On Perfect Number, Their Relation and Upper Bounds for Odd Perfect Number

A Dissertation for

MAT-651 Discipline Specific Dissertation

Credits: 16

Submitted in partial fulfilment of Masters Degree

M.Sc. in Mathematics

by

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APRIL 2024

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "On Perfect Number, Their Relation and Upper Bounds for Odd Perfect Number" is based on the results of investigations carried out by me in the Mathematics Discipline at School of Physical and Applied Sciences, Goa University under the Supervision of Dr. Manvendra Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University or its authorities will be not be responsible for the correctness of observations or other findings given the dissertation.

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This is to certify that the dissertation report "On Perfect Number, Their Relation and Upper Bounds for Odd Perfect Number" is a bonafide work carried out by Ms. Saloni Sonu Shet Mandrekar under my supervision in partial fulfilment of the requirements for the award of the degree of M.Sc. in Mathematics in the Mathematics Discipline at the School of Physical and Applied Sciences, Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The Paper have three chapters to study, each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER:
The Introductory stage of this Project report is based on overview of the perfect numbers, even perfect number and odd perfect number. Also the required function needed in the paper have been discussed in this chapter. With that the history of Perfect number from 300 have been mentioned.The Reference Material of this chapter have been taken from [1]

SECOND CHAPTER:
This chapter is based on Arithmetical Relation of perfect numbers which involve divisor function (σ) and Euler's Totient function (ϕ). Different properties of even perfect have been taken, applying in (ϕ) results have been derived. Also at end of the chapter some examples have been solved.Reference Material of this chapter have been taken from [2].

THIRD CHAPTER:
Odd perfect number study have been very vast but no one have discovered any odd perfect number yet. This paper consist Odd Perfect number, Diophantine Equations, and Upper Bounds. Here the upper bound for odd perfect number have been programmed in terms of minimum distinct number of prime factor of Odd perfect number. Reference Material of this chapter have been taken from [3] and also from [4].

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my Mentor, Dr. Manvendra Tamba, who was a continual source of inspiration. The time and efforts he provided throughout the year were very supporting. His useful advice and suggestions were really helpful to me during the project's completion. His vast knowledge, extensive experience, and professional competence in Algebra and Number Theory enabled me to successfully accomplish this project. This endeavour would not have been possible without his help and supervision.

ABSTRACT

The Project Report has been made on Perfect numbers which have not been introduced to us in any level of school or college, by studying the following results will be able to find more relations between perfect number and Arithmetical function,In this paper, we get some new formulas for generalized perfect numbers and their relationship between arithmetical functions ϕ, σ concerning Ore's harmonic numbers and by using these formulas we present some examples. We also obtain a new upper bound for odd multiperfect numbers. If N is an odd perfect number with k distinct prime divisors and P is its largest prime divisor, we find as a corollary that $10^{12}P^2N << 2^{4^k}$. Using this new bound, and extensive computations, we derive the inequality $k \geq 10$, also by finding upper bounds for odd perfect number will be useful in higher research to examine (non)existence of odd perfect number.

Keywords: Perfect number, 2-hyperperfect number, Euler's totient function, Ore harmonic number, Odd Perfect numbers, Mersenne prime, Diophantine equations.

Contents

Chapter 1

Perfect Number

1.1 Introduction

In number theory, a perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For instance, 6 has divisors 1, 2 and 3 (excluding itself), and $1 + 2 + 3 = 6$, so 6 is a perfect number. The next perfect number is 28, since $1 + 2 + 4 + 7 + 14 = 28$.

The sum of divisors of a number, excluding the number itself, is called its aliquot sum, so a perfect number is one that is equal to its aliquot sum. Equivalently, a perfect number is a number that is half the sum of all of its positive divisors including itself; in symbols,$\sigma_1(n) = 2n$ where σ_1 is the sum-of-divisors function and n is $\mathbf{N}$.

This definition is ancient, appearing as early as Euclid's Elements (VII.22) where it is called perfect, ideal, or complete number. Euclid also proved a formation rule (IX.36) whereby $q(q+1)/2$ is an even perfect number whenever q is a prime of the form $2^p - 1$ for positive integer $p-$what is now called a Mersenne prime. Two millennia later, Leonhard Euler proved that all even perfect numbers are of this form. This is known as the Euclid-Euler theorem.

It is not known whether there are any odd perfect numbers, nor whether infinitely many perfect numbers exist. The first few perfect numbers are 6, 28, 496 and 8128.[1]

1.2 History

The almost mystical regard for perfect numbers is as old as the mathematics concerning them. The Pythagoreans equated the perfect number 6 to marriage, health, and beauty on account of the integrity and agreement of its parts. Around 100 c.e., Nicomachus noted that perfect numbers strike a harmony between the extremes of excess and deficiency (as when the sum of a number's divisors is too large or small), and fall in the "suitable" order: 6, 28, 496, and 8128 are the only perfect numbers in the intervals between 1, 10, 100, 1000, 10000, and they end alternately in 6 and 8. Near the end of the twelfth century, Rabbi Josef b. Jehuda Ankin suggested that the careful study of perfect numbers was an essential part of healing the soul. Erycius Puteanus in 1640 quotes work assigning the perfect number 6 to Venus, formed from the triad (male, odd) and the dyad (female, even). Hrotsvit, a Benedictine in the Abbey of Gandersheim of Saxony and perhaps the earliest female German poet, listed the first four perfect numbers in her play Sapientia as early as the tenth century.

We should not leave unmentioned the principal numbers... those which are called "perfect numbers". These have parts which are neither larger nor smaller than the number itself, such as the number six, whose parts, three, two, and one, add up to exactly the same sum as the number itself. For the same reason twenty-eight, four hundred ninety-six, and eight thousand one hundred twenty-eight are called perfect numbers.

Saint Augustine (among others, including the early Hebrews) considered 6 to be a truly perfect number-God fashioned the Earth in precisely this many days (rather than at once) to signify the perfection of His work. Indeed, as recorded by Alcuin of York (who lived from 732 to 804 c.e.), the second origin was imperfect, as it arose from the deficient number $8 > 1 + 2 + 4$, this number counting the 8 souls in Noah's ark (Noah, his three sons, and their four wives, in *Genesis*, chapter 7) from which sprung the entire human race. Philo Judeus, in the first century c.e., called 6 the most productive of all numbers, being the smallest perfect number.

Throughout the centuries that followed, various mathematicians carefully studied perfect numbers (the continued extensive history is given by Dickson and also by Picutti). Up to the time of Descartes and Fermat, a sizeable pool of important results-as well as much misinformation-had been collected[1].

1.3 Even Perfect Number

Euclid proved that Perfect Number are of the form $2^{p-1}(2^p-1)$, where (2^p-1) is Mersenne primes and p is a prime.
For example, the first four perfect numbers are generated by this formula is given by
$p = 2$: $2^1(2^2 - 1)$=2x3=6
$p = 3$: $2^2(2^3 - 1)$=4x7=28
$p = 5$: $2^4(2^5 - 1)$=16x31=496
$p = 7$: $2^6(2^7 - 1)$=64x127=8128
Prime numbers of the form

$2^p - 1$ are known as Mersenne primes, after the seventeenth-century monk Marin Mersenne, who studied number theory and perfect numbers. For $2^p - 1$ to be prime, it is necessary that p itself be prime. However, not all numbers of the form $2^p - 1$ with a prime p are prime; for example, $2^9 - 1 = 19682 = 2 \times 9841$ is not a prime number. In fact, Mersenne primes are very rare, as of 2023 only 51 have been Known.

While Nicomachus had stated (without proof) that all perfect numbers were of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime (though he stated this somewhat differently), Ibn al-Haytham (Alhazen) circa AD 1000 was unwilling to go that far, declaring instead (also without proof) that the formula yielded only every even perfect number. It was not until the 18th century that Leonhard Euler proved that the formula $2^{p-1}(2^p - 1)$ will yield all the even perfect numbers. Thus, there is a one-to-one correspondence between even perfect numbers and Mersenne primes; each Mersenne prime generates one even perfect number, and vice versa. This result is often referred to as the Euclid-Euler theorem. It states that an even number is perfect if and only if it has the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number.

An exhaustive search by the GIMPS(Great Internet Mersenne Prime

Search) distributed computing project has shown that the first 51 even perfect numbers are $2^{p-1}(2^p - 1)$ for p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, 77232917, and 82589933. As of December 2018, 51 Mersenne primes are known, and therefore only 51 even perfect are known numbers the largest of which is $2^{82589932} * (2^{82589933} - 1)$ with 49,724,095 digits. It is not known whether there are infinitely many perfect numbers, nor whether there are infinitely many Mersenne primes.
Even perfect numbers (except 6) are of the form
$T_{2^p-1} = 1 + \frac{(2^p-2)\times(2^p+1)}{2} = 1 + 9 \times T_{(2^p-2)/3}$

with each resulting triangular number $T_7 = 28, T_{31} = 496, T_{127} = 8128$ (after subtracting 1 from the perfect number and dividing the result by 9) ending in 3 or 5, the sequence starting with $T_2 = 3, T_{10} = 55, T_{42} = 903, T_{2730} = 3727815, ...$ It follows that by adding the digits of any even perfect number (except 6), then adding the digits of the resulting number, and repeating this process until a single digit (called the digital root) is obtained, always produces the number 1. For example, the digital root of 8128 is 1, because 8 + 1 + 2 + 8 = 19, 1 + 9 = 10, and 1 + 0 = 1. This works with all perfect numbers. Every even perfect number is also a practical number[6].

1.4 Odd Perfect Number

It is unknown whether any odd perfect numbers exist, though various results have been obtained. In 1496, Jacques Lefèvre stated that Euclid's rule gives all perfect numbers, thus implying that no odd perfect number exists. Euler stated: "Whether ... there are any odd perfect numbers is a most difficult question". More recently, Carl Pomerance has presented a heuristic argument suggesting that indeed no odd perfect number should exist. All perfect numbers are also harmonic divisor numbers, and it has been conjectured as well that there are no odd harmonic divisor numbers other than 1. Many

of the properties proved about odd perfect numbers also apply to Descartes numbers, and Pace Nielsen has suggested that sufficient study of those numbers may lead to a proof that no odd perfect numbers exist[6].
Any odd perfect number N must satisfy the following conditions:

- $N > 10^{1500}$.
- N is not divisible by 105.
- N is of the form $N \equiv 1(mod\ 12)$ or $N \equiv 117(mod\ 468)$ or $N \equiv 81(mod\ 324)$.
- An odd perfect number has more than 300 digits
- An odd perfect number has at least 75 prime factors
- An odd perfect number has at least 9 distinct prime factors
- An odd perfect number has the largest prime factor must have at least 20 digits

1.5 Hyper-perfect number

In number theory, a $k-$hyperperfect number/Multiperfect Number is a natural number n for which the equality $n = k(\sigma(n))$ holds, where $\sigma(n)$ is the divisor function (i.e., the sum of all positive divisors of n). A hyper-perfect number is a $k-$hyperperfect number for some integer k. Hyper-perfect numbers generalize perfect numbers, which are 1-hyperperfect.

The first few numbers in the sequence of k-hyperperfect numbers are 6, 21, 28, 301, 325, 496, 697,, with the corresponding values of k being 1, 2, 1, 6, 3, 1, 12, ... The first few k-hyperperfect numbers that are not perfect are 21, 301, 325, 697, 1333, ...[6].

1.6 The σ-function

Also called as Divisor function is, the sum-of-divisors function $\sigma(n)$, an arithmetic function If $n = ab$, where a and b have no common divisor, every divisor of n is the product of a divisor of a and a divisor of b. In other words $\sigma(ab) = \sigma(a)\sigma(b)$.
A function satisfying this relation for all pairs (a, b) with no common divisor is called a multiplicative function.

Using the multiplicativity property, we get now a formula for $\sigma(n)$, where $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} .....p_l^{k_l}$ [7].

Chapter 2

On Perfect Numbers and their Arithmetical Relations

2.1 Introduction

In the last chapter we have discussed about the definition of Perfect Number, Even Perfect number, odd perfect number ect..The definitions and properties used in last chapter will be followed here as well. In this chapter we are going to get some new formulas for generalized perfect numbers and their relationship between arithmetical functions ϕ and σ concerning Ore's harmonic numbers and by using these formulas we present some examples.

2.1.1 Divisor function

In this chapter we are going to define perfect numbers as $\sigma(N) = 2N, (N \in Z_{>0})$ where σ is a divisor function, the symbol $Z_{>0}$ will stand for set of all non negative integer or set of all Natural Numbers.
As we know if p is a prime number and $k \geq 1$ than its only divisors are 1 and p. So, $\sigma(p) = p + 1$ and $\sigma(p^k) = 1 + p + p^2 + ... + p^k = \frac{p^{k+1}-1}{p-1}$
Furthermore, if $2^{n+1} - 1 = p, N = 2^n p$ is perfect, then we have, $\sigma(N) = (2^{n+1} - 1)(p + 1) = 2^{n+1}p$[7].

2.1.2 Euler's totient function

In number theory, Euler's totient function counts the positive integers up to a given integer n that are relatively prime to n. It is written using the Greek letter phi as $\phi(n)$, and may also be called Euler's phi function. In other words, it is the number of integers k in the range $1 \leq k \leq n$ for which the greatest common divisor $gcd(n, k)$ is equal to 1. The integers k of this form are sometimes referred to as totatives of n. For any prime p, we have $\phi(p) = p - 1$ where ϕ is Euler's totient function defined as the number of invertible elements in an complete residue system. Some properties of Euler's totient function which we will use to prove our results will be [7];

- If a and m are integers such that $(a, m) = 1$ then $a^{\phi(m)} \equiv 1(\text{mod m})$
- If $n \geq 1$ we have $\sum\limits_{d/n} \phi(d) = n$
- For$n \geq 1$ we have $\phi(n) = n \prod\limits_{p/n} (1 - \dfrac{1}{p})$
- if n is prime $\phi(n) = n - 1$
- for $n = p \times q$, and if p and q are primes then $\phi(n) = (p - 1)(q - 1)$.
- for $n = p \times q$, and if p and q are composite $\phi(n) = n(1 - \frac{1}{p})(1 - \frac{1}{q})$
- p is prime, a is integer $\phi(p^a)= p^a - p^{a-1}$.

2.1.3 Harmonic divisor number

In mathematics, a harmonic divisor number or Ore number is a positive integer whose divisors have a harmonic mean that is an integer. The first few harmonic divisor numbers are 1, 6, 28, 140, 270, 496, 672, 1638, 2970, 6200, 8128, 8190..

Harmonic divisor numbers were introduced by Oystein Ore, who showed that every perfect number is a harmonic divisor number and conjectured that there are no odd harmonic divisor numbers other than 1.

$H(n) = \frac{k}{\frac{1}{d_1}+\frac{1}{d_2}+.....+\frac{1}{d_k}}$, d are divisors of n and k is an integer, where $H(n)$ is called harmonic mean.
For example, the harmonic mean of 1 and 2 is $\frac{2}{1+\frac{1}{2}} = 4/3$. Such an integer is called Ore harmonic number or harmonic divisor number. According the theorem of Ore; every perfect number is harmonic [7].

2.1.4 Superperfect number

n is called superperfect number if $\sigma(\sigma(n)) = 2n$ and every even superperfect number n must be a power of 2, that is, $2^p - 1$ such that $2^p - 1$ is a Mersenne prime[6].

2.1.5 Unitary Perfect Number

A unitary perfect number is an integer which is the sum of its positive proper unitary divisors, not including the number itself. (A divisor d of a number n is unitary divisor if d and $\frac{n}{d}$ share no common factors). 9 and 165 are all of the odd unitary superperfect numbers [2].

2.1.6 $k-$hyperperfect numbers

$k-$hyperperfect numbers which is defined n as; $k-$hyperfect number if $n = 1+k[\sigma(n)-n-1]$ and so $\sigma(n) = \frac{k+1}{k}n+\frac{k-1}{k}$. Also there is a conjecture which states that all 2-hyperferfect numbers are of the form $n = (3^k - 1).(3^k - 2)$ where $3^k - 2$ is prime. Also all hyperperfect numbers less than 10^{11} [2].

2.1.7 Super-hyperperfect number

If $\sigma(\sigma(n)) = \frac{3}{2}(n+1)$, then n is called super-hyperperfect number. In study of generalized perfect numbers, there are some conjectures and numerical results conjectured that, if $n = 3^{p-1}$ where p and $\frac{3^p-1}{2}$ are primes, then n is a super-hyperperfect number [2].

2.1.8 multiplicatively $e-$perfect

n is called multiplicatively $e-$perfect if $T_e(n) = n^2$ where $T_e(n)$ denote the product of exponential divisors of n [2].

2.2 Main Results

Theorem 2.1 [2]: *If k>1, 1+2+4+...+k = 2k-1 where (2k-1) is prime and k(2k-1) is a perfect number, then*

$$\phi(k[2k-1]) = k(k-1) \tag{2.1}$$

Moreover, if $k[2k-1] = 2^n(2^{n+1}-1)$ *is an Euclid number, then*

$$\phi(2^n[2n+1-1]) = 2^n(2^n-1) \tag{2.2}$$

. *Proof*: Let $k \geq 1$, $1+2+4+...+k = 2k-1$ and $2k-1$ is prime.
Since $\phi(k[2k-1])$ is a multiplicative function and $(k, 2k-1) = 1$,
we can write $\phi(k[2k-1]) = \phi(k).\phi(2k-1)$. From hypothesis k is even and is of the form $2^n(n \geq 1)$. By using a property of Euler's totient function, we can write,

$$\phi(k) = k.(1-\frac{1}{2}) = \frac{k}{2} \tag{2.3}$$

Since 2k - 1 is prime,

$$\phi(2k-1) = 2k-2 \tag{2.4}$$

from 2.3 and 2.4, we obtain

$$\phi(k[2k-1]) = \phi(k).\phi(2k-1) = \tfrac{k}{2}(2k-2)$$
$$\phi(k[2k-1]) = k(k-1).$$

Any even perfect number is an Euclid number, that is, it is of the form $2^n(2^{n+1}-1)$ where $2^{n+1}-1$ is prime. So, if $k = 2^n$ and $2k-1 = 2^{n+1}-1$ is prime, then $k[2k-1]$ is an Euclid number. So, $\phi(2^n[2^{n+1}-1]) = 2^n(2^n-1)$.

Proposition 2.2[2]: *If $N > 6$ is a perfect number, then $\phi(N)$ is even and $\phi(N)$ is not a prime.*
Proof: Let N is a perfect number. From the result of Euclid theorem and theorem 2.1,
$N = k.(2k-1)$ and $\phi(k[2k-1]) = k(k-1)$.
Also, k=2^n and $2|2^n(2^n-1)$
$\implies 2|k.(k-1)$

so $\phi(N)$ is an even number and $\phi(6) = 2$ and from hypothesis $N > 6$ so $\phi(N) > 2$.
Moreover, $k|\phi(N)$ and $(k-1)|\phi(N)$ so $\phi(N)$ is not a prime.

Corollary 2.3[2]:*If N is a perfect number and $N = k.(2k-1)$, then there is at least a natural number n which satisfies $n|\phi(N)$ and $(n+1)|\phi(N)$.*
Proof: Since

$$N = k(2k-1)$$

$$\therefore \phi(k(2k-1)) = k(k-1)$$

let n=k-1

$$\implies n+1 = k$$

$$\implies \phi(k(2k-1)) = k(k-1) = (n+1).n$$

$$\implies (n+1)|\phi(k(2k-1)) \text{ and } n|\phi(k(2k-1))$$

Proposition 2.4[2]: *If $\phi(N) = b$ and N is a perfect number, then $\sqrt{4b+1}$ is a Mersenne prime.*
Proof: From the result of theorem 2.1, we can write $\phi(N) = k(k-1) = b$. and let $(2k-1)$ is mercene prime

$$\implies b = k^2 - k$$

$$b = (k - \frac{1}{2})^2 - \frac{1}{4}$$

$$b + \frac{1}{4} = (k - \frac{1}{2})^2$$

$$\pm\sqrt{b + \frac{1}{4}} = k - \frac{1}{2}$$

$$\implies k = \pm\sqrt{\frac{4b+1}{4}} + \frac{1}{2}$$

$$k = (\frac{1}{2}. \pm \sqrt{4b+1}) + \frac{1}{2}$$

$$k = \frac{1}{2}(1 \pm \sqrt{4b+1})$$

$$2k = 1 \pm \sqrt{4b+1}$$

$$\implies 2k - 1 = \pm\sqrt{4b+1}$$

$\because$ $(2k-1)$ is a Mersenne prime. So, it is positive. Thus, $\sqrt{4b+1}$ is a Mersenne prime.

Proposition 2.5[2]:*If* $N = 2^n(2^{n+1}-1)$ *is an Euclid number, then*

$$\phi(N^2) = N.\phi(N)$$

Proof:

$$\phi(N^2) = \phi(2^{2n}(2^{n+1}-1)^2)$$

$$= (2^{2n})(2^{n+1}-1)(2^{n+1}-1).\frac{1}{2}.(1 - \frac{1}{2^{n+1}-1})$$

$$= (2^{2n})(2^{n+1}-1)(2^n-1)$$

$$= (2^n)(2^{n+1}-1)[2^n(2^n-1)]$$

$$= N\phi(N)$$

Theorem 2.6[2]: *Let* p *is a Mersenne prime and* $p = 2^{n+1} - 1$. *If* $N = 2^n(2^{n+1}-1)$ *is a perfect number, then*

$$\sigma(N) = \frac{4p}{p-1}\phi(N)$$

or

$$\sigma(N) = \frac{2^{n+2}-2}{2^n-1}\phi(N)$$

Proof: From hypothesis and theorem 2.1, we can write

$$\phi(N) = \phi(2^n p) = 2^n(2^n-1)$$

So, $2^n = \frac{p+1}{2}$, then

$$2^n(2^n-1) = (\frac{p+1}{2})(\frac{p-1}{2})$$

$$\phi(N) = \frac{p^2-1}{4}$$

$$\sigma(N) = \sigma(2^n p) = \sigma(2^n).\sigma(p)$$

where $\sigma(2^n) = \dfrac{2^{n+1}-1}{2-1}$ and $\sigma(p) = p+1$,So

$$\sigma(N) = p.(p+1)$$

From $\phi(N)$ and $\sigma(N)$ we obtain,

$$\sigma(N) = \frac{4p}{p-1}.\phi(N)$$

or $p = 2^{n+1}-1$. So,

$$\sigma(N) = \frac{2^{n+2}-2}{2^n-1}.\phi(N)$$

Theorem 2.7[2]: *Let N is a perfect number and $\sigma(N)$ is harmonic. If N is of the form $2^n(2^{n+1}-1)$ where 2^{n+1}-1 is a Mersenne prime, then*

$$H(\sigma(N)) = \frac{(n+2)(2^{n+2}-2)}{2^{n+2}-1}$$

Proof: If N is a perfect number; this means $\sigma(N) = 2N$ or $\sigma(2^n(2^{n+1}-1)) = 2^{n+1}(2^{n+1}-1)$.

$$H(\sigma(N)) = H(2^{n+1}[2^{n+1}-1])$$

$$= [\frac{1}{2n+4}(1+\frac{1}{2}+...+\frac{1}{2^{n+1}}+\frac{1}{2^{n+1}-1}+\frac{1}{2(2^{n+1}-1)}+...+\frac{1}{2^{n+1}(2^{n+1}-1)})]^{-1}$$

$$= [\frac{1}{2(n+2)}.(\frac{2^{n+2}-1}{2^{n+1}}+\frac{2^{n+2}-1}{2^{n+1}(2^{n+1}-1)})]^{-1}$$

$$= [\frac{1}{2(n+2)}\frac{(2^{n+2}-1)2^{n+1}}{2^{n+1}(2^{n+1}-1)}]^{-1}$$

$$= \frac{(n+2)(2^{n+2}-2)}{2^{n+2}-1}$$

This completes the proof.

Corollary 2.8[2]: *If N is a perfect number, then $\sigma(N)$ is harmonic.*
Proof: Ore's harmonic number theorem tells us that every perfect number is harmonic. From Ore's theorem and theorem 2.7, N is harmonic. Also, $\sigma(N) = 2N$. So, $\sigma(N)$ is harmonic.

Theorem 2.9[2]: *Let $3^k - 2$ is prime. If n is a 2-hyperperfect number, then*

$$\phi(n) = n - 3^{2k-2}.$$

Proof: From hypothesis, we can take $n = 3^{k-1}.(3^k - 2)$. Also, $\phi(n)$ is a multiplicative function, and $(3^{k-1}, 3^k - 2) = 1$. So,

$$\phi(n) = \phi(3^{k-1}.(3^k - 2))$$

$$= \phi(3^{k-1}.\phi(3^k - 2))$$

$$= (3^{k-1} - 3^{k-2}).(3^k - 3)$$

$$= 3^{k-1}.[(3^k - 2) - 3^{k-1}]$$

$$= n - 3^{2k-2}$$

Theorem 2.10[2]: *If n is a super-hyperperfect number, then $\phi(\phi(n)) = \frac{2}{9}n$.*
Proof: From the definition of super-hyperperfect number we know that $n = 3^{p-1}$ where p and $\frac{3^p-1}{2}$ are primes and $(3^{p-1}, 2) = 1$. So,

$$\phi(\phi(3^{p-1})) = \phi((3^{p-1})(1 - \frac{1}{3}))$$

$$= \phi(3^{p-2}.2)$$

$$= \phi(3^{p-2}).\phi(2)$$

$$= 3^{p-2}(1 - \frac{1}{3})$$

$$= \frac{2.3^{p-1}}{9}$$

$$= \frac{2}{9}n$$

Theorem 2.11[2]:*If n is an even superperfect number, then*

$$\phi(\phi(n)) = \frac{n}{4}$$

Proof: If n is an even superperfect number, then n is of the form $2^p - 1$ where $2^p - 1$ is a Mersenne prime. So,

$$\phi(n) = \phi(2^{p-1})$$

$$= 2^{p-1}(1 - \frac{1}{2})$$

$$= 2^{p-2}$$

$$\phi(\phi(n)) = \phi(2^{p-2})$$

$$= 2^{p-2}.(1 - \frac{1}{2})$$

$$= \frac{2^{p-1}}{4}$$

$$\phi(\phi(n)) = \frac{n}{4}$$

2.3 Example

In this section, we introduced some examples related to our theorems.
Example 1[2]: *If $N = \frac{p+1}{4}(\phi(p) + \sigma(p))$ where $p = 2^{n+1} - 1$ is a Mersenne prime, then N is a perfect number.*
solution: Let $p = 2^{n+1} - 1$ is a Mersenne prime.
Also, by using properties $\sigma(p) = p + 1$ and $\sigma(p) = p - 1$,

$$\sigma(p) + \phi(p) = 2p.$$

Therefore,

$$N = \frac{p(p+1)}{2} = 2^n(2^{n+1} - 1)$$

So, according to theorem 2.1, N is a perfect number.

Example 2[2]: *Let p is a Mersenne prime, $b = \phi(N)$ and N is a perfect number. By using Proposition 2.4,If $\frac{N}{b} = \frac{2p}{p-1}$, then find $\phi(\frac{2N}{p+1})$.*
solution: Since $p = \sqrt{4b+1}$ so, $b = \frac{p^2-1}{4}$. Then,

$$N(p-1) = 2pb$$

$$N(p-1) = 2p(\frac{p^2-1}{4})$$
$$4N(p-1) = 2p(p^2-1)$$
$$\frac{2N}{p+1} = p$$
$$\phi(\frac{2N}{p+1}) = p-1$$

So, the result holds.

Example 3[2]: *Every $d = \frac{\phi(N)}{2}$ is a triangular number where N is an even perfect number.*
solution: This follows directly from theorem 2.1. If $m = 2n$ then $\phi(N) = m(m-1)$.
where $\frac{m(m-1)}{2}$, is a form of triangular number.
So, $\frac{\phi(N)}{2}$ is a triangular number.

Example 4[2]: *Let $\epsilon > 0$, N is a perfect number, $n \in N$.Then there is a perfect numbers which satisfies $\frac{\phi(N)}{N} < \frac{1}{2}$.*
solution: Assume $p = 2^{n+1} - 1$ is a Mersenne prime, $N = 2^n p$ is a perfect number,

$$\phi(N) = 2^n(2^n - 1)$$
$$= \frac{p^2-1}{4}$$
$$N = \frac{p(p+1)}{2}$$
$$\frac{\phi(N)}{N} = \frac{p-1}{2p}$$

$$= \frac{1}{2} - \frac{1}{2p}$$

Since $\epsilon = \frac{1}{2p}$, so $\frac{\phi(N)}{N} < \frac{1}{2}$

Example 5[2]:*If N is an even perfect number then, $\phi(N) > \frac{N}{4}$ and $\frac{\sigma(N)}{\phi(N)} < 8$.*
solution: We can verify these inequalities in two ways, Firstly, we write $N = 2^n.p$, $p = 2^{n+1} - 1$. So,

$$\phi(N) = N \prod_{p|n} (1 - \frac{1}{p})$$

$$> N \prod_{p|n} (1 - \frac{1}{2}) = N \prod_{p|n} \frac{1}{2} = N.2^{-2} > \frac{N}{4}$$

$$\sigma(N) = 2N$$

$$\phi(N) > \frac{N}{4}$$

$$\frac{\sigma(N)}{\phi(N)} < \frac{2N}{\frac{N}{4}} < 8$$

The second way is shorter. By using theorem 2.6

$$\frac{\sigma(N)}{\phi(N)} = \frac{2^{n+2} - 2}{2^n - 1} < 8$$

The inequality is true because $3 < 2^{n+1}$.

Chapter 3

Odd Perfect Numbers, Diophantine Equation and Upper Bounds

3.1 Introduction

In this chapter we are going to obtain a new upper bound for odd multiperfect numbers. If N is an odd perfect number with k distinct prime divisors and P is its largest prime divisor, we find as a corollary that $10^{12}P^2N < 2^{4^k}$. Using this newbound, we show that if p and q are distinct primes and $p^a q^b || N$, then there are reasonably sized bounds on a and b in terms of k, it follows that $gcd(\sigma(p^a), \sigma(q^b))$ has only moderately sized prime divisors. Taking advantage of the new information we derive the inequality $k \geq 10$.

One of the oldest unsolved problems in mathematics is whether there exists an odd perfect number N. There are many roadblocks to the existence of such a number. For instance, we now know that $N > 10^{1500}$ and N has at least 108 prime factors (counting multiplicity). If k is the number of distinct prime factors, then we have $k \geq 9$ and $N < 2^{4^k}$.

While working with odd perfect numbers has been mostly computational, the bound $N < 2^{4^k}$ is a purely theoretical result. Due to its doubly exponential growth it has not been used seriously in calculations. In this paper we find a way to make this upper bound an effective estimation tool. As an application, we are able to prove that an odd perfect number must have at least 10 distinct prime factors.

However, there are a great number of necessary conditions for their existence, which go through periodic improvements. The list of conditions given below but with recent improvements included.
Let N be an odd perfect number (if such exists). Write $N = \prod_{i=1}^{k} p_i$ where each p_i is prime, $p_1 < p_2 < ... < p_k$, and $k = \omega(N)$ is the number of distinct prime factors. The factors $p_i^{a_i}$ are called the prime components of N. Then N is of the form

$N = q^{\alpha} p_1^{2e_1} \cdots p_k^{2e_k}$,

where:

- $q, p_1, ..., p_k$ are distinct odd primes (Euler).
- $q \equiv \alpha \equiv 1 (mod\ 4)$ (Euler).
- The smallest prime factor of N is at most $\frac{k-1}{2}$.
- At least one of the prime powers dividing n exceeds 10^{62}.
- $N < 2^{(4^{k+1}-2^{k+1})}$
- $\alpha + 2e_1 + 2e_2 + 2e_3 + \cdots + 2e_k \geq \frac{99k-224}{37}$
- $qp_1p_2p_3 \cdots p_k < 2N^{\frac{17}{26}}$
- $\frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} < \ln 2$
  Furthermore, several minor results are known about the exponents $e_1, ..., e_k$.
- Not all $e_i \equiv 1 (mod\ 3)$.
- Not all $e_i \equiv 2 (mod\ 5)$.
- If all $e_i \equiv 1 (mod\ 3)$ or $2 (mod\ 5)$, then the smallest prime factor of N must lie between 10^8 and 10^{1000}.

- More generally, if all $2e_i+1$ have a prime factor in a given finite set S, then the smallest prime factor of N must be smaller than an effectively computable constant depending only on S.
- If $(e_1, ..., e_k) = (1, ..., 1, 2, ..., 2)$ with t ones and u twos, then $(t-1)/4 \leq u \leq 2t + \sqrt{\alpha}$
- $(e_1, ..., e_k) \neq (1, ..., 1, 3)$,$(1, ..., 1, 5)$, $(1, ..., 1, 6)$.
- If $e_1 = ... = e_k = e$, then e cannot be 3, 5, 24, 6, 8, 11, 14 or 18, $k \leq 2e^2 + 8e + 2$ and $N < 2^{4^{2e^2+8e+3}}$.

With such a number of conditions, it might seem that an odd perfect number could not exist [7].

3.2 A better upper bound

Let N be a positive integer. N is said to be perfect when $\sigma(N)/N = 2$. Multiperfect when $\sigma(N)/N \in \mathbb{Z}$. n/d is perfect when $\sigma(N)/N = n/d$. we will always assume $n, d \in \mathbb{Z}^+$. Note that n/d does not need to be in lowest terms. $N = \prod_{i=1}^{k} p_i^{e_i}$ where $p_1 < ... < p_k$ are prime devisors of N.

$\therefore$ the equation $\sigma(N)/N = n/d$ can be written as,

$$d \prod_{i=1}^{k} \left(\sum_{j=0}^{e_i} p_i^j \right) = n \prod_{i=1}^{k} p_i^{e_i}$$

This motivates us to look at the Diophantine equation

$$\mathbf{d} \prod_{i=1}^{k} \left(\sum_{j=0}^{e_i} x_i^j \right) = n \prod_{i=1}^{k} x_i^{e_i} \tag{3.1}$$

in k variables $x_1, ..., x_k$. It turns out that if we fix k and look for integer solutions with the x_i's greater than 1 and odd, then there are finitely many solutions. In fact, there is an explicit upper bound on $\prod_{i=1}^{k} x_i^{e_i}$ in terms of n, d, and k, but independent of the e_i. As we will generalize and improve these results, and as some of the proofs are scattered in the literature, we include all the needed pieces here [3].

Lemma 3.1[3]: *Let* $x_1, x_2, w \in R_{>0}$*, and assume* $w < 1$*. We have*

$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right) \geq \left(1-\frac{1}{wx_1}\right)\left(1-\frac{1}{w^{-1}x_2}\right) \tag{3.2}$$

if and only if $w \leq x_2/x_1$ *Furthermore, equality holds iff* $w = x_2/x_1$*. In particular, if* $x_1 \leq x_2$*, then strict inequality holds.*
Proof:

$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right) \geq \left(1-\frac{1}{x_2/x_1x_1}\right)\left(1-\frac{1}{(x_2/x_1)^{-1}x_2}\right)$$

This follows from basic algebraic manipulation. Note that we can characterize equality and strict inequality in (3.2) under the much weaker assumption that x_1, x_2, and w are nonzero. However, there is no need for this generality.

Lemma 3.2[3]: Let $1 < x_1 \leq x_2 \leq ... \leq x_n$ and $1 < y_1 \leq y_2 \leq ... \leq y_n$ *be nondecreasing sequences of real numbers satisfying*

$$\prod_{i=1}^{m} x_i \leq \prod_{i=1}^{m} y_i \tag{3.3}$$

for every m in the range $1 \leq m \leq n$. Then we have

$$\prod_{i=1}^{n}\left(1-\frac{1}{x_i}\right) \leq \prod_{i=1}^{n}\left(1-\frac{1}{y_i}\right) \tag{3.4}$$

where equality holds if and only if $x_i = y_i$ for every $i \geq 1$.
Proof: We wish to minimize

$$\prod_{i=1}^{n}\left(1-\frac{1}{y_i}\right) \tag{3.5}$$

subject to the constraints that the y_i form a nondecreasing sequence and satisfy (3.3) for each m. If we set $N = \prod_{i=1}^{n} x_i$, then lowering each y_i which is bigger than N down to N will only decrease. Thus, we see that any minimizing solution belongs to the compact set inside the box $[x_1, N]^n$ subject to the constraints given in (3.3) and the constraint that the y_i are nondecreasing. (The inequality $y_1 > 1$ is an open condition, but the closed condition $y_1 \geq x_1$ implies it.) Thus we may fix the y_i so that they in fact minimize (3.5).

Assume, by way of contradiction, that this minimizing solution does not agree with the x_i. We let r be the first index where $x_r \neq y_r$, and so from (3.3) we have $x_r < y_r$. As the y_i minimize (3.5), we see that $r < n$. We let t be the largest index where $y_t = y_{r+1}$.
Define the new sequence

$$z_i = \begin{cases} y_i, \text{ if } i \neq r, t \\ wy_r, \text{ if } i = r, \\ w^{-1}y_t, \text{ if } i = t \end{cases}$$

for some real number w with $0 < w < 1$ to be further specified shortly. Consider what happens if we replace the y_i's with the z_i's. First, we choose w so that it satisfies $w > y_{r-1}/y_r$ (where $y_r - 1 = 1$ in case $r = 1$), and also $w > y_t/y_{t+1}$ (where this condition is vacuously satisfied if $t = n$). With these assumptions on w, the new sequence z_i satisfies $1 < z_1 \leq z_2 \leq ... \leq z_n$. Second, the quantity (3.5) decreases by Lemma 3.1. Third, (3.3) still holds when $m < r$ or $m \geq t$, since in those cases $\prod_{i=1}^{m} y_i = \prod_{i=1}^{m} z_i$.

We can make (3.3) hold for an m in the interval $r \leq m < t$, if we have a strict inequality $\prod_{i=1}^{m} x_i < \prod_{i=1}^{m} y_i$ (by assuming $w > \prod_{i=1}^{m} x_i/y_i$ for each such m). Notice that this strict inequality does hold when $m = r$. However, if the strict inequality held for all m in the interval $r < m < t$, then this would contradict our assumption that the y_i's were a minimizing choice.
Thus, we must have an equality

$$\prod_{i=1}^{s} x_i = \prod_{i=1}^{s} y_i \tag{3.6}$$

for some s satisfying $r < s < t$. By the definition of r we also have the equality

$$\prod_{i=r}^{s} x_i = \prod_{i=r}^{s} y_i \tag{3.7}$$

Recall that $x_r < y_r$ and that $y_{r+1} = y_{r+2} = ... = y_s = ... = y_t$. Thus (3.7) turns into $\prod\limits_{i=r+1}^{s} x_i > \prod\limits_{i=r+1}^{s} y_i = y_s^{s-r}$ As the x_i are a non decreasing sequence, we have

$$x_{s+1} \geq x_s \geq \left(\prod_{i=r+1}^{s} x_i\right)^{1/(s-r)} > y_s = y_{s+1},$$

where the last equality holds since $r+1 \leq s < t$ (using the definition of t). But then (3.6) implies $\prod\limits_{i=1}^{s+1} x_i > \prod\limits_{i=1}^{s+1} y_i$, contradicting (3.3) when $m = s+1$. As we reached a contradiction in every case, this proves that the only minimizing solution is when $x_i = y_i$ for every $i \geq 1$.
Before putting the previous lemma to good use, we need one more straightforward result.

Lemma 3.3[3]: *Let $r \in \mathbb{Z}_{>0}$ and define $F_r : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ by the rule $F_r(x) = x^{2^r} - x^{2^{r-1}}$. The function F_r is monotonically increasing.*
Proof:

$$F_r(x) = x^{2^r} - x^{2^{r-1}}$$

$$F_r'(x) = \frac{d}{dx}(x^{2^r} - x^{2^{r-1}})$$

$$F_r'(x) = 2^r x^{2^r-1} - 2^{r-1} x^{2^{r-1}-1}$$

$$= 2^{r-1} x^{2^{r-1}-1}(2x^{2^{r-1}} - 1)$$

the equation becomes positive when $x \geq 1$

Lemma 3.4[3]: *Let $r, a, b \in \mathbb{Z}_{>0}$ and let $x_1, ..., x_r$ be integers with $1 < x_1 \leq ... \leq x_r$. If*

$$\prod_{i=1}^{r}(1 - \frac{1}{x_i}) \leq \frac{a}{b} < \prod_{i=1}^{r-1}(1 - \frac{1}{x_i}) \tag{3.8}$$

then $a \prod\limits_{i=1}^{r} x_i \leq (a+1)^{2^r} - (a+1)^{2^{r-1}}$
Proof: Work by induction on $r \geq 1$. Notice that $a < b$ in any case. When

$r = 1$ we have $x_1 \leq b/(b-a)$ which is maximized when $b = a+1$. Thus $ax_1 \leq a(a+1) = (a+1)^{2^1} - (a+1)^{2^0}$.

Now assume that $r \geq 2$ and also assume that the lemma holds for all integers smaller than r (and for any choices for a and b). Treating a as a fixed constant, we can assume that b has been chosen, along with integers $1 < x_1 \leq x_2 \leq ... \leq x_r$, so that $\prod_{i=1}^{r} x_i$ is maximal and (3.8) holds. Next, set $n_i = (a+1)^{2^{i-1}} + 1$ for $i < r$, and set $n_r = (a+1)^{2^{r-1}}$. We have $1 < n_1 < n_2 < ... < n_r$ and

$$\prod_{i=1}^{r}\left(1 - \frac{1}{n_i}\right) = \frac{a}{a+1} < \prod_{i=1}^{r-1}\left(1 - \frac{1}{n_i}\right)$$

Thus, from our maximality assumption,

$$\prod_{i=1}^{r} n_i \leq \prod_{i=1}^{r} x_i \tag{3.9}$$

If $ax_1 < an_1 = (a+1)^2 - 1$, then after multiplying (3.8) by $\frac{x_1}{x_1-1}$ we have

$$\prod_{i=2}^{r} (1 - \frac{1}{x_i}) \leq \frac{ax_1}{b(x_1-1)} < \prod_{i=2}^{r-1} (1 - \frac{1}{x_i})$$

The induction hypothesis implies $(ax_1)\prod_{i=2}^{r} x_i \leq (ax_1+1)^{2^{r-1}} - (ax_1+1)^{2^{r-2}} < (a+1)^{2^r} - (a+1)^{2^{r-1}}$. Thus we may as well assume $n_1 \leq x_1$.
If $ax_1x_2 < an_1n_2 = (a+1)^4 - 1$, then multiplying (3.8) by $\frac{x_1x_2}{(x_1-1)(x_2-1)}$ and performing a similar computation yields the upper bound we seek. Thus we may assume $n_1n_2 \leq x_1x_2$. Repeating this argument, we have $\prod_{i=1}^{m} n_i \leq \prod_{i=1}^{m} x_i$ for $1 \leq m < r$. But this also holds when $m = r$ by (3.9).
Lemma 3.2 now applies, so we have

$$\prod_{i=1}^{r} (1 - \frac{1}{x_i}) \geq \prod_{i=1}^{r} (1 - \frac{1}{n_i}) = \frac{a}{a+1};$$

but as (3.8) holds for some b we must have $b = a + 1$. Again, appealing to Lemma 3.2, we have $x_i = n_i$ for all $i \geq 1$. In this case, we compute

$$a \prod_{i=1}^{r} n_i = (a+1)^{2^r} - (a+1)^{2^{r-1}}$$

as required.

Remark: The bound given in the lemma is the best possible in case $b = a+1$. To simplify notation, throughout the paper we let F_r be defined as in Lemma 3.3. Thus, (3.8) says

$$a \prod_{i=1}^{r} x_i \leq F_r(a+1)$$

We also define $F_0(x) := x - 1$

Notation: Let X be a finite set of integers. We write $\Pi(X)$ for $\prod_{x \in X} x$, with the empty product equaling 1. We will also write $\Pi'(X)$ or $\prod_{x \in X}(x-1)$.

Lemma 3.5[3] *Let $k, n, d \in \mathbb{Z}_{>0}$. Suppose (3.1) holds for some choice of positive integer exponents $\{e_1, ..., e_k\}$, and odd integers $X = \{x_1, ..., x_k\}$ each greater than 1. Let S be a (possibly empty) subset of X. There exist sets $S', S'' \subseteq X$ satisfying $S \cap S' = \phi$, $\phi \neq S'' \subseteq S \cup S'$,* and such that if we let $w = |S'|, v = |S''|, T = (S \cup S') \backslash S''$, $\delta = d \prod_{x_i \in S''} \sum_{j=0}^{e_i} x_i^j$, and $\nu = n \prod_{x_i \in S''} x_i^{e_i}$ then:

(i) $\delta \prod_{x_i \in X|S''} (\sum_{j=0}^{e_i} x_i^j) = \nu \prod_{x_i \in X|S''} x_i^{e_i}$.

(ii) $\delta \Pi(T) < \dfrac{1}{\Pi(S'')\Pi'(S'')} F_{v+w}(d\Pi(S)+1)$.

Proof: We are assuming that the elements in X are odd, and so the fraction n/d, when written in lowest terms, has odd denominator. In particular,

$$\prod_{x_i \in S} \left(1 - \frac{1}{x_i}\right) \neq \frac{d}{n}. \tag{3.10}$$

Thus, we have two cases to consider.

Case 1: $\prod\limits_{x_i \in S} \left(1 - \frac{1}{x_i}\right) > \frac{d}{n}$

In this case, set $d' = d\Pi(S)$ and $n' = n\Pi'(S)$. From $\prod_{x_i \in S} \left(1 - \frac{1}{x_i}\right) > \frac{d}{n}$ we see $d'/n' < 1$. Further, we calculate

$$\prod_{x_i \in S} \left(1 - \frac{1}{x_i}\right) < \prod_{x_i \in X} \left(\frac{x_i - 1}{x_i - \frac{1}{x_i^{e_i}}}\right) = \prod_{x_i \in X} \frac{x_i^{e_i}}{\sum\limits_{j=0}^{e_i} x_i^j} = \frac{d}{n}$$

$\therefore$

$$\prod_{x_i \notin S} \left(1 - \frac{1}{x_i}\right) < \frac{d'}{n'} < 1. \tag{3.11}$$

This implies that there is a subset $S' \subseteq X|S$ such that if we write $S' = \{y_1, ..., y_w\}$ with $y_1 \leq ... \leq y_w$, then

$$\prod_{i=1}^{w} \left(1 - \frac{1}{y_i}\right) \leq \frac{d'}{n'} < . \prod_{i=1}^{w-1} \left(1 - \frac{1}{y_i}\right)$$

and so by Lemma 3.4 we have $d'\Pi(S') \leq F_w(d'+1)$. Using the definition of d' we rewrite this as

$$d\Pi(S)\Pi(S') \leq F_w(d\Pi(S) + 1). \tag{3.12}$$

Notice that we also have

$$\prod_{x_i \in S \cup S'} \left(1 - \frac{1}{x_i}\right) < \frac{d}{n} \tag{3.13}$$

This completes the construction of S' in Case 1.

Case 2: $\prod\limits_{x_i \in S} \left(1 - \frac{1}{x_i}\right) < \frac{d}{n}$

We set $S' = \phi$ (so $w = 0$). We note that inequalities (3.12) and (3.13) still hold in this case.

The construction of the set S'' is the same in both Case 1 and Case 2. We only need to know that inequalities (3.12) and (3.13) hold in both cases, so we continue with the general construction. Put $d'' = d\Pi(S)\Pi(S')$ and $n'' = n\Pi'(S)\Pi'(S')$. Inequality (3.13) is equivalent to $n''/d'' < 1$, which we will use shortly. We calculate

$$\prod_{x_i \in S \cup S'} \frac{1 - \frac{1}{x_i^{e_i+1}}}{1 - \frac{1}{x_i}} = \prod_{x_i \in S \cup S'} \frac{\sum_{j=0}^{e_i} x_i^j}{x_i^{e_i}} \leq \prod_{x_i \in X} \frac{\sum_{j=0}^{e_i} x_i^j}{x_i^{e_i}} = \frac{n}{d}$$

and hence

$$\prod_{x_i \in S \cup S'} \left(1 - \frac{1}{x_i^{e_i+1}}\right) \leq \frac{n''}{d''} < 1. \tag{3.14}$$

We pick a subset $S'' = \{z_1, ..., z_v\} \subseteq S \cup S'$ such that $z_1^{e(z_1)+1} \leq ... \leq z_v^{e(z_v)+1}$ and

$$\prod_{i=1}^{v} \left(1 - \frac{1}{z_i^{e(z_i)+1}}\right) \leq \frac{n''}{d''} < \prod_{i=1}^{v-1} \left(1 - \frac{1}{z_i^{e(z_i)+1}}\right).$$

(By $e(z_i)$ we mean the exponent corresponding to z_i.) By Lemma 3.4, and the inequality $n'' < d''$, we have

$$n'' \prod_{x_i \in S''} x_i^{e_i+1} \leq F_v(n'' + 1) \leq F_v(d'') = F_v(d\Pi(S)\Pi(S')) \tag{3.15}$$

This completes the construction of S''. We now only need to verify properties (i) and (ii). Property (i) is obvious,ccoming from equation (3.1). For property (ii),we compute

$$\delta\Pi(T) = d \prod_{x_i \in S''} \Big(\frac{x_i^{e_i+1} - 1}{x_i - 1}\Big) \frac{\Pi(S)\Pi(S')}{\Pi(S'')}, \quad \text{by the definition of } \delta \text{ and T}$$

$$= \frac{1}{\Pi(S'')\Pi'(S'')} d'' \prod_{x_i \in S''} (x_i^{e_i+1} - 1) \quad \text{by the definition of} d''$$

$$\leq \frac{1}{\Pi(S'')\Pi'(S'')} n'' \prod_{x_i \in S''} x_i^{e_i+1} \quad \text{by inequality (3.14)}$$

$$\leq \frac{1}{\Pi(S'')\Pi'(S'')} F_v(d\Pi(S)\Pi(S')) \quad \text{by inequality (3.15)}$$

$$\leq \frac{1}{\Pi(S'')\Pi'(S'')} F_v(F_w(d\Pi(S)+1)) \quad \text{by inequality (3.12)}$$

$$\leq \frac{1}{\Pi(S'')\Pi'(S'')} F_{v+w}(d\Pi(S)+1)$$

Thus, we have established the needed inequality.
We are now ready to put all of these lemmas together to prove an improved upper bound on odd multiperfect numbers. The improvement from previous results is in the denominator term.

Theorem 3.6[3]: *Let $k, n, d \in \mathbb{Z}_{>0}$. Suppose Equation (3.1) holds for some choice of positive integer exponents $\{e_1, ..., e_k\}$, and odd integers $X = \{x_1, ..., x_k\}$ each greater than 1. in this case*

$$\prod_{i=1}^{k} x_i^{e_i} < \frac{F_{2k}(d+1)}{n\Pi(X)\Pi'(X)} < \frac{(d+1)^{2^{2k}}}{n\Pi(X)\Pi'(X)}$$

proof: Let $X_0 = X$, $n_0 = n$, $d_0 = d$, and $S_0 = \phi$. Using Lemma 3.5, we can construct S_0', S_0'', w_0, v_0, ν_0, δ_0, and T_0 (using the same notation, just with the extra subscript) satisfying properties (i) and (ii). Thus

$$\delta_0 \prod_{x_i \in X_0 | S_0''} \left(\sum_{j=0}^{e_i} x_i^j \right) = \nu_0 \prod_{x_i \in X_0 | S_0''} x_i^{e_i}. \tag{3.16}$$

Putting $X_1 = X_0|S_0''$, $S_1 = T_0$, $n_1 = \nu_0$ and $d_1 = \delta_0$, we see from equation (3.16) that we can again use Lemma 3.5. Hence, we can construct S_1'', S_1', T_1', and so forth. We continue this process of repeatedly using Lemma 3.5, increasing the indices at every step. Since $S_i'' \neq \phi$, we see that $X_0 \supsetneq X_1 \supsetneq ...$, and so this process must terminate (in at least k steps), say $X_{r+1} = \phi$. Further, we see that $\sum_{i=0}^{r} w_i = \sum_{i=o}^{r} v_i = k$ (since for each element $x \in X$ there are unique indices $i \leq j$ such that x is added into S_i', and then put into S_j''). Using property (ii), repeatedly, we have

$$d_{r+1}\Pi(S_{r+1}) < \frac{1}{\Pi(S_r'')\Pi'(S_r'')} F_{w_r+v_r}(d_r\Pi(S_r)+1)$$

$$\leq \frac{1}{\Pi(S''_r)\Pi'(S''_r)} F_{w_r+v_r}\left(\frac{1}{\Pi(S''_{r-1})\Pi'(S''_{r-1})} F_{w_{r-1}+v_{r-1}}(d_{r-1}\Pi(S_{r-1})+1)+1\right)$$

$$< \frac{1}{\Pi(S''_r)\Pi'(S''_r)} F_{w_r+v_r}\left(\frac{1}{\Pi(S''_{r-1})\Pi'(S''_{r-1})} (d_{r-1}\Pi(S_{r-1})+1)^{2^{w_{r-1}+v_{r-1}}}+1\right)$$

Note that $w_r \geq 1$, and $v_i \geq 1$ for each i. Also observe that $(d_{r-1}\Pi(S_{r-1})+1)^{2^{w_{r-1}}} \geq \Pi(S''_{r-1})$,$\dot{\text{T}}$hus

$$F_{w_r+v_r}\left(\frac{1}{\Pi(S''_{r-1})\Pi'(S''_{r-1})} (d_{r-1}\Pi(S_{r-1})+1)^{2^{w_{r-1}+v_{r-1}}}+1\right)$$

$$\leq F_{w_r+v_r}\left(\frac{1}{\Pi(S''_{r-1})} (d_{r-1}\Pi(S_{r-1})+1)^{2^{w_{r-1}+v_{r-1}}}\right)$$

$$\leq \frac{1}{\Pi(S''_{r-1})^2} F_{w_r+v_r}\left((d_{r-1}\Pi(S_{r-1})+1)^{2^{w_{r-1}+v_{r-1}}}\right)$$

$$= \frac{1}{\Pi(S''_{r-1})^2} F_{w_r+v_r+w_{r-1}+v_{r-1}}(d_{r-1}\Pi(S_{r-1})+1)$$

Continuing our inequality from before, and repeating the ideas used in the computations above, we have

$$d_{r+1}\Pi(S_{r+1}) < \frac{1}{\Pi(S''_r)\Pi'(S''_r)\Pi(S''_{r-1})^2} F_{w_{r-1}+w_r+v_{r-1}+v_r}(d_{r-1}\Pi(S_{r-1})+1)$$

$$< \frac{1}{\Pi(S''_r \cup S''_{r-1})\Pi'(S''_r \cup S''_{r-1})} F_{w_{r-1}+w_r+v_{r-1}+v_r}(d_{r-1}\Pi(S_{r-1})+1)$$

$$< ..... < \frac{1}{\Pi(\cup_{i=0}^r S''_i)\Pi'(\cup_{i=0}^r S''_i)} F_{\sum\limits_{i=0}^{r}(w_i+v_i)}(d_0\Pi(S_0)+1)$$

$$= \frac{1}{\Pi(X)\Pi'(X)} F_{2k}(d_0\Pi(S_0)+1)$$

now, $d_0 = d, S_0 = S_{r+1} = \phi$, and $d_{r+1} = d\prod_{x_i \in X} \dfrac{x_i^{e_i+1}-1}{x_i-1} = n\prod_{x_i \in X} x_i^{e_i}$ plugging this values in above equation we obtain the theorem.

Remark: It is believed that there are an infinite number of even perfect numbers. Such numbers necessarily have exactly two distinct prime factors. Thus, the previous theorem should (at least conjecturally) prove false if we do not stipulate that the x_i are odd, even if we force the x_i to be prime. In any case, we do have the infinite family of solutions to (3.1) when $k = 2$, $x_1 = 2$, $x_2 = 2^m - 1$, for $n = 2$, $d = 1$, $e_1 = m - 1$, and $e_2 = 1$, when we remove the hypothesis that the x_i are odd. There also exist more exotic infinite families like $x_1 = 3$, $x_2 = 3$, $x_3 = 3^m - 1$, for $n = 2$, $d = 1$, $e_1 = 1$, $e_2 = m - 1$, and $e_3 = 1$.

Remark: The hypotheses in the previous theorem are weak enough to capture the so-called "spoof" odd perfect number constructed by Descartes; $N = 3^2 7^2 11^2 13^2 22021^1$, where $22021 = 19^2 61$ is treated as a prime, there are no other spoofs of this sort with $k \leq 7$. On the other hand, our conditions are not weak enough to capture spoofs involving negative integers, such as $N = 2^3 3^2 (-5)^1 (-13)^1$.

Corollary 3.7[3]: *Using the assumptions and notations of Theorem 3.6, and setting* $N = \prod_{i=1}^{k} x_i^{e_i}$*, the following chain of inequalities holds:*

$$N < d\frac{F_{2k}(d+1)}{F_{k+1}(d+1)} < d(d+1)^{(2^k-1)^2}. \tag{3.17}$$

In particular, when N is an odd multiperfect number we achieve

$$N < 2^{(2^k-1)^2} \tag{3.18}$$

Proof: By hypothesis, we have the equality

$$\prod_{i=1}^{k} \frac{x_i^{e_i+1} - 1}{x_i^{e_i}(x_i - 1)} = \frac{n}{d}$$

which we can rewrite in the form

$$\prod_{i=1}^{k} \left(1 - \frac{1}{x_i^{e_i+1}}\right) = \frac{n\Pi'(X)}{d\Pi(X)}.$$

Applying Lemma 3.4 with $a = n\Pi'(X)$ and $b = d\Pi(X)$, we see that

$$n\Pi'(X)\prod_{i=1}^{k} x_i^{e_i+1} = n\Pi'(X)\Pi(X)N \leq F_k(n\Pi'(X)+1).$$

After dividing both sides by $n\Pi'(X)\Pi(X)$, and using the fact that $d\Pi(X) \geq n\Pi'(X)+1$, we arrive at the inequality

$$N \leq d\frac{(n\Pi'(X)+1)^{2^k} - (n\Pi'(X)+1)^{2^{k-1}}}{(n\Pi'(X)+1)^2 - (n\Pi'(X)+1)} \tag{3.19}$$

If $n\Pi'(X)+1 < (d+1)^{2^k}$, then as the quantity on the right-hand side of (3.19) is a strictly increasing function in terms of $n\Pi'(X)+1$, we obtain the first inequality in (3.17). On the other hand, $n\Pi'(X)+1 \geq (d+1)^{2^k}$, then using the bound found in Theorem 3.6, we again achieve the first inequality in (3.17).

Now, we prove the second inequality in (3.17). As $(y^{ab}-1)/(y^a-1) = y^{ab-a} + y^{ab-2a} + ... + y^a + 1 < y^{ab-a+1}$ (for $y \geq 2$), we calculate

$$\begin{aligned}
&d\frac{(d+1)^{2^{2k}} - (d+1)^{2^{2k-1}}}{(d+1)^{2^{k+1}} - (d+1)^{2^k}} \\
&= d(d+1)^{2^{2k-1}-2^k}\frac{(d+1)^{2^{2k-1}}-1}{(d+1)^{2^k}-1} \\
&< d(d+1)^{2^{2k-1}-2^k+2^{2k-1}-2^k+1} = d(d+1)^{(2^k-1)^2}
\end{aligned}$$

For the last statement, take $d = 1$.

Corollary 3.8[3]: *Let N be an odd perfect number with k distinct prime factors. If P is the largest prime factor of N, then $10^{12}P^2N < 2^{4^k}$.*

Proof: Since N is perfect we take $n = 2$ and $d = 1$. Clearly, $2(P-1) > P$. It is known, that the second largest prime factor of N is bigger than 10^4, and the third largest prime factor is bigger than 10^2. The corollary now follows by specializing the main result of Theorem 3.6 to this case.

In the past, these types of theoretical upper bounds on odd perfect numbers were of little use in calculations due to their doubly exponential growth.

However, in the next section we find a way to exploit the existence of very large prime divisors of N (when they occur), which then makes the upper bound a feasible computational tool.

3.3 Using the GCD algorithm

In [4] it was proved that if N is an odd perfect number with k distinct prime factors, p is a Fermat prime, and $p^a || N$ with a large, then the special prime factor of N is large. In particular, we can use this fact in conjunction with Corollary 3.8 to obtain an upper bound on the size of the special prime, and hence on the size of a. To do so, we first need to recall a few well-known results.

Let $\Phi_n(x)$ be the nth cyclotomic polynomial (i.e. the minimal polynomial over Q for a primitive nth root of unity), we have the partial factorization

$$p^n - 1 = \prod_{d|n} \Phi_d(p)$$

and so

$$\sigma(p^a) = \frac{p^{a+1}-1}{p-1} = \prod_{d|(a+1), d>1} \Phi_d(p) \tag{3.20}$$

We are further interested in the factorization of $\Phi_d(p)$. If c and d are integers with $d > 1$ and $gcd(c, d) = 1$ we write $o_d(c)$ for the multiplicative order of c modulo d. If p is prime, we write v_p for the valuation associated to p. In other words, for $n \in Z_+$ we have $p^{v_p(n)} || n$.

Lemma 3.9[4]: *Let $m > 1$ be an integer, and let q be prime. Write $m = q^b n$ with $gcd(q, n) = 1$*
If $b = 0$, then

$$\Phi_m(x) \equiv 0 (mod q)$$

is solvable if and only if $q \equiv 1 (mod\ m)$. The solutions are those x with $o_q(x) = m$. Furthermore, $v_q(\Phi_m(x)) = v_q(x^m - 1)$ for such solutions.

If $b \neq 0$, then

$$\Phi_m(x) \equiv 0(modq)$$

is solvable if and only if $q \equiv 1(mod\ n)$. The solutions are those x with $o_q(x) = n$. Furthermore, if $m > 2$, then $v_q(\Phi_m(x)) = 1$ for such solutions.

Lemma 3.10[4]: *Let p and q be primes, $q \geq 3$, and $a \in Z_+$. Then*

$$v_q(\sigma(p^a)) = \begin{cases} v_q(p^{o_q(p)} - 1) + v_q(a+1), & \text{if } o_q(p)|(a+1)\text{and } o_q(p) \neq 1, \\ v_q(a+1) \quad \text{if } o_q(p) = 1, \\ 0 \quad\quad \text{otherwise.} \end{cases}$$

In our work we will want a prime divisor q of $\Phi_d(p)$ with $o_q(p) = d$.

Lemma 3.11[4]: *Let N be an odd perfect number, $p^a||N$ with p prime, and let q be a Fermat prime. Then:*

$$v_q(\sigma(p^a)) = \begin{cases} v_q(p+1) + v_q(a+1), & \text{if } \pi = p \equiv -1(modq), \\ v_q(a+1) \quad \text{if } p \equiv 1(modq), \\ 0 \quad\quad \text{otherwise.} \end{cases}$$

The next three lemmas state some limitations on the exponent a. In particular, it cannot have too many prime divisors or be too large.

Lemma 3.12[3]: *Let p be a prime and let N be an odd perfect number with k distinct prime divisors. If $p^a||N$ with $a \in Z_{>0}$, then $\sigma_0(a+1) \leq k$.*
Lemma 3.13[3]: *Let p be a Fermat prime, let N be an odd perfect number, and let π be the special prime factor. Let $a, m \in Z_{>0}$. If $p^m|(\pi+1)$ and $p^a||N$ with $a < 3m$, then $\pi \nmid \sigma(p^a)$.*

Lemma 3.14[3] *Let q be a Fermat prime, let N be an odd perfect number with k distinct prime factors, let π be the special prime factor, and suppose $q^a|N$ for some $a \in Z_{>0}$. If $a > (k-1)(k-2)$, then $2q^{a-(k-2)(k-3)-[(k-2)/2]}|(\pi+1)$.*

Using above lemma's and conjunction ie Let N be an odd perfect number with k distinct prime factors. If P is the largest prime factor of N, then

$10^{12}P^2N < 2^{4^k}$, we have.

Proposition 3.15[3]: *Let p and q be odd, distinct primes less than 180, and let $T = \{3, 5, 17\}$. Let N be an odd perfect number with $k \leq 9$ distinct prime divisors. Suppose $p^a||N, q^b||N$, and $10^{100} < p^a < q^b$ for some $a, b \in Z_{>0}$.*

(i) If $p, q \in T$, then $a < \frac{1}{10}(\frac{4^9 log(2)+147 log(q)-12 log(10)}{log(p)} + 147)$.

(ii) If $p \in T, q \notin T$, then $a < \frac{1}{6}(\frac{4^9 log(2)+12 log(q)-12 log(10)}{log(p)} + 102)$.

(iii) If $p \notin T, q \in T$, then $a < \frac{1}{6}(\frac{4^9 log(2)+102 log(q)-12 log(10)}{log(p)} + 12)$.

(iv) If $p \notin T, q \notin T$, then $a < \frac{1}{4}(\frac{4^9 log(2)+12 log(q)-12 log(10)}{log(p)} + 12)$.

Proof: A quick computer search, restricting to odd primes p and q less than 180, demonstrates that if $q^{p-1} \equiv 1 (mod p^n)$, then $n \leq 3$ except for the pair $(p, q) = (3, 163)$. For those pairs $(p, q) \neq (3, 163)$, we have $v_p(\sigma(q^b)) \leq k+3$. This inequality also holds for the exceptional pair $(p, q) = (3, 163)$ since the multiplicative order of 163 modulo 3 is 1 and 163 cannot be the special prime. Similarly, $v_q(\sigma(p^a)) \leq k+3$. (We also note that there is only one pair with $q^{p-1} \equiv 1 (mod p^3)$ for which similar reasoning does not allow us to use the slightly better bound $k+2$, namely, $(p, q) = (3, 53)$. But such an improvement is not significant, and we will not pursue it here.)
By Corollary 3.8 we find

$$2^{4^9} > 10^{12} 2N \geq 10^{12} 2p^a q^b \left(\frac{\sigma(p^a q^b)}{2p^{k+3} q^{k+3}} \right) > 10^{12} \frac{p^{4a-12}}{q^{12}}$$

Note that the term inside the large parentheses consists of divisors of N relatively prime to p and q (possibly with extra factors in the denominator), which is how we obtain the second inequality. Solving for a yields the bound in part (iv).

$$2^{4^9} > 10^{12} \frac{p^{4a-12}}{q^{12}}$$

$$log(2^{4^9}) > log(10^{12}\frac{p^{4a-12}}{q^{12}})$$

$$4^9 log(2) > log(10^{12}) + log(\frac{p^{4a-12}}{q^{12}})$$

$$4^9 log(2) > 12log(10) + log(p^{4a-12}) - log(q^{12})$$

$$4^9 log(2) > 12log(10) + (4a-12)log(p) - 12log(q)$$

$$4^9 log(2) - 12log(10) + 12log(q) > (4a-12)log(p)$$

$$\frac{4^9 log(2) - 12log(10) + 12log(q)}{log(p)} > (4a-12)$$

$$\frac{4^9 log(2) - 12log(10) + 12log(q)}{log(p)} + 12 > 4a$$

$$\frac{1}{4}\left(\frac{4^9 log(2) - 12log(10) + 12log(q)}{log(p)} + 12\right) > a$$

(note: Other Inequalities can be solved in similar way once the bound is generated, which will be provided below).

Now suppose for a moment that p is a Fermat prime in T. Since $k \leq 9$, we have $(k-1)(k-2) \leq 56$. But $17^{80} < 10^{100}$ so $a > 80$. Lemma 3.14 then yields

$$2p^{a-45}|2p^{a-(k-2)(k-3)-\lfloor(k-2)/2\rfloor}|(\pi+1)$$

From $a > 80$ we obtain $3(a-45) > a$, so by Lemma 3.13 we conclude that $\pi \mid \sigma(p^a)$. If q is a Fermat prime in T, then by the same analysis we obtain $\pi \nmid \sigma(q^b)$ and $2q^{b-45}|(\pi+1)$. In either case, we see that neither p nor q can be the special prime. In case (iii), when q is a Fermat prime in T (and p is not), we compute

$$2^{4^9} > 10^{12}P^2N \geq 10^{12}\pi^2 p^a q^b \frac{\sigma(p^a q^b)}{2p^{k+3}q^{k+3}} > 10^{12}q^{4b-102}p^{2a-12} > 10^{12}p^{6b-12}p^{-102}$$

When p is a Fermat prime in T (but q is not) we similarl find

$$2^{4^9} > 10^{12}P^2N \geq 10^{12}\pi^2 p^a q^b \frac{\sigma(p^a q^b)}{2p^{k+3}q^{k+3}} > 10^{12}p^{6b-102}p^{-12}$$

Finally, in case (i) we know $2p^{a-45}q^{b-45}|(\pi+1)$. So we have

$$2^{4^9} > 10^{12}P^2N \geq 10^{12}\pi^3 p^a q^b \frac{\sigma(p^a q^b)}{2p^{k+3}q^{k+3}} > 10^{12}p^{10b-147}p^{-147}$$

The extra π comes from N, since $\pi \nmid \sigma(p^a q^b)$ in this case. Solving for a in the above inequalities yields the stated bounds.
The numbers in this proposition are not chosen to be the strongest possible, but rather to be convenient for the case $k = 9$. If some of the hypotheses are strengthened then the proposition will work (with modified bounds) for larger k and larger primes. The real strength of the proposition is in the fraction out front. We can use, with little loss in computational speed, the more uniform bound

$$\frac{1}{\epsilon}\left(\frac{4^k log(2)}{log(p)} + C(k,q)\right),$$

where $C(k,q)$ is some constant depending only on k and q, and where ϵ is 4, 6, or 10 depending on the number of Fermat primes among $\{p,q\}$. Similar statements apply to the following proposition.

Proposition 3.16[3]: *Let p and q be odd, distinct primes less than 180, and let $T = \{3,5,17\}$. Let N be an odd perfect number with $k \leq 9$ distinct prime divisors. Suppose $p^a||N, q^b||N$, and $10^{100} < p^a < q^b$ for some $a,b \in Z_{>0}$.*

(i) If $p,q \in T$, then $b < \frac{1}{5}(\frac{4^9 log(2)-(5a-147)log(p)-12log(10)}{log(q)} + 147)$.

(ii) If $p \in T, q \notin T$, then $b < \frac{1}{2}(\frac{4^9 log(2)-(4a-102)log(p)-12log(10)}{log(q)} + 12)$.

(iii) If $p \notin T, q \in T$, then $b < \frac{1}{5}(\frac{4^9 log(2)-(a-12)log(p)-12log(10)}{log(q)} + 135)$.

(iv) If $p,q \notin T$, then $a < \frac{1}{2}(\frac{4^9 log(2)-(2a-12)log(p)-12log(10)}{log(q)} + 12)$.

Proof: One does an analysis as in the previous proposition. The only difficulty is deciding to use the lower bound $10^{12}P^2N \geq 10^{12}\pi^2 p^a q^b \sigma(p^a q^b)/(p^{k+3}q^{k+3})$ when $p \in T$ and q is not, and to use $10^{12}P^2N \geq 10^{12}\pi^3 p^a q^b \sigma(q^b)/p^{k+3}$ when

$q \in T$ and p is not. In case (i) we find

$$2^{4^9} > 10^{12}P^2N \geq 10^{12}\pi^3 p^a q^b \frac{\sigma(p^a q^b)}{2p^{k+3}q^{k+3}} > 10^{12}p^{5a-147}q^{5b-147}$$

Solving for b gives the needed bound. The other cases are similar and are left to the reader.

Remark. One also has the inequality $b > a\frac{log(p)}{log(q)}$ in all cases, as $p^a < q^b$. When searching through candidate odd perfect numbers N, one often can reduce to the case when N is divisible by a prime power $p^a||N$ with a large. Using congruence conditions, when a is large enough one can show that there exists a very big prime factor Q_1 of N. Historically, it was considerations that odd perfect numbers must have seven, and then eight, distinct prime factors, later the insight which improved the number of distinct prime factors to nine was that not only do congruence conditions yield a very large prime factor Q_1, but there must also be another large prime $Q_1' > 10^{11}$ (which, in practice, is not quite as large as Q_1) that divides $\sigma(p^a)$. If one can show the existence of a third large prime, further improvements can be made.
The propositions above are the key tool to finding a possible third large prime divisor. First, reduce to the case where we have two different prime powers $p^a||N$ and $q^b||N$ with a, b large. The sizes of a and b are bounded above. We know that there should be a large prime divisor Q_1' of $\sigma(p^a)$ and a large prime divisor Q_2' for $\sigma(q^b)$. Our aim is to show that Q_1' and Q_2' are not equal. Thus, we compute $gcd(\sigma(p^a), \sigma(q^b))$ (for a, b limited to the ranges given in the propositions above), and find that there are no common large primes. This computation was run in Mathematica, on a single core, over the course of a few months. We summarize the results of this computation as follows.

Theorem 3.17[3]: *Let p and q be odd, distinct primes, less than 105. Let N be an odd perfect number with $k \leq 9$ distinct prime divisors. Suppose $p^a||N$, $q^b||N$, and $10^{100} < p^a < q^b$ for some $a, b \in Z_{>0}$. Then the largest prime which divides both $\sigma(p^a)$ and $\sigma(q^b)$ is smaller than 10^{11}.*

The number 10^{11} was chosen to be compatible with the bounds developed

in [4], and it was fortunate that it was sufficiently large to preclude the existence of counterexamples. While the existence of large common prime divisors is very scarce, there are still some close calls, such as 27866489501 dividing $gcd(\sigma(p^a), \sigma(q^b))$ with $p = 59$, $a = 2874$, $q = 7$, and $b = 15394$. Another close call occurred with the prime 17622719441, for inputs $p = 103$, $a = 3598$, $q = 61$, and $b = 11833$. Even so, if we were to reduce 10^{11} to 10^{10} then we could still easily deal with those examples that arise. (The two just mentioned are the only two larger than 10^{10}).
The assumption in Theorem 3.17 that N is an odd perfect number is used in two ways. First, we can use the bounds given in the previous propositions. Second, we may limit the exponents a and b even further (according to whether or not p or q can be the special prime).

3.4 Improvements of the inequality k $\geq$ 10

Lemma 3.18[3]: *Let p be an odd prime and let $q \in \{3, 5, 17\}$. If $q^{p-1} \equiv 1(mod\ p^2)$, then either $(q, p) = (3, 11), (q, p) = (17, 3)$, or $q^{o_p(q)} - 1$ has a prime divisor greater than 10^{14}.*
Proof: If $p > 10^{14}$, then p is the needed prime divisor.
If $p < 10^{14}$, then there are only twelve pairs (p, q) with $q^{p-1} \equiv 1(modp^2)$. Two cases are exceptional, and they appear in the statement of this lemma. For each of the other cases we compute all the prime divisors of $q^{o_p}(q) - 1$ less than 10^{14}, and see that the remaining cofactor is not 1. The cases are mentioned in [4,lemma 9].

Proposition 3.19[3]: *Let N be an odd perfect number with k, k_1, and k_2 having their usual meanings. Suppose $q \in \{3, 5, 17\}$ is a known prime divisor of N, $q^n || N, q \neq \pi$, and $\pi \nmid \sigma(q^n)$. Suppose $p_1, ..., p_{k_1-1}$ are the other known prime factors of N, besides q. For each $i = 1, 2, ..., k_1 - 1$ define*

$$\epsilon_i = \begin{cases} 0 \quad if O'_{p_i}(q) = 0, \\ max(s + t - 1, 1) \quad if O'_{p_i}(q) \neq 0, \end{cases} \tag{3.21}$$

where $s = v_{p_i}(\sigma(q^{o_{p_i}(q)-1}))$ *and* $t \in Z_{>0}$ *is minimal so that* $p_i^t > 100$
$V = \prod_{i=1}^{k_1-1} p_i^{\epsilon_i}$ *Suppose* π *is among the* k_2 *unknown prime factors. Finally, assume that all unknown prime factors are greater than 100.*
if

$$min(\sigma(q^n)/V), \sigma(q^{100}/V) > 1,$$

then $k_2 > 1$. *In that case, if*

$$min\left(10^{14}, \left(\frac{\sigma(q^n)}{V}\right)^{\dfrac{1}{k_2-1}}, \left(\frac{\sigma(q^{100})}{V}\right)^{\dfrac{1}{k_2-1}}\right) > 1,$$

then $\sigma(q^n)$ *has a prime divisor among the unknown primes at least as big as this minimum.*

Proof: We only do the case when $q = 3$, since the other cases are similar. First suppose $\sigma(q^n) = (q^{n+1} - 1)/(q - 1)$ is at most divisible by $p_i^{e_i}$ for the known primes, and square-free for the unknown primes. Then since $\pi, q \nmid \sigma(q^n)$, the largest unknown prime divisor of $\sigma(q^n)$ is at least

$$\left(\frac{\sigma(q^n)}{V}\right)^{\frac{1}{k_2-1}}$$

unless this quantity is ≤ 1 (in which case there might be no unknown factors). So we may assume there is some prime $p|N$, $o_p'(q) \neq 0$, so that $\sigma(q^n)$ is divisible by p^2 if p is unknown, or p^{e+1} if p is known (and ϵ is the corresponding ϵ_i), with p maximal among such primes. We may also assume that if $p^2|(q^{n+1} - 1)$ and p is unknown, then $p|(n + 1)$. (This is where 10^{14} comes into the minimum.) Thus, in either case, $p^t|(n + 1)$ where $p^t > 100$ (taking $t = 1$ if p is an unknown prime). Then we have

$$\sigma\left(q^{p^t-1}\right)|\sigma(q^n)$$

Thus it suffices to find a large divisor of $(q^{p^t} - 1)/(q - 1)$. The quantity $(q^{p^t} - 1)/(q - 1)$ is only divisible by primes larger than p, or p itself to the first power. (In this case, q being Fermat means the quantity is not divisible by p, and we could replace $max(s+t-1,1)$ by $s+t-1$ in the definition of ϵ_i. But to keep similar notations later when we take q to be an arbitrary prime, we do not use this fact.) But then, by the maximality condition on p, $(q^{p^t}-1)/(q-1)$ is not divisible by more than $p_i^{e_i}$ for known primes and the first power for all the unknown primes. So the analysis we used in the first paragraph goes through by only changing $n+1$ to p^t Finally, note that $p^t > 100$, so we have the appropriate bound.

The most useful case when we will use is when n is very large and k_1 is close to k. So, in practice, we will usually end up with 10^{13} as the lower bound on a divisor of $\sigma(q^n)$.

Proposition 3.20[3]: *Let N be an odd perfect number, and let $q < 1000$ be a prime divisor of N with $q^n || N$. Suppose $b, k, k_1, k_2, l_1, l_2, k_1'$, and l_1'have same meaning as defined above. Let T be the set of known primes with unknown component, different from q, and $\not\equiv (mod q)$. Let*

$$\tau = n - b - \sum_{p \in T, o_q'(p) \neq 0} \left(v_q(p^{o_q(p)} - 1) + \lfloor \frac{k_1' + k_2}{\sigma_0(o_q(p))} \rfloor \right) - (k_1' - l_1' + k_2)(k_1' + k_2 - 1). \tag{3.22}$$

If $\tau > 0$, then one of the unknown primes is not congruent to 1 (mod q). Further, in this case, one of the unknown primes is at least as large as $min(q^{\tau'-2}, 10^{1000})$ where

$$\tau' = min\lceil \left(n - b - \sum_{p \in T, o_q'(p) \neq 0} \left(v_q(p^{o_q(p)} - 1) + \lfloor \frac{k_1' + k_2 - m}{\sigma_0(o_q(p))} \rfloor \right) - (k_1' - l_1' + k_2 - m)(k_1' + k_2 - m - 1) - m \lfloor \frac{k_1' + k_2 - m}{2} \rfloor \right) / m \rceil.$$

Lemma 3.21[3]: Let q be a prime with $7 \leq q < 1000$. Suppose $a^{q-1} \equiv 1 (mod q^n)$ for some $n \in Z_{>0}$ and some positive integer a with $(q-1)|a$. Then $a \geq min(q^{n-4}, 10^{1000})$.
(note: the above result is obtain by machine computation, more details are available is [4]).

Lemma 3.22[3]: Let p and q be primes with $10^2 < p < 10^{11}$ and $7 \leq q < 180$. If $q^{p-1} \equiv 1(mod\ p^2)$, then $\sigma(q^{o_p(q)-1})$ is divisible by two primes greater than 10^{11}.

Proof: The paper [5] lists all 61 pairs (p, q) satisfying the conditions of the lemma.

For 56 of those pairs, we have two explicit prime factors of $A = \sigma(q^{o_p(q)-1})$, each greater than 10^{11}. For another 3 pairs, we have one such prime factor P for which $P^2 \nmid A$. Further, after computing all prime factors of A/P less than 10^{11} we find that the cofactor is not 1, so there must exist some other prime factor greater than 10^{11}.

The remaining two pairs are $(p, q) = (1025273, 41)$ and $(q, p) = (122327, 157)$. In the first case $o_p(q) = 2^3.128159$. Exhaustively removing all of the prime factors of $41^{128159} - 1$ less than 10^{11}, we find a non-trivial cofactor. Similarly, $41^{128159} + 1$ also has a prime divisor larger than 10^{11}. Both of these primes divide A. This deals with the first case. The second case is dealt with similarly, as $o_p(q) = 2.1973$.

3.5 Abundance and deficiency

Let $n \in Z_+$. Recall the multiplicative function $\sigma_{-1}(n) = \sum_{d|n} d^{-1}$ we introduced earlier. This function can alternatively be written using the formula $\sigma_{-1}(n) = \sigma(n)/n$, and so $\sigma(n)/n = 2$ if and only if $\sigma_{-1}(n) = 2$. A number n is called *abundant* when $\sigma_{-1}(n) > 2$ and deficient when $\sigma_{-1}(n) < 2$. We can use abundance and deficiency computations to limit choices on possible prime factors of an odd perfect number N. First, we extend the definition of σ_{-1} by setting

$$\sigma_{-1}(p^\infty) = \lim_{a \to \infty} \sigma_{-1}(p^a) = \frac{p}{p-1}$$

Lemma 3.23[4]: *Let p and q be odd primes. If $1 \leq a < b \leq \infty$, then $1 < \sigma_{-1}(p^a) < \sigma_{-1}(p^b)$. If $a, b \in [1, \infty]$ and $p < q$, then $\sigma_{-1}(q^b) < \sigma_{-1}(p^a)$.*

Lemma 3.24[4]: *Let N be an odd perfect number. Suppose $p_1, ..., p_{k_1}$ are the known prime factors of N, $p_i^{a_i}|N$, and $k_1 < k = \omega(N)$. If $\Pi = \prod_{i=1}^{k_1} \sigma_{-1}(p_i^{a_i}) < 2$ then the smallest unknown prime is*

$$p_{k_1+1} \geq \frac{\Pi}{2 - \Pi}$$

Proof: We find

$$2 = \sigma_{-1}(N) \geq \left(\prod_{i=1}^{k_1} \sigma_{-1}(p_i^{a_i}) \right) \sigma_{-1}(p_{k_1+1}) = \Pi \frac{p_{k_1+1} + 1}{p_{k_1+1}}$$

where the inequality in the middle follows from above lemma. Noting $\Pi \geq 1$, we obtain $\frac{2}{\Pi} \geq 1 + \frac{1}{p_{k_1+1}}$. Therefore $\frac{2-\Pi}{\Pi} \geq \frac{1}{p_{k_1+1}}$ and taking reciprocals gives us the result, since $2 - \Pi > 0$.

Note that in the lemma if $\Pi > 2$, then $\prod_{i=1}^{k_1} p_i^{a_i}$ is abundant, hence N is abundant. If $\Pi = 2$, then $\prod_{i=1}^{k_1} p_i^{a_i}$ is already an odd perfect number.
The following lemma is the true key to our search for odd perfect numbers, as simple as the proof is (after wading through the hypotheses). This is because we built up machinery in the last few sections to find bounds for large prime divisors of N.

Lemma 3.25[4]: *Let N be an odd perfect number. Let $p_1, ..., p_k$ be the prime divisors of N, and let a_i be such that $p_i^{a_i}||N$. Fix the numbering on the indices so that $p_1, ..., p_{l_1}$ are the primes with known prime component, $p_{l_1+1}, ..., p_{k_1}$ are the other known primes, and $p_{k_1+1} < ... < p_k$ are the unknown primes. Suppose among the unknown primes we have bounds $p_k > P_1 > 1, ..., p_{k-v+1} > P_v > 1$, with $v < k_2$. For each $u = 0, 1, ..., v$ set*

$$\Delta_u = \left(\prod_{i=1}^{l_1} \sigma_{-1}(p_i^{a_i}) \right) \left(\prod_{i=l_1+1}^{k_1} \frac{p_i}{p_i - 1} \right) \left(\prod_{i=l_1+1}^{u} \frac{P_i}{P_i - 1} \right)$$

Finally, suppose $k_2 > 0$.

If $\Delta_u < 2$, then the smallest unknown prime is

$$p_{k_1+1} \leq \frac{\Delta_u(k_2 - u)}{2 - \Delta_u} + 1.$$

$\therefore$

$$p_{k_1+1} \leq min_{u \in [0,v], \Delta_u < 2} \frac{\Delta_u(k_2 - u)}{2 - \Delta_u} + 1.$$

Proof: We compute

$$2 = \sigma_{-1}(N) = \prod_{i=1}^{k} \sigma_{-1}(p_i^{a_i})$$

$$\leq \left(\prod_{i=1}^{l_1} \sigma_{-1}(p_i^{a_i})\right)\left(\prod_{i=l_1+1}^{k-u} \sigma_{-1}(p_i^{\infty})\right)\left(\prod_{i=k-u+1}^{k} \sigma_{-1}(p_{k-i+1}^{\infty})\right)$$

$$= \Delta_u \prod_{i=l_1+1}^{k-u} \sigma_{-1}(p_i^{\infty}) \leq \prod_{i=0}^{k-u-k_1-1} \sigma_{-1}((p_{k_1+1} + i)^{\infty})$$

$$= \Delta_u \frac{p_{k_1+1} + k - u - k_1 - 1}{p_{k_1+1} - 1}$$

$$= \Delta_u \left(1 + \frac{k_2 - u}{p_{k_1+1} - 1}\right)$$

Now, recall that $u \leq v < k2$ which implies $k_2 - u \geq 1$. Also $0 < \Delta_u < 2$, so we solve the main inequality as we did in the previous lemma, finding

$$p_{k_1+1} \leq \frac{\Delta_u(k_2 - u)}{2 - \Delta_u} + 1$$

One major difference between this lemma and previous lemma is that if $\Delta_u > 2$, then that doesn't necessarily imply N is abundant. (It is true that if $k_1 = k$ and $\Delta_0 < 2$, then N is deficient, however.) This means that we might end up with $\Delta_0 > 2$, and hence we have no upper bound on p_{k_1+1}.

Proposition 3.26[3]: *Let N be an odd perfect number and let $7 \leq q < 180$ be a known prime divisor of N, with $q^n || N$. Let τ, τ' be as seen in above Proposition, suppose all the hypotheses of that proposition are met, and* *let p be the guaranteed unknown prime.Let $p_1, ..., p_{k_1-1}$ be the known primes different from q. Let ϵ_i be defined as before,and put $V = \prod_{i=1}^{k_i-1} p_i^{\epsilon_i}$. Finally, assume that all unknown prime factors are greater than 100.*

If $min(\sigma(q^{\tau'-4})/V), \sigma(q^{96}/V) > 1$,then $k_2 > 1$. In that case, if

$$min\left(10^{11}, \left(\frac{(q^{96})}{(q-1)V}\right)^{\frac{1}{k_2-1}}, \left(\frac{(q^{\tau'-4})}{(q-1)V}\right)^{\frac{1}{k_2-1}}\right) > 1,$$

then $\sigma(q^n)$ has a prime divisor, different from p, among the unknown primes, at least as big as the above minimum.

Proof: We do the case $q \in 7, 11, 13$ to the range $7 \leq q < 180$, assume if

$$min\left(10^{11}, \left(\frac{(q^{96})}{(q-1)V}\right)^{\frac{1}{k_2-1}}, \left(\frac{(q^{\tau'-4})}{(q-1)V}\right)^{\frac{1}{k_2-1}}\right) > 1,$$

First note that if $d|(n+1)$, then $\sigma(q^{d-1})|\sigma(q^n)$ and so it suffices to show that $\sigma(q^{d-1})$ has a prime divisor larger than the above minimum, different from p, for some $d|(n+1)$.

by proposition [3.19] we may assume that at most ϵ_i copies of p_i divide $\sigma(q^{d-1})$, for some d either greater than 100 or equal to $n+1$. Furthermore, because 10^{11} occurs in the above minimum, we may assume that the only unknown prime greater than 10^{11} that may divide $\sigma(q^n)$ is p. Then by Lemma [3.18] and the fact that the unknown primes are greater than 100, we may assume $\sigma(q^n)$ is square-free for unknown primes, except possibly p.

From the proof for Proposition 3.20, we have $p^{q-1} \equiv 1(modq^{\tau'})$ and $\tau \leq n$. Write $q^d - 1 = (q-1)mp^c$ with $c \in N$,$m \in Z_+$, and $gcd(p, m) = 1$. Powering this equation to the $(q-1)st$ power, we have

$((q-1)m)^{q-1} \equiv ((q-1)mp^c)^{q-1} = (q^d - 1)^{q-1} \equiv 1(modq^{min(\tau', 100)})$.

By Lemma 3.5, $(q-1)m \geq min(q^{\tau'-4}, q^{96}, 10^{50}) = min(q^{\tau'-4}, 10^{50})$.6 (noting $q^{96} < 10^{1000}$) Thus

$$\frac{m}{V} \geq min\left(\frac{q^{\tau'-4}}{(q-1)V}, \frac{q^{96}}{(q-1)V}\right)$$

Since m/V is at least as big as the part of $\sigma(q^d - 1)$ made up from the unknown primes, different from p, "if $min(\sigma(q^{\tau'-4})/V), \sigma(q^{96}/V) > 1$,then

$k_2 > 1$ root of the minimum we have the appropriate lower bound.
Note that "if $min(\sigma(q^{\tau'-4})/V), \sigma(q^{96}/V) > 1$ holds but $k_2 = 1$, then we reach a contradiction.So we add this contradiction to the list found in Algorithm.

The final change in our implementation of the algorithm is that when applying above lemma to find upper bounds on the next unknown prime, we use Theorem 3.17 above to contribute more large primes. However, Theorem 3.17 does not automatically prove the existence of a third large prime. In practice, we have one prime Q_1 coming from congruence conditions relative to p, and another prime $Q'_1 > 10^{11}$ dividing $\sigma(p^a)$ with $Q'_1 \neq Q_1$. Similarly, we have one prime Q_2 from congruence conditions relative to q, and another prime $Q'_2 > 10^{11}$ dividing $\sigma(q^b)$ with $Q'_2 \neq Q_2$. Theorem 3.17 asserts $Q'_1 \neq Q_2$, but it could still be the case that $Q_2 \neq Q'_1$ and $Q_1 = Q'_2$.
When computing the bounds coming from above lemma we are thus lead to consider two situations, which we describe now. Suppose our algorithm has reached a point where we have a list $q_1^{a_1}, q_2^{a_2}, ..., q_n^{a_n}$ of infinite prime powers in a suspected odd perfect number. Let Q_i be the large prime coming from q_i from the congruence conditions in [4, Proposition 7] or Proposition 3.20 (according to whether $q_i \in 3, 5, 17$, or not). Similarly, let Q'_i be the second large prime coming from q_i, using Proposition 3.19 or 3.23. We drop from our list any q_i for which $Q'_i < 10^{11}$, so that Theorem 3.17 will apply.
There are now two main cases. We apply above lemma in both cases, and then use the lesser of the two bounds achieved. One option is that $Q_1 = Q_2 = ... = Q_n$. In this case, as each $Q'_i \neq Q_i = Q_1$ and $Q'_i \neq Q'_j$ by Theorem 3.17, we can apply above lemma with the bounds $P_1 = Q_1, P_2 = 10^{11}, P_3 = 10^{11}, ..., P_{n+1} = 10^{11}$. The second option is that the Q_i are not all equal. Let Q_1 be the largest element in $Q_1, Q_2, ..., Q_n$, and let Q_n be the smallest. The worst possible case would be that among the distinct primes $Q'_1, Q'_2, ..., Q'_n$ the two largest primes are equal to Q_1 and Q_n, and the rest are close to 10^{11}. Thus, we apply above lemma with the list of bounds $P_1 = Q_1, P_2 = Q_n, P_3 = 10^{11}, ..., P_n = 10^{11}$.
With all of these changes in place, we rerun the algorithm described below The computation takes just over one day on a single core, covering a little over thirty million cases, and we achieve:

Theorem 3.27[3] : *There are no odd perfect numbers with less than 10 distinct prime factors.*

Remark: There are a few problem cases requiring special treatment, which we describe now. These cases also illustrate some of the benefits and limitations in the changes we made above.
Initially, the plan had been to make the primes $p = 101, 103$ become infinite when $p^a > 10^{50}$. However, with this choice the prime power 103^∞ never satisfies the bounds in Proposition 3.20. This left cases such as $3^\infty 5^\infty 19^\infty 103^\infty 1399^\infty p^\infty$ (where $13689227 \leq p \leq 13691033$). Here we needed another large prime, since the bounds on the next unknown prime were already bigger than 10^{11}. This problem was solved by expanding the factorization table for $p = 101, 103$ up to the level $p^a \leq 10^{150}$.
This still leaves five problem cases:

- $3^4 11^\infty 5^1 73^\infty 2633^\infty 1157609^\infty$
- $3^4 11^\infty 5^1 73^\infty 2633^\infty 1157621^\infty$
- $3^4 11^\infty 5^1 73^\infty 2633^\infty 1157627^\infty$
- $3^4 11^\infty 5^1 97^\infty 263^\infty 575513^\infty$
- $3^4 11^\infty 5^1 103^\infty 227^\infty 349667^\infty$

We will only discuss the first case, as the other four are dealt with similarly. In that case, the next unknown prime p is given inside an interval $249075961044 < p < 498151922091$. This interval contains more than nine billion primes, which is too many to check one at a time. The reason for the extremely large interval is that the upper (and lower) bound on the interval is larger than 10^{11}, and thus falls outside the scope of the bounds in Proposition 3.24. When this project was begun, the number 10^{11} was the bound initially chosen when proving Lemma 3.22, and thus subsequently used in many other lemmas and propositions.
There are at least two ways to deal with this case. First, one could redo the computation of Theorem 3.17 to include the pairs

$(p,q) \in (11, 2633), (73, 2633), (2633, 11), (2633, 73)$,
and then modify Lemma 3.22, and all subsequent results, to include the new prime 2633. A second option is to again modify Lemma 3.22, and all subsequent results, but this time just for the primes 11 and 73, by replacing the bound 10^{11} with 10^{12}.

3.6 Algorithm

We start by needing a prime divisor of N. Later we develop machinery which can yield a lower and upper bound on a prime divisor of N. In our case we find $2 < p_1 < k+2$. So for example, if $k = 4$, then $p = 3$ or 5. By considering the Eulerian form, we see that the cases

$$3^2||N, 3^4||N, 3^6||N, 3^8||N, ..., 5^1||N, 5^2||N, 5^4||N, 5^5||N, ..$$

are the only ones possible. There is a benefit and cost to considering each of these cases individually. The cost is that there are an infinite number of cases, and hence we simply cannot consider them all. The benefit is that in each individual case we do not have to rely on the results of develop machinery computation below to find bounds on p_2. For example,in the case
$3^2||N$, since $13 = \sigma(3^2)|2N$, we find that $13|N$, and so we can take $p^2 = 13$. (Note that the subscript does not mean that 13 is the second smallest prime divisor of N. Only that 13 is the second prime divisor we found for N in this case.)
As a matter of terminology, we think of each of the cases

$$3^2||N, 3^4||N, 3^6||N, 3^8||N, ..., 5^1||N, 5^2||N, 5^4||N, 5^5||N, ...$$

as branches on a tree, with each branch providing new factors for our algorithm and hence branching further. As the tree branches out, we eventually arrive at cases which are contradictory in some way. However, we still have to deal with the fact that there are an infinite number of branches. To get around this problem, we combine all the branches with large powers of primes into one composite branch. In other words, if p is a prime divisor of N, we combine all the cases $p^n||N$, for large n, together into one case.More

precisely, we let B be a large integer (which will be around the size 10^{50}) which we fix at the beginning of the algorithm, and then we combine all the branches p^n together, for all $n \geq n_0$, where n_0 is minimal so that $p^{n_0} > B$. On this combined branch we are not assuming $p^n || N$ for any specific n, but rather we just assume $p^a | N$ for some $a \geq n_0$. In this way, we deal with all the remaining cases at once. As a matter of notation we label this conglomerated branch by p^∞.
For example, if we take $k = 4$ and $B = 50$, then we have five initial branches

$$3^2, 3^\infty, 5^1, 5^2, 5^\infty.$$

The first case $3^2 || N$ branches further into two sub-branches 13^1, 13^∞, and we can continue this branching process. When we are on a branch with p^∞ we say p is an infinite prime (not to be confused with the infinite primes of algebraic number theory). Notice that infinite primes do not provide more factors for the factor chain, since we don't have $\sigma(p^n) | 2N$ for any specific n, so we have to rely on the intervals of our machinery algorithm to find bounds for the next prime. If we set B too low, then the primes on our branches become infinite too quickly, and we may have the case that the intervals of machinery are very large, or even that there is no upper bound for the next prime! (This corresponds to the case when $\Delta_0 > 2$ in Lemma 3.25.) If we make B large enough, the intervals will always have upper bounds, and the algorithm will only have to consider a finite number of cases.

At each stage in the algorithm there will be prime divisors of N that are known and some that are unknown, meaning that the prime divisors are either specified by the algorithm or they are not, respectively. This set of known primes will change at every stage of the algorithm as it runs through different cases, and so the known and unknown primes are constantly changing. We let k_1 be the number of known, distinct prime divisors of N (at any given stage), and let $k_2 = k - k_1$ be the number of unknown, distinct prime divisors. Among the known prime divisors of N, some of the prime components are also known (again, known being a technical term meaning specified by the algorithm). In other words, if p is a known prime divisor of N and if our algorithm yields some $n \in Z_+$ so that $p^n||N$, we say p^n is a known prime component. We let l_1 be the number of known prime components of N, and let $l_2 = k - l_1$ be the number of unknown prime components.

A word of warning: In some theorems we will assume p is a prime with $p^n||N$, but this doesn't even mean that p is a known prime, let alone that the prime component is known. This is because, while p^n is, by hypothesis, a component of N, the prime p and the number n might not have been specified by the algorithm. Throughout, we will only use the phrases "known prime" and "known component" to mean known to us through our algorithm, rather than by hypothesis.

More formally, the known components are those prime components which occur on the branch we are on, which are not infinite. The known primes are the primes we have branched upon, along with the primes coming from σ of the known components. For example, if we are on the branch $3^\infty 5^4$, then the known primes are $3, 5, 11, 71$ (the primes 11 and 71 come from $\sigma(5^4)|2N$), and the only known component is 5^4. In this case we say that 3 and 5 are on while 11 and 71 are off. In other words, the on primes are exactly the known primes for which we have started the branching process. Note that $k_1 - l_1$ is exactly the number of (known) primes which are infinite or off. In this example, since 11 is the smallest off prime we continue the branching process first on this prime, rather than 71. When there are no off primes we use the interval bounds of machinery to arrive at more primes, as explained earlier. Whenever we reach a contradiction, we go to the next available branch. To clarify the previous exposition, we do the case when k = 4, B = 50. The following is the entire output, which is explained after the printout.

$3^2 => 13^1$
$13^1 => 2^1 7^1 N : 9 < p_3 < 11$
$13^\infty : 3 < p_3 < 9$
$5^1 => 2^1 3^1 : 15 < p_4 < 17F$
$5^2 => 31^1 A$
$5^\infty : 42 < p_4 < 46, SF1 : 42 < p_4 < 46$

$7^2 => 3^1 19^1 D$

$7^\infty : 10 < p_4 < 12$

$11^\infty A$

$3^\infty : 3 < p_2 < 11$

$5^1 => 2^1 3^1 : 8 < p_3 < 13 F$

$5^2 => 31^1, SF1 : 22 < p_4 < 26$

$5^\infty : 14 < p_3 < 17 N : 14 < p_3 < 17$

$7^2 => 3^1 19^1 N : 11 < p_3 < 13$

$7^\infty : 7 < p_3 < 16$

$11^\infty : 22 < p_4 < 27, SF1 : 22 < p_4 < 27$

$13^1 => 2^1 7^1 : 15 < p_4 < 18$

$17^\infty F$

$13^\infty : 16 < p_4 < 20$

$17^1 => 2^1 3^2 F$

$17^\infty A$

19^∞No contradiction. The number B is too small

We start with the case $3^2||N$. Since $13 = \sigma(3^2)|N$, we go to the sub-branch (represented by the indentation on the second line) with $13||N$. On this branch we could further branch off on the new prime 7 coming from $\sigma(13)$, but the letter N means that there are no primes in the interval given by the bounds of Machinery, which is a contradiction. So, we backtrack to the next possible branch, which is $3^2 13^\infty$. On this branch we have no off primes, and so we again use the interval bounds, and find $3 < p_3 < 9$, hence $p_3 = 5$ or 7. The next four cases are all contradictory, as represented by the different letters near the ends of the lines. All of the different contradictions will be explained in Implementation part. The rest of the output is self-explanatory except the very last branch $3^\infty 7^\infty 13^\infty 19^\infty$, which doesn't yield a contradiction. Thus the algorithm terminates unsuccessfully because the bound B was chosen too small. One must increase B and rerun the program to successfully complete this case [4].

3.7 An implementation

First, we do not allow the bound B to increase within the algorithm. Allowing the computer to vary B fully automates the algorithm at the expense of unnecessary complexity. We fix the number B at the outset, and only increase it manually if needed..

Second, the use of Lemma 3.25 allows for stronger upper bounds on intervals for primes. In the terminology of that lemma, our implementation always has $v \in [0, 2]$, the exact number depending on if we find large prime divisors for N from proved results.

Third, some of the contradictions are different. Here is a complete list of the contradictions in our implementation:

- MT There are too many total factors.
- MS There are too many copies of a single prime with known component
- S There is an off prime smaller than an on prime coming from interval computations.
- A The number is abundant.
- D There are k known primes, and $\Delta_u < 2$, hence N is deficient.
- F The special prime π belongs to a known component, but the hypotheses of Proposition [4, Proposition 7] hold showing π must be in an unknown component due to a Fermat prime
- N There are no primes in the interval given by Lemmas 3.24 and 3.25, or there are primes in the interval but they are already known, on primes.
- SF1 There are $k-1$ known primes, and the interval formula gives an upper bound of $p_k < C$, but we know from the fact that a large power of a small Fermat prime divides N that some unknown prime is larger than C by [4, Proposition 7].

- SF2 Similar to SF1 except we have a contradiction between the interval formula and Proposition 3.19.
- SNF1 Similar to SF1, except we have a contradiction from a small non-Fermat prime, using Proposition 3.20.
- SNF2 Similar, using Proposition 3.26

The first seven contradictions are all standard, while the last four are new. There were other contradictions we might have included, but they either rely heavily on the extensive computations of others or do not present a significant increase in the speed of the algorithm [4].

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