + F_{k,n-1}^{(i)} + F_{k,n}^{(i)} \\ kS_{k,n}^{(i)} &= kF_{k,0}^{(i)} + kF_{k,1}^{(i)} + kF_{k,2}^{(i)} + \cdots + kF_{k,n}^{(i)} \end{aligned}$$

$$\begin{aligned} kS_{k,n}^{(i)} + S_{k,n}^{(i)} &= kF_{k,0}^{(i)} + F_{k,2}^{(i)} + F_{k,3}^{(i)} + \cdots + F_{k,n+1}^{(i)} + F_{k,n}^{(i)} \\ &= kF_{k,0}^{(i)} + S_{k,n}^{(i)} - F_{k,0}^{(i)} - F_{k,1}^{(i)} + F_{k,n}^{(i)} + F_{k,n+1}^{(i)} \\ \Rightarrow S_{k,n}^{(i)} &= \frac{1}{k} \left((k-1)F_{k,0}^{(i)} - F_{k,1}^{(i)} + F_{k,n}^{(i)} + F_{k,n+1}^{(i)} \right) \end{aligned}$$

5.2.3 Generating function of the k-Fibonacci difference sequences

Let $f^{(i)}(x)$ be the generating function of the k-Fibonacci difference sequence $F_k^{(i)}$, that is $f^{(i)}(x) = F_{k,0}^{(i)} + F_{k,1}^{(i)}x + F_{k,2}^{(i)}x^2 + \cdots$

Then we will prove the following formula for the generating function.

$$f^{(i)}(x) = \frac{F_{k,0}^{(i)} + (F_{k,1}^{(i)} - kF_{k,0}^{(i)})x}{1 - kx - x^2} \tag{5.12}$$

Proof.

$$\begin{aligned} f^{(i)}(x) &= F_{k,0}^{(i)} + F_{k,1}^{(i)}x + F_{k,2}^{(i)}x^2 + F_{k,3}^{(i)}x^3 + F_{k,4}^{(i)}x^4 + \cdots \\ kxf^{(i)}(x) &= kF_{k,0}^{(i)}x + kF_{k,1}^{(i)}x^2 + kF_{k,2}^{(i)}x^3 + kF_{k,3}^{(i)}x^4 + \cdots \\ x^2 f^{(i)}(x) &= F_{k,0}^{(i)}x^2 + F_{k,1}^{(i)}x^3 + F_{k,2}^{(i)}x^4 + \cdots \end{aligned}$$

$$\begin{aligned} f^{(i)}(x)(1 - kx - x^2) &= F_{k,0}^{(i)} + (F_{k,1}^{(i)} - kF_{k,0}^{(i)})x \\ \Rightarrow f^{(i)}(x) &= \frac{F_{k,0}^{(i)} + (F_{k,1}^{(i)} - kF_{k,0}^{(i)})x}{1 - kx - x^2} \end{aligned}$$

because the recurrence formula (5.7). From the formula (5.10), it is
$F_{k,0}^{(i)} = \sum_{j=0}^{i} \binom{i}{j}(k-1)^{i-j}F_{k,-j}$ and $F_{k,1}^{(i)} = \sum_{j=0}^{i} \binom{i}{j}(k-1)^{i-j}F_{k,1-j}$, with $F_{k,-n} = (-1)^{n-1}F_{k,n}$.

□

As an example, we indicate the generating functions of the first five k-Fibonacci differences sequences:

$$\begin{aligned} f^{(1)}(x) &= \frac{1 - x}{1 - kx - x^2} \\ f^{(2)}(x) &= \frac{(k-2) + 2x}{1 - kx - x^2} \end{aligned}$$

$$f^{(3)}(x) = \frac{(k^2-3k+4)+(k-4)x}{1-kx-x^2}$$
$$f^{(4)}(x) = \frac{(k^3-4k^2+8k-8)+(k^2-4k+8)x}{1-kx-x^2}$$
$$f^{(5)}(x) = \frac{(k^4-5k^3+13k^2-20k+16)+(k^3-5k^2+12k-16)x}{1-kx-x^2}$$

5.3 On the sequence of initial values $\{F_{k,0}^{(n)}\}_{n\in\mathbb{N}}$

In this section we will study the sequence $\{F_{k,0}^{(n)}\}_{n\in\mathbb{N}}, k \geq 2$ and then apply to the problem of the polynomial interpolation.
From the formula of the negative k-Fibonacci numbers, we can obtain the following corollary of the formula (5.10).

Corollary 5.3.0.1. *If $n = 0, i = n$ and $k > 1$*

$$F_{k,0}^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (k-1)^{n-j}(-1)^{j+1} F_{k,j} \tag{5.13}$$

Theorem 5.3.0.2. *$\{F_{k,0}^{(n)}\}_{n\in\mathbb{N}}$ is a generated k-Fibonacci sequence that verifies the recurrence relation*

$$F_{k,0}^{(n+1)} = (k-2)F_{k,0}^{(n)} + kF_{k,0}^{(n-1)} \tag{5.14}$$

with initial conditions$F_{k,0}^{(0)} = 0$ and $F_{k,0}^{(1)} = 1$

Proof. For the proof we will apply the Binet identity (5.2).

$$\begin{aligned}(k-2)F_{k,0}^{(n)} + kF_{k,0}^{(n-1)} &= (k-2)\sum_{j=0}^{n} \binom{n}{j} (k-1)^{n-j}(-1)^{j+1} F_{k,j} \\ &\quad + k\sum_{j=0}^{n-1} \binom{n-1}{j} (k-1)^{n-1-j}(-1)^{j+1} F_{k,j}\end{aligned}$$

$$
\begin{aligned}
&= (k-2)\sum_{j=0}^{n}\binom{n}{j}(k-1)^{n-j}(-1)^{j+1}\left(\frac{\sigma_1^j-\sigma_2^j}{\sigma_1-\sigma_2}\right)\\
&\quad + k\sum_{j=0}^{n-1}\binom{n-1}{j}(k-1)^{n-1-j}(-1)^{j+1}\left(\frac{\sigma_1^j-\sigma_2^j}{\sigma_1-\sigma_2}\right)\\
&= \frac{(k-2)}{\sigma_1-\sigma_2}\left[-\sum_{j=0}^{n}\binom{n}{j}(k-1)^{n-j}(-1)^j\sigma_1^j+\sum_{j=0}^{n}\binom{n}{j}(k-1)^{n-j}(-1)^j\sigma_2^j\right]\\
&\quad + \frac{k}{\sigma_1-\sigma_2}\left[-\sum_{j=0}^{n-1}\binom{n-1}{j}(k-1)^{n-1-j}(-1)^j\sigma_1^j\right.\\
&\quad \left.+\sum_{j=0}^{n-1}\binom{n-1}{j}(k-1)^{n-1-j}(-1)^j\sigma_2^j\right]\\
&= \frac{(k-2)}{\sigma_1-\sigma_2}\{-(k-1-\sigma_1)^n+(k-1-\sigma_2)^n\}\\
&\quad + \frac{k}{\sigma_1-\sigma_2}\left\{-(k-1-\sigma_1)^{n-1}+(k-1-\sigma_2)^{n-1}\right\}\\
&= \frac{1}{\sigma_1-\sigma_2}\left\{-(k-1-\sigma_1)^{n-1}[(k-2)(k-1-\sigma_1)+k]\right.\\
&\quad \left.-(k-1-\sigma_2)^{n-1}[(k-2)(k-1-\sigma_2)+k]\right\}\\
&= \frac{1}{\sigma_1-\sigma_2}\left\{-(k-1-\sigma_1)^{n-1}[k^2-2k-\sigma_1 k+2\sigma_1+2]\right.\\
&\quad \left.-(k-1-\sigma_2)^{n-1}[k^2-2k-\sigma_2 k+2\sigma_2+2]\right\}
\end{aligned}
$$

but $k^2-2k-\sigma_r k+2\sigma_r+2=(k-1-\sigma_r)^2$ for $r=1,2$

$$
\begin{aligned}
&= \frac{1}{\sigma_1-\sigma_2}\left\{-(k-1-\sigma_1)^{n+1}+(k-1-\sigma_2)^{n+1}\right\}\\
&= \frac{1}{\sigma_1-\sigma_2}\left\{-\sum_{j=0}^{n+1}\binom{n+1}{j}(k-1)^{n+1-j}(-1)^j\sigma_1^j+\sum_{j=0}^{n+1}\binom{n+1}{j}(k-1)^{n+1-j}(-1)^j\sigma_2^j\right\}\\
&= \frac{1}{\sigma_1-\sigma_2}\left\{\sum_{j=0}^{n+1}\binom{n+1}{j}(k-1)^{n+1-j}(-1)^{j+1}(\sigma_1^j-\sigma_2^j)\right\}\\
&= \sum_{j=0}^{n+1}\binom{n+1}{j}(k-1)^{n+1-j}(-1)^{j+1}F_{k,j}\\
&= F_{k,0}^{(n+1)}
\end{aligned}
$$

□

The first terms of the sequence $\{F_{k,0}^{(n)}\}$ are:

$$\{0,1,k-2,k^2-3k+4,k^3-4k^2+8k-8,k^4-5k^3+13k^2-20k+16,\dots\} \qquad (5.15)$$

For $k=2,3,\dots,11$ all these sequences are cited in OEIS. If $k=3$, the coefficients are $\{0,1,1,4,7,\dots\}$ that we use later, in an example of polynomial interpolation.

5.3.1 Binet identity

It is trivial to prove that $F_{k,0}^{(n)}=\frac{(\sigma_1-1)^n-(\sigma_2-1)^n}{\sigma_1-\sigma_2}$ with $\sigma_{1,2}=\frac{k\pm\sqrt{k^2+4}}{2}$.
Developing this formula and taking into account the formula (5.2)

$$\begin{aligned} F_{k,0}^{(n)} &= \frac{(\sigma_1-1)^n-(\sigma_2-1)^n}{\sigma_1-\sigma_2} = \frac{1}{\sigma_1-\sigma_2}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(\sigma_1^j-\sigma_2^j) \\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}F_{k,j} \end{aligned}$$

This form of the identity $F_{k,0}^{(n)}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}F_{k,j}$ is simpler than the (5.13). In symbolic form, $F_{k,0}^{(n)}=[1-F_{k,0}]^{(n)}$.

5.3.2 Generating function of the sequence $\{F_{k,0}^{(n)}\}$

Let $f_0(x)$ be the generating function of the sequence $\{F_{k,0}^{(n)}\}$. Then,

$$\begin{aligned} f_0(x) &= F_{k,0}^{(0)}+F_{k,0}^{(1)}x+F_{k,0}^{(2)}x^2+F_{k,0}^{(3)}x^3+\cdots \\ (k-2)xf_0(x) &= (k-2)F_{k,0}^{(0)}x+(k-2)F_{k,0}^{(1)}x^2+(k-2)F_{k,0}^{(2)}x^3+\cdots \\ kx^2f_0(x) &= kF_{k,0}^{(0)}x^2+kF_{k,0}^{(1)}x^3+\cdots \end{aligned}$$

and taking into account $F_{k,0}^{(0)} = 0$, $F_{k,0}^{(1)} = 1$ and the recurrence relation (5.14), $f_0(x)(1-(k-2)x-kx^2) = F_{k,0}^{(0)} + (F_{k,0}^{(1)} - (k-2)F_{k,0}^{(0)})x = x$

$$\Rightarrow f_0(x) = \frac{x}{1-(k-2)x-kx^2}$$

5.3.3 *k*-Fibonacci Newton interpolation

Let us consider the $n+1$ points $(x_j, F_{k,j})$, $j = 0, 1, 2, \ldots, n$ with $x_j < x_{j+1}$ and let suppose we wish to find a polynomial $P_n(k,x)$ that takes the value $F_{k,j}$ for $x = x_j$. It is the same thing that to say that we must find a polynomial that passes through for the points $(x_j, F_{k,j})$ for $j = 0, 1, 2, \ldots, n$. Let $h_j = x_{j+1} - x_j$, The k-Fibonacci Newton polynomial interpolation is

$$P_n(k,x) = F_{k,0} + \frac{F_{k,0}^{(1)}}{1!}\frac{x-x_0}{h_0} + \frac{F_{k,0}^{(2)}}{2!}\frac{x-x_0}{h_0}\frac{x-x_1}{h_1} + \cdots$$

or in reduced form,

$$P_n(k,x) = F_{k,0} + \sum_{i=1}^{n}\frac{F_{k,0}^{(i)}}{i!}\prod_{j=0}^{i-1}\frac{x-x_j}{h_j},$$

Where $F_{k,0}^{(i)}$ is given by the formula (5.13). This formula can be simplified by if $x_j = j$ and takes the more practical form

$$P_n(k,x) = F_{k,0} + \sum_{i=1}^{n}\frac{F_{k,0}^{(i)}}{i!}\prod_{j=0}^{i-1}(x-j) \tag{5.16}$$

If $x_{j+1} - x_j = h\,\forall j$, the error is given by $\varepsilon = \frac{F_{k,0}^{(n+1)}}{(n+1)!}\frac{1}{h^{n+1}}\prod_{j=0}^{n}(x-x_j)$. Thus, the maximum error will occur at some point in the interval between two successive nodes.
If $x \neq 0, h \neq 1$, but $x_j - x_{j-1} = h$, the change variable $t_j = \frac{x_j - x_0}{h}$ transforms the sequence

$\{x_0, x_1, x_2, \dots\}$ in $\{0, 1, 2, \dots\}$.
If $F_{k,0}^{(0)} \neq 0$, we must apply the formula (5.6).
For example

$$\begin{aligned}
P_4(k,x) = F_{k,0} &+ \frac{1}{1!}F_{k,0}^{(1)}(x-x_0) + \frac{1}{2!}F_{k,0}^{(2)}(x-x_0)(x-x_1) \\
&\frac{1}{3!}F_{k,0}^{(3)}(x-x_0)(x-x_1)(x-x_2) \\
&\frac{1}{4!}F_{k,0}^{(4)}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \\
&= x + \frac{1}{2}(k-2)x(x-1) + \frac{1}{6}(k^2-3k+4)x(x-1)(x-2) \\
&\quad + \frac{1}{24}(k^3-4k^2+8k-8)x(x-1)(x-2)(x-3)
\end{aligned}$$

If for instance $k=3$, then $P_4(3,x) = \frac{1}{24}(7x^4 - 26x^3 + 41x^2 + 2x)$. The error is bounded by $\varepsilon = \frac{19}{5!} max|x(x-1)(x-2)(x-3)(x-4)| = 0.0562981$ (using Mathematica 8.0).
If we apply directly the classical Newton interpolation, then

x_j	$F_{3,j}$	$\Delta(F_{3,j})$	$\Delta^2(F_{3,j})$	$\Delta^3(F_{3,j})$	$\Delta^4(F_{3,j})$
0	0				
		1			
1	1		1		
		2		4	
2	3		5		7
		7		11	
3	10		16		
		23			
4	33				

The number of the first diagonal line are the numerators of the coefficients of the Newton polynomial:

$$\begin{aligned}
P_4(3,x) &= 0 + \frac{1}{1!}x + \frac{1}{2!}x(x-1) + \frac{4}{3!}x(x-1)(x-2) + \frac{7}{4!}x(x-1)(x-2)(x-3) \\
&= \frac{1}{24}(7x^4 - 26x^3 + 41x^2 + 2x).
\end{aligned}$$

Chapter 6

<u>k-FIBONACCI NUMBERS</u>

6.1 Introduction

The content of this topic is taken from the reference [4]. Fibonacci numbers possess wonderful and amazing properties; though some are simple and known, others find broad scope in research work. Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy and Vajda. The Fibonacci numbers F_n are the terms of the sequence $\{0,1,1,2,3,5,8,\dots\}$ wherein each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$. Also the ratio of two consecutive Fibonacci numbers converges to the Golden mean, $\phi = \frac{1+\sqrt{5}}{2}$. The Fibonacci numbers and Golden mean find numerous applications in modern science and have been extensively used in number theory, applied mathematics,physics, computer science, and biology.

The well known Fibonacci sequence is defined as $F_0 = 0, F_1 = 1$,

$$F_n = F_{n-1} + F_{n-2} \ \forall\, n \geq 2$$

In a similar way, Lucas sequence is defined as $L_0 = 2, L_1 = 1$,

$$L_n = L_{n-1} + L_{n-2} \ \forall\, n \geq 2$$

The second order Fibonacci sequence has been generalized in several ways. Some authors have preserved the recurrence relation and altered the first two terms of the sequence while others have preserved the first two terms of the sequence and altered the recurrence relation slightly. The k-Fibonacci sequence introduced by Falcon and Plaza depends only on one integer parameter k and is defined as follows:

$$F_{k,0} = 0, \;\; F_{k,1} = 1,$$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{where} \;\; n \geq 1, \; k \geq 1.$$

The first few terms of this sequence are $\{0, 1, k, k^2 + 1, \dots\}$
The particular cases of the k-Fibonacci sequence are as follows.
If $k = 1$, the classical Fibonacci sequence is obtained: $\{F_n\}_{n\in\mathbb{N}} = \{0, 1, 1, 2, 3, 5, \dots\}$
If $k = 2$, the Pell sequence is obtained: $\{P_n\}_{n\in\mathbb{N}} = \{0, 1, 2, 5, 12, 29, \dots\}$
Motivated by the study of k-Fibonacci numbers, the k-Lucas numbers have been defined in a similar fashion as:

$$L_{k,0} = 2, \;\; L_{k,1} = 1,$$

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad \text{where} \;\; n \geq 1, \; k \geq 1.$$

The first few terms of this sequence are $\{2, 1, k^2 + 2, \dots\}$
The particular cases of the k-Lucas sequence are as follows.
If $k = 1$, the classical k-Lucas sequence is obtained: $\{2, 1, 3, 4, 7, \dots\}$
If $k = 2$, the Pell-Lucas sequence is obtained: $\{2, 2, 6, 14, 34, \dots\}$
In the 19th century, the French mathematician Binet devised two remarkable analytical

formulas for the Fibonacci and Lucas numbers. The same idea has been used to develop Binet formulas for other recursive sequences as well.The wellknown Binet's formulas for k-Fibonacci numbers and k-Lucas numbers are given by

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \quad L_{k,n} = r_1^n + r_2^n$$

where r_1, r_2 are roots of equation $r^2 - kr - 1 = 0$ which are given by

$$r_1 = \frac{k + \sqrt{k^2+4}}{2}, \qquad r_2 = \frac{k - \sqrt{k^2+4}}{2}$$

We also note that $r_1 + r_2 = k, \quad r_1 r_2 = -1, \quad r_1 - r_2 = \sqrt{k^2+4}$

There are a huge number of simple as well as generalized identities available in the Fibonacci related literature in various forms. Some properties for common factors of Fibonacci and Lucas numbers are studied by Thongmoon. The k-Fibonacci numbers which are of recent origin were found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge partition, serving as an example between geometry and numbers. Also authors established some new properties of k-Fibonacci numbers and k-Lucas numbers in terms of binomial sums. Falcon and Plaza studied 3-dimensional k-Fibonacci spirals considering geometric point of view. Some identities for k-Lucas numbers may be found. In many properties of k-Fibonacci numbers are obtained by easy arguments and related with so-called Pascal triangle. The aim of the present paper is to establish connection formulas between k-Fibonacci and k-Lucas numbers, thereby deriving some results out of them. In the following section we investigate some products of k-Fibonacci numbers and k-Lucas numbers. Though the results can be established by induction method as well, Binet's formula is mainly used to prove all of them.

6.2 On the products of k-Fibonacci and k-Lucas numbers

Theorem 6.2.0.1. $F_{k,2n}L_{k,2n} = F_{k,4n}$, *where* $n \geq 1$

Proof.

$$\begin{aligned}
F_{k,2n}L_{k,2n} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right]\left[r_1^{2n} + r_2^{2n}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n} + (r_1 r_2)^{2n} - (r_1 r_2)^{2n} - r_2^{4n}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n} - r_2^{4n}\right] = F_{k,4n}
\end{aligned}$$

□

Theorem 6.2.0.2. $F_{k,2n}L_{k,2n+1} = F_{k,4n+1} - 1$, *where* $n \geq 1$

Proof.

$$\begin{aligned}
F_{k,2n}L_{k,2n+1} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right]\left[r_1^{2n+1} + r_2^{2n+1}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n+1} + (r_1 r_2)^{2n} r_2 - (r_1 r_2)^{2n} r_1 - r_2^{4n+1}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n+1} - r_2^{4n+1}\right] + \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)}(r_2 - r_1) \\
&= F_{k,4n+1} - (-1)^{2n} = F_{k,4n+1} - 1
\end{aligned}$$

□

Theorem 6.2.0.3. $F_{k,2n}L_{k,2n+2} = F_{k,4n+2} - k$, *where* $n \geq 1$

Proof.

$$\begin{aligned}
F_{k,2n}L_{k,2n+2} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right]\left[r_1^{2n+2} + r_2^{2n+2}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n+2} + (r_1 r_2)^{2n} r_2^2 - (r_1 r_2)^{2n} r_1^2 - r_2^{4n+2}\right]
\end{aligned}$$

$$= \frac{1}{r_1 - r_2}\left[r_1^{4n+2} - r_2^{4n+2}\right] - \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)}(r_1^2 - r_2^2)$$
$$= F_{k,4n+2} - (-1)^{2n} k = F_{k,4n+2} - k$$

□

Theorem 6.2.0.4. $F_{k,2n}L_{k,2n+3} = F_{k,4n+3} - (k^2+1)$, *where* $n \geq 1$

Proof.

$$F_{k,2n}L_{k,2n+3} = \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right]\left[r_1^{2n+3} + r_2^{2n+3}\right]$$
$$= \frac{1}{r_1 - r_2}\left[r_1^{4n+3} + (r_1 r_2)^{2n} r_2^3 - (r_1 r_2)^{2n} r_1^3 - r_2^{4n+3}\right]$$
$$= \frac{1}{r_1 - r_2}\left[r_1^{4n+3} - r_2^{4n+3}\right] - \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)}(r_1^3 - r_2^3)$$
$$= F_{k,4n+3} - (-1)^{2n}\left[r_1^2 + r_2^2 + r_1 r_2\right]$$
$$= F_{k,4n+3} - (L_{k,2} - 1) = F_{k,4n+3} - (k^2+1)$$

□

Theorem 6.2.0.5. $F_{k,2n-1}L_{k,2n+1} = F_{k,4n} + 1$, *where* $n \geq 1$

Proof.

$$F_{k,2n-1}L_{k,2n+1} = \left[\frac{r_1^{2n-1} - r_2^{2n+1}}{r_1 - r_2}\right]\left[r_1^{2n+1} + r_2^{2n+1}\right]$$
$$= \frac{1}{r_1 - r_2}\left[r_1^{4n} + r_1^{2n-1} r_2^{2n+1} - r_1^{2n+1} r_2^{2n-1} - r_2^{4n}\right]$$
$$= \frac{1}{r_1 - r_2}\left[r_1^{4n} - r_2^{4n}\right] + \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)}\left[\frac{r_2}{r_1} - \frac{r_1}{r_2}\right]$$
$$= F_{k,4n} - (r_1 r_2)^{2n-1} = F_{k,4n} + 1$$

□

Theorem 6.2.0.6. $F_{k,2n+1}L_{k,2n} = F_{k,4n+1} + 1$, *where* $n \geq 1$

Proof.

$$\begin{aligned}
F_{k,2n+1}L_{k,2n} &= \left[\frac{r_1^{2n+1}-r_2^{2n+1}}{r_1-r_2}\right]\left[r_1^{2n}+r_2^{2n}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+1}+(r_1r_2)^{2n}r_1-(r_1r_2)^{2n}r_2-r_2^{4n+1}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+1}-r_2^{4n+1}\right]+\frac{(r_1r_2)^{2n}}{(r_1-r_2)}(r_1-r_2) \\
&= F_{k,4n+1}+(-1)^{2n}=F_{k,4n+1}+1
\end{aligned}$$

□

Theorem 6.2.0.7. $F_{k,2n+2}L_{k,2n}=F_{k,4n+2}+k,$ *where* $n\geq 1$

Proof.

$$\begin{aligned}
F_{k,2n+2}L_{k,2n} &= \left[\frac{r_1^{2n+2}-r_2^{2n+2}}{r_1-r_2}\right]\left[r_1^{2n}+r_2^{2n}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+2}+(r_1r_2)^{2n}r_1^2-(r_1r_2)^{2n}r_2^2-r_2^{4n+2}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+2}-r_2^{4n+2}\right]+(-1)^{2n}\left[\frac{r_1^2-r_2^2}{r_1-r_2}\right] \\
&= F_{k,4n+2}+[r_1+r_2]=F_{k,4n+2}+k
\end{aligned}$$

□

Theorem 6.2.0.8. $F_{k,2n+2}L_{k,2n+1}=F_{k,4n+3}-1,$ *where* $n\geq 1$

Proof.

$$\begin{aligned}
F_{k,2n+2}L_{k,2n+1} &= \left[\frac{r_1^{2n+2}-r_2^{2n+2}}{r_1-r_2}\right]\left[r_1^{2n+1}+r_2^{2n+1}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+3}+(r_1r_2)^{2n+1}r_1-(r_1r_2)^{2n+1}r_2-r_2^{4n+3}\right] \\
&= \frac{1}{r_1-r_2}\left[r_1^{4n+3}-r_2^{4n+3}\right]+(-1)^{2n+1} \\
&= F_{k,4n+3}-1
\end{aligned}$$

□

Theorem 6.2.0.9. $F_{k,m}L_{k,n} = F_{k,m+n} - (-1)^m F_{k,n-m}$, *for* $n \geq m+1$, $m \geq 0$

Proof.

$$\begin{aligned}
F_{k,m}L_{k,n} &= \left[\frac{r_1^m - r_2^m}{r_1 - r_2}\right]\left[r_1^n + r_2^n\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{m+n} + r_1^m r_2^n - r_1^n r_2^m - r_2^{m+n}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{m+n} - r_2^{m+n}\right] + \frac{1}{r_1 - r_2}\left[r_1^m r_2^n - r_1^n r_2^m\right] \\
&= F_{k,m+n} - (r_1 r_2)^m \left[\frac{r_1^{n-m} - r_2^{n-m}}{r_1 - r_2}\right] = F_{k,m+n} - (-1)^m F_{k,n-m}
\end{aligned}$$

□

For different value of m, we have different results:

If $m = 0$ then $F_{k,0}L_{k,n} = F_{k,n} - F_{k,n} = 0, \quad n \geq 1$

If $m = 1$ then $F_{k,1}L_{k,n} = F_{k,n+1} + F_{k,n-1}, \quad n \geq 2$

or $L_{k,n} = F_{k,n+1} + F_{k,n-1}$

If $m = 2$ then $F_{k,2}L_{k,n} = F_{k,n+2} - F_{k,n-2}, \quad n \geq 3$

$$\text{or } L_{k,n} = \frac{F_{k,n+2} + F_{k,n-2}}{k} \text{ and so on}$$

Theorem 6.2.0.10. $F_{k,n}L_{k,2n+m} = F_{k,3n+m} - (-1)^n F_{k,n+m}$, *for* $n \geq 1$, $m \geq 0$

Proof.

$$\begin{aligned}
F_{k,n}L_{k,2n+m} &= \left[\frac{r_1^n - r_2^n}{r_1 - r_2}\right]\left[r_1^{2n+m} + r_2^{2n+m}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{3n+m} + r_1^n r_2^{2n+m} - r_1^{2n+m} r_2^n - r_2^{3n+m}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{3n+m} - r_2^{3n+m}\right] + (r_1 r_2)^n \left[\frac{r_2^{n+m} - r_1^{n+m}}{r_1 - r_2}\right] \\
&= F_{k,3n+m} - (-1)^n F_{k,n+m}
\end{aligned}$$

□

For different values of m, we have various results:

If $m = 0$ then $F_{k,n}L_{k,2n} = F_{k,3n} - (-1)^n F_{k,n}, \quad n \geq 1$

If $m = 1$ then $F_{k,n}L_{k,2n+1} = F_{k,3n+1} - (-1)^n F_{k,n+1}, \quad n \geq 1$ and so on.

Similarly we have the following result.

Theorem 6.2.0.11. $F_{k,2n+m}L_{k,n} = F_{k,3n+m} + (-1)^n F_{k,n+m}$, *for* $n \geq 1,\ m \geq 0$

Proof.

$$\begin{aligned}
F_{k,2n+m}L_{k,n} &= \left[\frac{r_1^{2n+m} - r_2^{2n+m}}{r_1 - r_2}\right]\left[r_1^n + r_2^n\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{3n+m} + r_2^n r_1^{2n+m} - r_2^{2n+m} r_1^n - r_2^{3n+m}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{3n+m} - r_2^{3n+m}\right] + (r_1 r_2)^n \left[\frac{r_1^{n+m} - r_2^{n+m}}{r_1 - r_2}\right] \\
&= F_{k,3n+m} + (-1)^n F_{k,n+m}
\end{aligned}$$

□

Theorem 6.2.0.12. $F_{k,2n}L_{k,2n+m} = F_{k,4n+m} - F_{k,m}$, *for* $n \geq 1,\ m \geq 0$

Proof.

$$\begin{aligned}
F_{k,2n}L_{k,2n+m} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right]\left[r_1^{2n+m} + r_2^{2n+m}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n+m} + r_1^{2n} r_2^{2n+m} - r_2^{2n} r_1^{2n+m} - r_2^{4n+m}\right] \\
&= \frac{1}{r_1 - r_2}\left[r_1^{4n+m} - r_2^{4n+m}\right] + (r_1 r_2)^{2n}\left[\frac{r_2^m - r_1^m}{r_1 - r_2}\right] \\
&= F_{k,4n+m} - F_{k,m}
\end{aligned}$$

□

For different values of m, we have various results:

If $m = 0$ then $F_{k,2n}L_{k,2n} = F_{k,4n}, \quad n \geq 1$

If $m = 1$ then $F_{k,2n}L_{k,2n+1} = F_{k,4n+1} - 1, \quad n \geq 1$ and so on.

Theorem 6.2.0.13. $F_{k,2n+m}L_{k,2n} = F_{k,4n+m} + F_{k,m}$, *for* $n \geq 1,\ m \geq 0$

Proof.

$$\begin{aligned} F_{k,2n+m}L_{k,2n} &= \left[\frac{r_1^{2n+m} - r_2^{2n+m}}{r_1 - r_2}\right]\left[r_1^{2n} + r_2^{2n}\right] \\ &= \frac{1}{r_1 - r_2}\left[r_1^{4n+m} + r_2^{2n}r_1^{2n+m} - r_1^{2n}r_2^{2n+m} - r_2^{4n+m}\right] \\ &= \frac{1}{r_1 - r_2}\left[r_1^{4n+m} - r_2^{4n+m}\right] + (r_1 r_2)^{2n}\left[\frac{r_1^m - r_2^m}{r_1 - r_2}\right] \\ &= F_{k,4n+m} + F_{k,m} \end{aligned}$$

□

For different values of m we have various results:

If $m = 0$ then $F_{k,2n}L_{2n} = F_{k,4n}, \quad n \geq 1$

If $m = 1$ then $F_{k,2n+1}L_{2n} = F_{k,4n+1} + 1, \quad n \geq 1$

If $m = 2$ then $F_{k,2n+2}L_{2n} = F_{k,4n+2}, \quad n \geq 1$ and so on.

Chapter 7

ANALYSIS AND CONCLUSIONS

In **Chapter 2** the new sequence defined using non-linear second order recurrence relation has most of the identities satisfied by Fibonacci sequence. However the congruence properties of this sequence are different from those of Fibonacci sequence.

In **Chapter 3** we have defined a new extension of Lucas sequence i.e. T_n. We see that gcd of two lucasenne sequence is again a lucasenne sequence under some condition, negative extension of T_n is totally opposite as P_n. Besides this, we have obtained T_n divides P_n under some condition. Then we computed cycles of T_n mod m and observed that the tail period $t(m)$ of T_n & P_n are similar and we obtain pisano period of L_n mod m. We observe that some properties related to pisano periods of L_n are same as that of F_n. Also we obtained some relations between P_n and T_n. Moreover, we generalise the non-linear second order recurrence relation and gcd property for the sequence of the type $X_0 = 0, X_1 = 1, X_{n+2} = X_{n+1} + X_n$ and $Y_n = 2^{X_n} - 1$.

In **Chapter 4** we have defined a new extension of pell sequence i.e. D_n. We see that D_n divides D_{rn} for $r \in \mathbb{N}$ and gcd of any two pellene sequence is again a pellene sequence. Along with this we see two consecutive pellene sequence are co-prime. We obtained a non-linear second order recurrence relation for D_n which differs from that of P_n, T_n. Also some congruence properties satisfied by D_n are same as that of P_n and T_n. Observe that tail period $t(m)$ of D_n mod m differs than T_n, P_n mod m. Moreover, we obtained a non-linear second order recurrence relation for $B_n = 2^{A_n} - 1$ and it is observed that $(B_n, B_{n+1}) = 1$. Also we have obtained some divisibility, gcd properties of extensions of some generalised fibonacci sequence.

In **Chapter 5** We have extended the concept of difference relation to the k–Fibonacci numbers having found several formulas for these new numbers. Then we have studied the sequence of the initial numbers of the successive k–Fibonacci difference sequences and apply to the problem of the polynomial interpolation.

And In **Chapter 6** we investigate some products of k-Fibonacci and k-Lucas numbers. We also present some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula.

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