The Riemann-Stieltjes integral: functions, geometry and some Applications

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Declaration by student

I hereby declare that the data presented in this Dissertation report entitled, "The Riemann-Stieltjes Integral: functions, geometric interpretation and some applications" is based on the results of investigations carried out by me in the Department of Mathematics at the School of Physical and Applied Sciences under the Supervision of Dr. M.Kunhanandan and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University or its authorities will be not be responsible for the correctness of observations / experimental or other findings given the dissertation.
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Preface

This report has been presented in order to satisfy the requirements for the MAT-651 Discipline specific Dissertation subject of the program Master of Science in Mathematics for the academic year 2023-2024.

The Riemann-Stieltjes integral: Functions, Geometric interpretation and some Applications" is the subject of this report. There are five chapters in it. After gaining an understanding of the Riemann-Stieltjes integral, we examine its geometric interpretation, some applications,and generalization to the Lebesgue-Stieltjes integral.

Chapter 1:
This chapter serves as an introduction, outlining the significance of the Riemann-Stieltjes integral as well as some preliminaries.

Chapter 2:
The definition and a discussion of some of the features of the Riemann-Stieltjes integral are covered in chapter 2. The impact of various integrator functions on the integral is examined. The Mean value theorems and the Fundamental theorem of Calculus for the Riemann-Stieltjes integral are studied.

Chapter 3:
The software "GeoGebra" is used to interpret the Riemann-Stieltjes integral geometrically.

Chapter 4:
Some applications of the Riemann-Stieltjes integral in Probability theory and Number theory are discussed.

Chapter 5:
The generalization of Riemann-Stieltjes integral to Lebesgue-Stiltjes integral, by integrating over measurable sets instead of compact intervals is studied in this chapter.

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Abstract

This report is a study of the existence, characterization and some of the properties of the Riemann-Stieltjes integral. The mean value theorems and the Fundamental theorem of Calculus for the Riemann-Stieltjes integral are studied. Next, we interprete the integral geometrically and consider the impact of the integrator functions on the integral. Further, we give some applications in number theory. We also discuss some applications in Probability theory, where the Riemann-Stieltjes integral yields a general formula for the expectation, independent of it's underlying distribution, to demonstrate the integral's versatility. The integral is then further generalized to Lebesgue-Stieltjes integral.

Keywords: Riemann-Stieltjes integral, integrator functions, geometric interpretation, Mean value theorems, Fundamental theorem of calculus for Riemann-Stieltjes integral.

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Chapter 1

Notations and abbreviations

$f \in \mathcal{C}[a,b]$	f is continuous on $[a,b]$
$m^*(E)$	Outer measure of E
$m(E)$	Measure of E
$S(P,f,\alpha)$	Riemann-Stieltjes sum of f with respect to α
$f \in R_\alpha[a,b]$	f is Riemann-Stieltjes integrable on $[a,b]$ w.r.t. α
$\int_a^b f d\alpha$	Riemann-Stieltjes integral of f w.r.t. α from a to b
$f \in R[a,b]$	f is Riemann integrable on $[a,b]$
$\overline{I}(f,\alpha)$	Upper integral of f w.r.t. α

Chapter 2

Introduction

The Riemann–Stieltjes integral is a generalization of the Riemann integral, named after Bernhard Riemann and Thomas Joannes Stieltjes. The definition of this integral was first published in 1894 by Stieltjes.

This generalization is useful in applications and also served as a foundation for the corresponding extension of the Lebesgue integration theory. Stieljes was a Dutch-born French mathematician who had an interesting career path to mathematics. He died in 1894 at the age of 38, too early to see his work published and his integral gain prominence.

The main distinction of the Stietjes integral $\int_a^b f d\alpha$ from the Riemann integral is that it depends on two functions: One, the integrand f and then a function α that replaces the identity map (that is, $\alpha(x) = x$) in the increment of the underlying variable. The Riemann-Stieltjes integral is called so not because Riemann had anything to do with it but rather that the integral uses the framework of the Riemann integral. A version called Lebesgue-Stieltjes integral exists in Lebesgue theory of integration. The Riemann integral is thus a special case of the Riemann-Stieltjes integral.

Stieltjes integral allows us to generalize the notion of area to the situation when the "length" of interval $[t_1, t_2]$ is given by $\alpha(t_2) - \alpha(t_1)$ and area of the rectangle $[t_1, t_2] \times [0, h]$ (where h is the height of the rectangle), is thus $h[\alpha(t_2) - \alpha(t_1)]$. Notably, this includes negative "lengths" but this is no problem because the Riemann integral anyway computes the signed area. The Stieltjes integral is quite useful in probability. There α is usually the cumulative distribution function of a random variable X.

The use of the Stieltjes integral permits treating all the various kinds of distributions of X — namely, discrete, continuous and mixtures thereof — under the same umbrella.

We note that Rudin's book presents a different definition of the Stieltjes integral which is based on Darboux's approach to Riemann integration. This streamlines the analysis somewhat but forces us to work with α monotone or, at best, of bounded variation. The applications mentioned earlier are not always of this kind and so we prefer to work with the Stieltjes integral in the Riemann sense. However, the two definitions are equivalent whenever the integrator function is monotonically increasing and the same has been proved in the following chapter.

2.1 Preliminaries

Definition 1 (Upper bound of a set)**.** Let X be a subset of $\mathbb{R}$. u is said to be an upper bound for the set X if $u \geqslant x$ for all $x \in X$.

Definition 2 (Lower bound of a set)**.** Let X be a subset of $\mathbb{R}$. v is said to be an lower bound for the set X if $v \leqslant x$ for all $x \in X$.

Definition 3 (Supremum of a set)**.** Let X be a subset of $\mathbb{R}$ that is bounded above. A real number M is called the sup of X if the following conditions hold:

i M is an upper bound for X.

ii $s \in X, \;\; s < M \implies s$ is not an upper bound for X.

Definition 4 (Infimum of a set)**.** Let X be a subset of $\mathbb{R}$ that is bounded below. A real number m is called the inf of X if the following conditions hold:

i m is a lower bound for X.

ii $s \in X, \;\; s > m \implies s$ is not a lower bound for X.

Definition 5 (A characterization of Supremum)**.** Let X be a non-empty subset of $\mathbb{R}$ that is bounded below. Then $M = \sup X$ if and only if the following two conditions hold:

i M is an upper bound for X.

ii Given $\epsilon > 0$, there exists $a \in X$ such that $M - \epsilon < a$.

Definition 6 (A characterization of infimum)**.** Let X be a non-empty subset of $\mathbb{R}$ that is bounded above. Then $m = \inf X$ if and only if the following two conditions hold:

i m is a lower bound for X.

ii Given $\epsilon > 0$, there exists $b \in X$ such that $b < m + \epsilon$.

Definition 7 (Covering of a set). Let E be a subset of $\mathbb{R}$. A collection of subsets $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ is called a covering of the set E if $E \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$.

Definition 8 (Continuity at a point in a closed and bounded interval). Let $[a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be defined. f is said to be continuous at a point $c \in (a, b)$ if for $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(c)| < \epsilon$

f is continuous at a if for $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in (a, a + \delta)$ implies $|f(x) - f(a)| < \epsilon$. f is continuous at b if for $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in (b - \delta, b)$ implies $|f(x) - f(b)|$.

Definition 9 (Uniformly continuous function). A function $f : [a, b] \to \mathbb{R}$ is said to be uniformly continuous on $[a, b]$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x,\ y \in [a, b]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Definition 10 (Differentiable function). Let $f\, [a, b] \to \mathbb{R}$ be a function. Suppose $c \in (a, b)$. f is differentiable at $x = c$ if $\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$ exists.

Theorem 1 (Lagrange's mean value theorem on $\mathbb{R}$). *Let $[a, b]$ be a closed bounded interval and $f : [a, b] \to \mathbb{R}$ be a function such that*

i f is continuous on $[a, b]$.

ii f is differentiable on the open interval (a, b),

then there exists $c \in (a, b)$ such that $f(b) - f(a) = (b - a) f'(c)$

Chapter 3

Riemann-Stieltjes Integral

3.1 Riemann-Stieltjes integral

Definition 11 (Bounded function)**.** Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to be bounded on X if there exist $m, M \in \mathbb{R}$ such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Equivalently, there exists $k \geqslant 0$ such that $|f(x)| \leqslant k$ for all $x \in [a, b]$.

We consider $f, \alpha : [a, b] \to \mathbb{R}$, that is, unless otherwise stated, all functions f, g, α, β etc. will be assumed to be real valued functions defined and bounded on $[a, b]$.

Definition 12 (Monotonically increasing function)**.** A function $f : [a, b] \to \mathbb{R}$ is said to be monotonically increasing on $[a, b]$ if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in [a, b]$

Definition 13 (Partition of an interval)**.** By a partition of the interval $[a, b]$, we mean a set of points $P = \{a = x_0, x_1, \cdots, x_n = b\}$ where $x_0 < x_1 < \cdots < x_n$.

Definition 14 (Refinement of a partition)**.** A partition Q of $[a, b]$ is said to be a refinement of the partition P on $[a, b]$ if $P \subseteq Q$.

Definition 15 (Riemann-Stieltjes sum)**.** Let $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a, b]$ and let t_k be a point in the sub-interval $[x_{k-1}, x_k]$. A sum of the form $S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta\alpha_k$ is called the Riemann-Stiletjes sum of f with respect to α.

Definition 16 (Riemann-Stieltjes Integral)**.** We say f is Riemann-Stieltjes integrable with respect to α on $[a, b]$ if there exists a real number A having the following property:

Given $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that for every choice of the points t_k in $[x_{k-1}, x_k]$ and for any partition P finer that P_ϵ we have

$$|S(P, f, \alpha) - A| < \epsilon,$$

where $S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta\alpha_k$.

When such a number A exists, it is uniquely determined and is denoted by $\int_a^b f d\alpha$ or by $\int_a^b f(x)\, d\alpha(x)$. We also say that the Riemann-Stieltjes integral exists. The functions f and α are referred to as the integrand and the integrator respectively. In the special case when $\alpha(x) = x$, we write $S(P, f)$ instead of $S(P, f, \alpha)$ and $f \in R$ instead of $f \in R_\alpha$. The integral is then called a Riemann integral and is denoted by $\int_a^b f dx$ or by $\int_a^b f(x)\, dx$. The numerical value of $\int_a^b f(x)\, d\alpha(x)$ depends only on f, α, a and b, and does not depend on the symbol x. The letter x is a "dummy variable" and may be replaced by any other convinient symbol.

3.2 Linearity

It is very easy to prove that the Riemann-Stieltjes integral is linear with respect to the integrand and the integrator.

3.2.1 Linearity w.r.t. integrand

Theorem 2. *Let $f, g \in R_\alpha[a,b]$. Then $c_1 f + c_2 g \in R_\alpha[a,b]$ for all $c_1, c_2 \in \mathbb{R}$ and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f d\alpha + c_2 \int_a^b g d\alpha$$

Proof. Let $h = c_1 f + c_2 g$

Given a partition P of $[a,b]$, we can write

$$\begin{aligned} S(P,h,\alpha) &= \sum_{k=1}^{n} h(t_k)\Delta\alpha_k \\ &= \sum_{k=1}^{n} \left[(c_1 f + c_2 g)(t_k)\right]\Delta\alpha_k \\ &= \sum_{k=1}^{n} \left(c_1 f(t_k) + c_2 g(t_k)\right)\Delta\alpha_k \\ &= c_1 \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\Delta\alpha_k \\ &= c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha) \end{aligned}$$

Let $\epsilon > 0$

$f \in R_\alpha[a,b] \implies$ there exists a partition P_ϵ of $[a,b]$ such that for any partition $P \supseteq P_\epsilon$,

$$\left| S(P,f,\alpha) - \int_a^b f d\alpha \right| < \epsilon$$

$g \in R_\alpha[a,b] \implies \exists$ a partition P'_ϵ of $[a,b]$ such that for any partition $P \supseteq P'_\epsilon$,

$$\left|S(P,g,\alpha) - \int_a^b g d\alpha\right| < \epsilon$$

Let $P''_\epsilon = P_\epsilon \cup P'_\epsilon$

$\implies$ For any partition $P \supseteq P''_\epsilon$, we have

$$\left|S(P,f,\alpha) - \int_a^b f d\alpha\right| < \epsilon \text{ and } \left|S(P,g,\alpha) - \int_a^b g d\alpha\right| < \epsilon$$

$$\begin{aligned}\left|S(P,h,\alpha) - c_1\int_a^b f d\alpha - c_2\int_a^b g d\alpha\right| &= \Big|c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha) \\ &\quad - c_1\int_a^b f d\alpha - c_2\int_a^b g d\alpha\Big| \\ &\leqslant |c_1|\left|S(P,f,\alpha) - \int_a^b f d\alpha\right| \\ &\quad + |c_2|\left|S(P,g,\alpha) - \int_a^b g d\alpha\right| \\ &< |c_1|\epsilon + |c_2|\epsilon \\ &= (|c_1| + |c_2|)\epsilon\end{aligned}$$

$$\therefore (c_1 f + c_2 g) \in R_\alpha[a,b].$$

□

3.2.2 Linearity w.r.t. integrator

Theorem 3. *If* $f \in R_\alpha [a,b]$ *and* $f \in R_\beta [a,b]$, *then* $f \in R_{(c_1\alpha + c_2\beta)}$ *on* $[a,b]$ *for all* $c_1, c_2 \in \mathbb{R}$ *and we have*

$$\int_a^b f d\left(c_1\alpha + c_2\beta\right) = c_1 \int_a^b f d\alpha + c_2 \int_a^b f d\beta$$

Proof. Let $\epsilon > 0$

$f \in R_\alpha [a,b] \implies$ there exists a partition P_1 of $[a,b]$ such that for a partition $P \supseteq P_1$,

$$\left| S\left(P, f, \alpha\right) - \int_a^b f d\alpha \right| < \epsilon$$

$f \in R_\beta [a,b] \implies$ there exists a partition P_2 of $[a,b]$ such that for a partition $P \supseteq P_2$,

$$\left| S\left(P, f, \beta\right) - \int_a^b f d\alpha \right| < \epsilon$$

Let $P_\epsilon = P_1 \cup P_2$

Let $\eta = c_1\alpha + c_2\beta$

$$\begin{aligned} S\left(P, f, \eta\right) &= \sum_{k=1}^{n} f\left(t_k\right) \Delta\eta_k \\ &= \sum_{k=1}^{n} f\left(t_k\right) \left(\eta\left(x_k\right) - \eta\left(x_{k-1}\right)\right) \\ &= \sum_{k=1}^{n} f\left(t_k\right) \left(\left(c_1\alpha + c_2\beta\right)\left(x_k\right) - \left(c_1\alpha + c_2\beta\right)\left(x_{k-1}\right)\right) \\ &= \sum_{k=1}^{n} f\left(t_k\right) \left(c_1\Delta\alpha_k + c_2\Delta\beta_k\right) \\ &= c_1 \sum_{k=1}^{n} f\left(t_k\right) \Delta\alpha_k + c_2 \sum_{k=1}^{n} f\left(t_k\right) \Delta\beta_k \end{aligned}$$

$$
\begin{aligned}
\left|S(P,f,\eta) - c_1\int_a^b f d\alpha - c_2\int_a^b f d\beta\right| &= \left|c_1 S(P,f,\alpha) + c_2 S(P,f,\beta)\right. \\
&\quad \left. - c_1\int_a^b f d\alpha - c_2\int_a^b f d\beta\right| \\
&= \left|c_1\left(S(P,f,\alpha) - \int_a^b f d\alpha\right)\right. \\
&\quad \left. + c_2\left(S(P,f,\beta) - \int_a^b f d\beta\right)\right| \\
&= |c_1|\left|S(P,f,\alpha) - \int_a^b f d\alpha\right| \\
&\quad + |c_2|\left|S(P,f,\beta) - \int_a^b f d\beta\right| \\
&< |c_1|\,\epsilon + |c_2|\,\epsilon \\
&= (|c_1| + |c_2|)\,\epsilon
\end{aligned}
$$

□

3.3 Additivity

Our next result is somewhat analogous to the previous two theorems and it tells us that the integral is additive with respect to the interval of integration.

Theorem 4. *Assume that $c \in (a,b)$. If two of the three integrals in the equation given below exist, then the third also exists and we have*

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

Proof. If P is a partition of $[a,b]$ such that $c \in P$, let $P' = P \cap [a,c]$ and $P'' = P \cap [c,b]$ denote the corresponding partitions of $[a,c]$ and $[c,b]$ respectively. Without loss of generality, let us assume that $c = x_m$ for some $1 \leqslant m < n$

$$\begin{aligned} S(P, f, \alpha) &= \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k \\ &= \sum_{k=1}^{m} f(t_k)\,\Delta\alpha_k + \sum_{k=m+1}^{n} f(t_k)\,\Delta\alpha_k \\ &= S(P', f, \alpha) + S(P'', f, \alpha) \end{aligned}$$

Assume that $\int\limits_a^c f d\alpha$ and $\int\limits_c^b f d\alpha$ exist.

Then, given $\epsilon > 0$, there exists a partition P'_ϵ of $[a, c]$ such that

$$\left| S(P', f, \alpha) - \int\limits_a^c f d\alpha \right| < \frac{\epsilon}{2}$$

whenever the partititon P' is finer than P'_ϵ and there exists a partition P''_ϵ of $[c, b]$ such that

$$\left| S(P'', f, \alpha) - \int\limits_c^b f d\alpha \right| < \frac{\epsilon}{2}$$

whenever the partition P'' is finer than P''_ϵ.

Then $P_\epsilon = P'_\epsilon \cup P''_\epsilon$ is a partition of $[a, b]$. Now, if partition P is finer than P_ϵ, then $P' \supseteq P'_\epsilon$ and $P'' \supseteq P''_\epsilon$

$$\left|S(P,f,\alpha)-\left(\int_a^c fd\alpha+\int_c^b fd\alpha\right)\right| = \left|S(P',f,\alpha)+S(P'',f,\alpha)-\int_a^c fd\alpha-\int_c^b fd\alpha\right|$$

$$\leqslant \left|S(P',f,\alpha)-\int_a^c fd\alpha\right|+\left|S(P'',f,\alpha)-\int_c^b fd\alpha\right|$$

$$< \epsilon+\epsilon$$

$$= 2\epsilon$$

$\therefore \int_a^b fd\alpha$ exists and is equal to $\int_a^c fd\alpha + \int_c^b fd\alpha$ □

Definition 17. If $a < b$, we define $\int_b^a fd\alpha = -\int_a^b fd\alpha$ whenever $\int_a^b fd\alpha$ exists. We also define $\int_a^a fd\alpha = 0$

3.4 Integration by parts

A notable connection exists between the integrand and the integrator in a Riemann-Stieltjes integral. The existence of $\int_a^b fd\alpha$ implies the existence of $\int_a^b \alpha df$and the converse is also true. Morever, we have a simple relation between the two integrals.

Theorem 5. *If* $f \in R_\alpha[a,b]$, *then* $\alpha \in R_f\,[a,b]$ *and we have*

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

Proof. Let $\epsilon > 0$ be given.

Since $f \in R_\alpha[a,b]$, there exists a partition P_ϵ of $[a,b]$ such that for a partition $P' \supseteq P_\epsilon$, we have

$$\left|S(P',f,\alpha)-\int_a^b fd\alpha\right| < \epsilon$$

$$\begin{aligned} S(P, f, \alpha) &= \sum_{k=1}^{n} \alpha(t_k)\, \Delta f_k \\ &= \sum_{k=1}^{n} \alpha(t_k)\left[f(x_k) - f(x_{k-1})\right] \\ &= \sum_{k=1}^{n} \alpha(t_k)\, f(x_k) - \sum_{k=1}^{n} \alpha(t_k)\, f(x_{k-1}) \end{aligned}$$

Let $A = f(b)\,\alpha(b) - f(a)\,\alpha(a)$

$$\begin{aligned} \sum_{k=1}^{n} \left[f(x_k)\,\alpha(x_k) - f(x_{k-1})\,\alpha(x_{k-1})\right] &= f(x_1)\,\alpha(x_1) + \cdots + f(x_n)\,\alpha(x_n) \\ &\quad - \left[f(x_0)\,\alpha(x_0) + \cdots f(x_{n-1})\,\alpha(x_{n-1})\right] \\ &= f(x_n)\,\alpha(x_n) - f(x_0)\,\alpha(x_0) \\ &= f(b)\,\alpha(b) - f(a)\,\alpha(a) \end{aligned}$$

$$\begin{aligned} A - S(P, f, \alpha) &= \sum_{k=1}^{n} f(x_k)\,\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\,\alpha(x_{k-1}) \\ &\quad - \sum_{k=1}^{n} \alpha(t_k)\, f(x_k) + \sum_{k=1}^{n} \alpha(t_k)\, f(x_{k-1}) \\ &= \sum_{k=1}^{n} f(x_k)\left[\alpha(x_k) - \alpha(t_k)\right] + \sum_{k=1}^{n} f(x_{k-1})\left[\alpha(t_k) - \alpha(x_{k-1})\right] \end{aligned}$$

The two sums on the right hand side can be combined into a single sum of the form $S(P', f, \alpha)$ where P' is the partition of $[a, b]$ obtained by taking the points x_k and t_k together.

$\implies P'$ is finer than P.

$$\therefore \left| A - S(P, \alpha, f) - \int_a^b f d\alpha \right| < \epsilon, \text{ where } P \supseteq P_\epsilon$$

$$\left| \left(A - \int_a^b f d\alpha \right) - S(P, \alpha, f) \right| < \epsilon$$

$$\Longrightarrow \int_a^b \alpha df \text{ exists, and } \int_a^b \alpha df = A - \int_a^b f d\alpha$$

□

3.5 Change of variable in a Riemann-Stieltjes integral

Theorem 6. *Let $f \in R_\alpha[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c)$, $b = g(d)$. Let h and β be the composite functions defined as follows:*
$h(x) = f[g(x)]$, $\beta(x) = \alpha[g(x)]$ if $x \in S$.
Then h is Riemann-Stieltjes integrable with respect to β on S and we have

$$\int_a^b f d\alpha = \int_c^d h d\beta$$

That is,

$$\int_{g(c)}^{g(d)} f(t)\, d\alpha(t) = \int_c^d f[g(x)] d\alpha[g(x)]$$

Proof. Suppose g is strictly increasing on S.

$\implies c < d$

$\implies g$ is one-one and has a strictly increasing, continuous inverse g^{-1} defined on $[a, b]$.

Therefore for every partition $P = \{y_0, y_1 \cdots y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, x_1 \cdots, x_n\}$ of $[a, b]$ with $x_k = g(y_k)$.

In fact, we can write

$P' = g(P)$ and $P = g^{-1}(P')$. Futhermore, a refinement of P produces a corresponding refinement of P' and the converse holds.

Let $\epsilon > 0$ be arbitrary.

$f \in R_\alpha [a, b] \implies \exists$ a partition P_ϵ of $[a, b]$ such that for $P' \supseteq P_\epsilon$, we have

$$\left| S(P', f, \alpha) - \int_a^b f d\alpha \right| < \epsilon$$

P'_ϵ is a partition of $[a, b]$.

Let $P_\epsilon = g^{-1}(P'_\epsilon)$ be the corresponding partition of $[c, d]$.

Let $P = \{y_0, y_1, \cdots, y_n\}$ be a partition of $[c, d]$ finer than P_ϵ.

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k) \Delta\beta_k$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$, Then $P' = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$ finer than P'_ϵ

Morever, we have

$$\begin{aligned} S(P,h,\beta) &= \sum_{k=1}^{n} f[g(u_k)]\{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\} \\ &= \sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} \\ &= S(P', f, \alpha) \qquad\qquad (\because t_k \in [x_{k-1}, x_k]) \end{aligned}$$

□

Note:

For $\alpha(x) = x$, we have

$$\int\limits_{g(c)}^{g(d)} f(t)\,dt = \int\limits_{c}^{d} f[g(x)]\,dg(x)$$

3.6 Reduction to Riemann Integral

The next theorem tells us that we can replace the term $d\alpha$ by $\alpha'(x)\,dx$ in $\int_a^b f(x)\,d\alpha(x)$ whenever α has a continuous derivative α'.

Theorem 7. *Assume $f \in R_\alpha[a,b]$ and that α has a continuous derivative α' on $[a,b]$. Then the Riemann integral $\int_a^b f(x)\,d\alpha(x)$ exists and we have*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\,\alpha'(x)\,dx$$

Proof. Let $g(x) = f(x)\,\alpha'(x)$ and $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Then the Riemann sum of g with respect to partition P is

$$\begin{aligned} S(P, g) &= \sum_{k=1}^{n} g(t_k) \Delta x_k \\ &= \sum_{k=1}^{n} f(t_k) \alpha'(t_k) \Delta x_k \end{aligned}$$

The same partition P and the same choice of the t_k can be used to form the Riemann-Stieltjes sum

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta\alpha_k$$

Now, using the mean value theorem on the function α, we have there exists $v_k \in (x_{k-1}, x_k)$ such that $\alpha(x_k) - \alpha(x_{k-1}) = (x_k - x_{k-1}) \alpha'(v_k)$, that is, $\Delta\alpha_k = \alpha'(v_k) \Delta x_k$

$$\begin{aligned} S(P, f, \alpha) - S(P, g) &= \sum_{k=1}^{n} f(t_k) \alpha'(v_k) \Delta x_k - \sum_{k=1}^{n} f(t_k) \alpha'(t_k) \Delta x_k \\ &= \sum_{k=1}^{n} f(t_k) [\alpha'(v_k) - \alpha'(t_k)] \Delta x_k \end{aligned}$$

Now, f is bounded $\implies$ there exists $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a, b]$

α' is continuous on $[a, b]$ $\implies$ α' is uniformly continuous on $[a, b]$

Therefore for $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 \leqslant |x - y| < \delta \implies |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}$$

Consider a partition P'_ϵ with $||P'_\epsilon|| < \delta$.

Then for any partition $P \supseteq P'_\epsilon$ we have

$$|\alpha'(v_k) - \alpha'(t_k)| < \frac{\epsilon}{2M(b-a)} \text{for all } k \in \{1, \cdots, n\}$$

$$\text{Therefore } |S(P, f, \alpha) - S(P, g)| < \sum_{k=1}^{n} M \cdot \frac{\epsilon}{2M(b-a)} \cdot (b-a) = \frac{\epsilon}{2} \tag{1}$$

Also, since $f \in R_\alpha[a, b]$, there exists a partition P''_ϵ such that

$$P \supseteq P''_\epsilon \implies \left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \frac{\epsilon}{2} \tag{2}$$

Let $P_\epsilon = P'_\epsilon \cup P''_\epsilon$

For $P \supseteq P_\epsilon$, we have

$$\begin{aligned}
\left| S(P, g) - \int_a^b f d\alpha \right| &= \left| S(P, g) - S(P, f, \alpha) + S(P, f, \alpha) - \int_a^b f d\alpha \right| \\
&\leqslant |S(P, g) - S(P, f, \alpha)| + \left| S(P, f, \alpha) - \int_a^b f d\alpha \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \cdots \text{(from (1) and (2))} \\
&= \epsilon
\end{aligned}$$

$$\text{Therefore} \quad \left| S(P, g) - \int_a^b f d\alpha \right| < \epsilon$$

□

3.7 Step functions as integrator

Let us first note that, if $\alpha : [a, b] \longrightarrow \mathbb{R}$ is given by $\alpha(x) = c$ for all $x \in [a, b]$ for some $c \in \mathbb{R}$, that is, α is a constant function. Then ,

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta\alpha_k = 0$$

This is true for any partition P of $[a, b]$

$\Longrightarrow \int_a^b f d\alpha$ exists and is equal to 0.

Next, suppose α is constant except for a jump discontinuity at some point $c \in (a, b)$, then the integral $\int_a^b f d\alpha$ may or may not exist and, if it exists, it's value need not be 0. We have the following theorem :

Theorem 8. *Given $a < c < b$, define α on $[a, b]$ as follows:*

The values $\alpha(a), \alpha(b), \alpha(c)$ are arbitrary.

$$\alpha(x) = \begin{cases} \alpha(a) & ; a \leqslant x < c \\ \alpha(b) & ; c < x \leqslant b \end{cases}$$

Let f be defined on $[a, b]$ in such a way that atleast one of the functions f or α is continuous from the left at c and atleast one is continuous from the right at c. Then $f \in R_\alpha[a, b]$ and we have

$$\int_a^b f d\alpha = f(c)[\alpha(c+) - \alpha(c-)]$$

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.

Without loss of generality, we can assume that c is a point in the partition, say $c = x_m$, for some $x \in \{1, 2, \cdots, m-1\}$, then every term in $S(P, f, \alpha)$ is 0 except the two terms arising from the subintervals $[x_{m-1}, c]$ and $[c, x_{m-1}]$.

$$\begin{aligned} S(P, f, \alpha) &= \sum_{k=1}^{n} f(t_k) \Delta\alpha_k \\ &= \sum_{k=1}^{n} f(t_k) \{\alpha(x_k) - \alpha(x_{k-1})\} \\ &= f(t_m) \{\alpha(c) - \alpha(c-)\} + f(t_{m+1}) \{\alpha(c+) - \alpha(c)\} \end{aligned}$$

where $t_m \leqslant c \leqslant t_{m+1}$.

This equation can also be written as follows:

$$\Delta = [f(t_m) - f(c)] [\alpha(c-) - \alpha(c)] + [f(t_{m+1}) - f(c)] [\alpha(c+) - \alpha(c)]$$

where $\Delta = S(P, f, \alpha) - f(c) [\alpha(c+) - \alpha(c-)]$.

Hence we have

$$|\Delta| \leqslant |f(t_m) - f(c)| \, |\alpha(c-) - \alpha(c)| + [f(t_{m+1}) - f(c)| \, |\alpha(c+) - \alpha(c)|$$

If f is continuous at c, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $||P|| < \delta$ implies $|f(t_m) - f(c)| < \epsilon$ and $|f(t_{m+1}) - f(c)| < \epsilon$.

In this case, we obtain the inequality

$$|\Delta| \leqslant \epsilon \, |\alpha(c-) - \alpha(c)| + \epsilon \, |\alpha(c+) - \alpha(c)|$$

But this inequality holds whether or not f is continuous at c.

For example, if f is discontinuous from the right and from the left at c, then

$\alpha(c) = \alpha(c-)$ and $\alpha(c) = \alpha(c+)$ and we get $\Delta = 0$.

One the other hand, if f is continuous from the left and discontinuous from the right at c, we must have $\alpha(c) = \alpha(c+)$ and we get $\Delta \leqslant \epsilon\,|\alpha(c) - \alpha(c-)|$.

Similarly, if f is continuous from the right and discontinuous from the left at c, we have $\alpha(c) = \alpha(c-)$ and $|\Delta| \leqslant \epsilon\,|\alpha(c+) - \alpha(c)|$.

Hence the last displayed inequality holds in either case.

This proves the theorem.

$\square$

Note:

The result also holds if $c = a$, provided that we write $\alpha(c)$ for $\alpha(c-)$ and it holds for $c = b$ if we write $\alpha(c)$ for $\alpha(c+)$.

Theorem 8 tells us that the value of a Riemann-Stieltjes integral can be altered by changing the value of f at a single point. The following example shows that the existence of the integral can also be affected by such a change.

Consider the interval $[-1, 1]$.

$$\text{Let} \quad \alpha(x) = \begin{cases} 0 & ; x \neq 0 \\ -1 & ; x = 0 \end{cases}$$

, $f(x) = 1$ for all $x \in [-1, 1]$.

In this case, Theorem 8 $\Longrightarrow \int_{-1}^{1} f d\alpha = 0$.

But if we redefine f as follows:

$$f(x) = \begin{cases} 1 & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$$

we can easily see that $\int_{-1}^{1} f d\alpha$ does not exist. In fact, when P is a partition of $[-1,1]$ which includes 0 as a point of subdivision, we have

$$\begin{aligned} S(P,f,\alpha) &= f(t_k)[\alpha(x_k) - \alpha(0)] + f(t_{k-1})[\alpha(0) - \alpha(x_{k-2})] \\ &= f(t_k) - f(t_{k-1}) \end{aligned}$$

where $x_{k-2} \leqslant t_{k-1} \leqslant 0 \leqslant t_k \leqslant x_k$.

Now, $S(P,f,\alpha) = 0$,1, or -1 depending on the choice of t_k and t_{k-1}

$\therefore \int_{-1}^{1} f d\alpha$ does not exist in this case.

Definition 18 (Step function)**.** A function α defined on $[a,b]$ is called a step function if there exists a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that α is constant on each subinterval (x_{k-1}, x_k). The number $\alpha(x_k+) - \alpha(x_k-)$ is called the jump discontinuity at x_k if $1 < k < n$. The jump at x_1 is $\alpha(x_1+) - (x_1)$ and the jump at x_n is $\alpha(x_n) - (x_n-)$.

When we integrate with respect to the step function, we get the following relation between Riemann-Stieltjes integrals and finite sums :

Theorem 9. *Let α be a step function defined on $[a,b]$ with jump α_k at x_k, where $x_1, \cdots, x_n$ are described as in the above definition. Let f be defined on $[a,b]$ in such a way that not both f and α are discontinuous from the right or from the left* *at each x_k. Then $\int_a^b f d\alpha$ exists and we have*

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k$$

Proof. We divide the interval $[a, b]$ into $n+1$ subintervals as follows:

$$\begin{aligned}[a,b] &= \left[a, \frac{x_0+x_1}{2}\right] \cup \left[\frac{x_0+x_1}{2}, \frac{x_1+x_2}{2}\right] \cup \cdots \\ &\cdots \cup \left[\frac{x_{n-2}+x_{n-1}}{2}, \frac{x_{n-1}+x_n}{2}\right] \cup \left[\frac{x_{n-1}+x_n}{2}, x_n\right] \\ &= [a, p_1] \cup [p_1, p_2] \cup \cdots \cup [p_{n-1}, p_n] \cup [p_n, x_n]\end{aligned}$$

where $p_k = \dfrac{p_{k-1}+p_k}{2}$ for all $k = 1, 2, ..., n$

By the theorem on the additivity of Riemann-Stieltjes integrals,

$$\int_a^b f d\alpha = \int_a^{p_1} f d\alpha + \int_{p_1}^{p_2} f d\alpha + \cdots + \int_{p_{n-1}}^{p_n} f d\alpha + \int_{p_n}^{x_n} f d\alpha$$

$\int_a^{p_1} f(x)\, d\alpha(x) = 0$ since α is constant on $[a, p_1]$.

$$\begin{aligned}\text{Therefore} \int_a^b f d\alpha &= f(x_1)[\alpha(x_1+) - \alpha(x_1-)] + f(x_2)[\alpha(x_2+) - \alpha(x_2)] + \cdots \\ &\cdots + f(x_{n-1}+)[\alpha(x_{n-1}+) - \alpha(x_{1-})] \\ &= \sum_{k=1}^{n} f(x_k)\, \alpha_k\end{aligned}$$

□

The greatest integer function is one of the simplest step functions. Its value at x is the greatest integer which is less than orequal to x and is denoted by $[x]$.

Thus, $[x]$ is the unique integer satisfying the inequalities $[x] \leqslant x < [x] + 1$.

Theorem 10. *Every finite sum can be written as Riemann-Stieltjes integral. In fact, given a sum* $\sum\limits_{k=1}^{n} a_k$*, define* f *on* $[0, n]$ *as follows:*

$$f(x) = \begin{cases} a_k & ; k-1 < x \leqslant k, \quad k = 1, \cdots, n \\ o & ; x = 0 \end{cases}$$

Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)\, d[x]$$

where $[x]$ *is the greatest integer less than or equal to* x.

Proof. The greatest integer function is a step function, continuous from the right and having jump 1 at each integer. The function f is continuous from the left at $1, 2, \cdots, n$.

Now, applying the previous theorem, we get the required result. □

3.8 Monotonically Increasing Integrators

The further theory of Riemann-Stieltjes integration will now be developed for monotonically increasing integrators, and we shall see later that for many purposes this is just as general as studying the theory for integrators which are of bounded variation.

When α is increasing, the differences $\Delta\alpha_k$ which appear in the Riemann-Stieltjes sums are all nonnegative. This simple fact plays a vital role in the development of the theory.

Definition 19 (Upper and Lower Sums)**.** Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and let

$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$.

The number $U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\,\Delta\alpha_k$ is called the Upper Riemann-Stieltjes sum of f with respect to α for the partition P.

The number $L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\,\Delta\alpha_k$ is called the Lower Riemann-Stieltjes sum of f with respect to α for the partition P.

Note:

We know that $m_k(f) \leqslant M_k(f)$ for all k.

If α is monotonically increasing on $[a, b]$, then $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) \geqslant 0$

$$m_k(f)\,\Delta\alpha_k \leqslant M_k(f)\,\Delta\alpha_k \qquad \text{for all } k$$

$\Longrightarrow$ Upper sums are always greater than or equal to the lower sums.

Furthermore, if $t_k \in [x_{k-1}, x_k]$, then

$$m_k(f) \leqslant f(t_k) \leqslant M_k(f)$$

$$m_k(f)\,\Delta\alpha_k \leqslant f(t_k)\,\Delta\alpha_k \leqslant M_k(f)\,\Delta\alpha_k$$

$$\Longrightarrow L(P, f, \alpha) \leqslant S(P, f, \alpha) \leqslant U(P, f, \alpha)$$

$$L(P, f, \alpha) \leqslant S(P, f, \alpha) \leqslant U(P, f, \alpha)$$

The above inequalities need not necessarily hold when α is not an increasing function.

For a monotonic increasing function α, with refinement in partition, lower sum increases and upper sum decreases. Also, the lower sum can never exceed the upper sum for any arbitrary partitions.

Theorem 11. *Assume that α is monotonically increasing on $[a, b]$. Then*

(i) If P' is finer than P, we have

$U(P', f, \alpha) \leqslant U(P, f, \alpha)$ *and* $L(P, f, \alpha) \leqslant L(P', f, \alpha)$

(ii) For any two partitions P_1 and P_2, we have

$L(P_1, f, \alpha) \leqslant U(P_2, f, \alpha)$.

Proof. (*i*) Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.
It is enough to prove the statement when P' contains one more point than P, say the point c.

Without loss of generality assume that c is the ith subinterval of P.
Therefore we have

$$\begin{aligned} U(P, f, \alpha) &= \sum_{k=1}^{n} M_k(f)\, \Delta\alpha_k \\ &= \sum_{k=1, k\neq i}^{n} M_k(f)\, \Delta\alpha_k + M_i(f)\, \Delta\alpha_i \end{aligned}$$

Let $M' = \sup_{x\in[x_{i-1},c]} f(x)$ and $M'' = \sup_{x\in[c,x_i]} f(x)$
Then $M' \leqslant M_i(f)$ and $M'' \leqslant M_i(f)$

$$U(P', f, \alpha) = \sum_{k=1, k\neq i}^{n} M_k(f)\,\Delta\alpha_k + M'[\alpha(c) - \alpha(x_{i-1})]$$
$$+ M''[\alpha(x_i) - \alpha(c)]$$

$$\begin{aligned} U(P', f, \alpha) - U(P, f, \alpha) &= M'[\alpha(c) - \alpha(x_{i-1})] + M''[\alpha(x_i) - \alpha(c)] - M_i(f)\,\Delta\alpha_i \\ &\leqslant M_i(f)[\alpha(c) - \alpha(x_{i-1})] \\ &= M_i(f)\,\Delta\alpha_i - M_i(f)\,\Delta\alpha_i \\ &= 0 \end{aligned}$$

$\implies U(P, f, \alpha) \geqslant U(P', f, \alpha)$

$\implies$ Upper sum decreases with refinements.

Similarly, we can prove that lower sum increases with refinement in partition.

(ii) Let $P = P_1 \cup P_2$

$$\implies L(P_1, f, \alpha) \leqslant L(P, f, \alpha) \leqslant U(P, f, \alpha) \leqslant U(P_2, f, \alpha)$$

□

Note:

Using the above theorem, for monotonic increasing α, we have

$$m[\alpha(b) - \alpha(a)] \leqslant L(P_1, f, \alpha) \leqslant U(P_2, f, \alpha) \leqslant M[\alpha(b) - \alpha(a)]$$

where $M = \sup_{x \in [a,b]} f(x)$, $\quad m = \inf_{x \in [a,b]} f(x)$ and P_1, P_2 are any partitions of $[a,b]$.

Definition 20 (Upper and Lower Integrals)**.** Assume that α is monotonically increasing on $[a,b]$. The Upper Riemann-Stieltjes integral of f with respect to α is defined as follows:

$\overline{\int_a^b} f d\alpha = \inf\{U(P, f, \alpha) : P \in \mathcal{P}[a,b]\} = \overline{I}(f, \alpha)$.

The lower Riemann-Stieltjes integral of f with respect to α is defined as follows:

$\underline{\int_a^b} f d\alpha = \sup\{L(P, f, \alpha) : P \in \mathcal{P}[a,b]\} = \underline{I}(f, \alpha)$.

Note: We sometimes write $\overline{I}(f, \alpha)$ and $\underline{I}(f, \alpha)$ for the upper and lower integrals. In the special case where $\alpha(x) = x$, the upper and lower sums are denoted by $U(P, f)$ and $L(P, f)$ and are called upper and lower Riemann sums. The corresponding integrals, denoted by $\overline{\int_a^b} f(x)\, dx$ and $\underline{\int_a^b} f(x)\, dx$, and are called upper and lower Riemann integrals. They were first introduced by J. G. Darboux(1875).

Theorem 12. *Assume that α is monotonic increasing on $[a,b]$. Then*

$$\underline{I}(f, \alpha) \leqslant \overline{I}(f, \alpha)$$

Proof. Let $\epsilon > 0$.

$$\overline{I}(f, \alpha) = \inf_P U(P, f, \alpha)$$

By the characterization of infimum, there exists a partition P_1 of $[a,b]$ such that

$$U(P_1, f, \alpha) < \overline{I}(f, \alpha)$$

Now,

$$L(P, f, \alpha) \leqslant U(P, f, \alpha) \quad \forall \text{ partitions } P$$

$$L(P_1, f, \alpha) \leqslant U(P, f, \alpha) < \overline{I}(f, \alpha) + \epsilon \;\; \forall \text{ partitions } P$$

$\therefore \overline{I}(f, \alpha) + \epsilon$ is an upper bound to all lower sums $L(P, f, \alpha)$.

$$(f, \alpha) \leqslant \overline{I}(f, \alpha) + \epsilon \underline{I}$$

$\because \epsilon$ is arbitrary, $\therefore \underline{I}(f, \alpha) \leqslant \overline{I}(f, \alpha)$

□

An example in which $\underline{I}(f, \alpha) < \overline{I}(f, \alpha)$.

Let $\alpha(x) = x$ and define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & ; x \in [0,1] \cap \mathbb{Q} \\ 0 & ; x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[0, 1]$, we have $M_k(f) = 1$ and $m_k(f) = 0$.

Every subinterval contains both, rationals and irrationals,

Therefore $U(P, f, \alpha) = 1$ and $L(P, f, \alpha) = 0$ for all partitions P.

$\implies \int_a^{\underline{b}} f dx = 1$ and $\int_{\underline{a}}^{b} f dx = 0$

Observe that the same result holds if

$$f(x) = \begin{cases} 0 & ; x \in [0,1] \cap \mathbb{Q}^c \\ 1 & ; x \in [0,1] \cap \mathbb{Q} \end{cases}$$

3.9 Riemann's Condition

If we are to expect the upper and lower sums to be equal then we must also expect the upper sums to become arbitrarily close to the lower sums. As a result, it appears logical to seek those functions f for which the difference $U(P,f,\alpha) - L(P,f,\alpha)$ can be made arbitrarily small.

Definition 21. If for every $\epsilon > 0$, there exists a partition P_ϵ such that for any partition P finer than P_ϵ we have

$$0 \leqslant U(P,f,\alpha) - L(P,f,\alpha) < \epsilon,$$

then we say that f satisfies Riemann's condition with respect to α on $[a,b]$.

Theorem 13. *Assume that α is monotonic increasing on $[a,b]$. Then the following statements are equivalent :*

(i) $f \in R_\alpha[a,b]$

(ii) f *satisfies Riemann's condition with respect to* α *on* $[a,b]$

(iii) $\underline{I}(f,\alpha) = \bar{I}(f,\alpha)$

Proof. To prove that $(i) \implies (ii)$.

Suppose $f \in R_\alpha[a,b]$

$\implies$ For $\epsilon > 0$, there exists a partition P_ϵ of $[a,b]$ such that for $P \supseteq P_\epsilon$, we have

$$\left| S(P,f,\alpha) - \int_a^b f\,d\alpha \right| < \epsilon$$

If $\alpha(a) = \alpha(b)$, that is, if α is constant on $[a, b]$ then (ii) is trivially satisfied. So we can assume that $\alpha(a) < \alpha(b)$.

Given $\epsilon > 0$, we can choose a partition P_ϵ such that for any partition P finer than P_ϵ and all choices of t_k and t'_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - A\right| < \frac{\epsilon}{3} \text{ and}$$

$$\left|\sum_{k=1}^{n} f(t'_k)\,\Delta\alpha_k - A\right| < \frac{\epsilon}{3} \text{ where } A = \int_a^b f d\alpha$$

$$\therefore \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - A + \sum_{k=1}^{n} f(t'_k)\,\Delta\alpha_k - A\right| + \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k\right| + \left|\sum_{k=1}^{n} f(t'_k)\,\Delta\alpha_k\right| < \frac{2\epsilon}{3}$$

$$\left|\sum_{k=1}^{n} \left[f(t'_k) - f(t'_k)\right]\Delta\alpha_k\right| < \frac{2\epsilon}{3}$$

$$M_k(f) = \sup_{x\in[x_{k-1},x_k]} f(x) \qquad \text{and} \qquad m_k(f) = \inf_{x\in[x_{k-1},x_k]} f(x)$$

$$M_k(f) - m_k(f) = Sup\{f(x) - f(') : x, x' \in [x_{k-1}, x_k]\}$$

$\therefore$ For $h > 0, \exists t_k, t'_k \in [x_{k-1}, x_k]$ such that

$$f(t_k) - f(t'_k) > M_k(f) - m_k(f) - h$$

$$M_k(f) - m_k(f) < f(t_k) - f(t'_k) + h$$

$$\text{Choose} \quad h = \frac{\epsilon}{3\left[\alpha\left(b\right) - \alpha\left(a\right)\right]}$$

$$\begin{aligned}
\left|U\left(P, f, \alpha\right) - L\left(P, f, \alpha\right)\right| &= \left|\sum_{k=1}^{n}\left[M_k\left(f\right) - m_k\left(f\right)\right]\Delta\alpha_k\right| \\
&< \left|\sum_{k=1}^{n}\left[f\left(t_k\right) - f\left(t_k'\right)\right]\Delta\alpha_k\right| + \left|h\Delta\alpha_k\right| \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3\left[\alpha\left(b\right) - \alpha\left(a\right)\right]}\left[\alpha\left(b\right) - \alpha\left(a\right)\right] \\
&= \epsilon
\end{aligned}$$

Next, to show that $(i) \implies (ii)$

Suppose f satisfies Riemann's condition w.r.t. α on $[a, b]$ such that for any partition $P \supseteq P_\epsilon$, we have

$$U\left(P, f, \alpha\right) - L\left(P, f, \alpha\right) < \epsilon$$

$$U\left(P, f, \alpha\right) < L\left(P, f, \alpha\right) + \epsilon$$

$\therefore$ for such partition P, we have

$$\overline{I}\left(f, \alpha\right) \leqslant U\left(P, f, \alpha\right) < L\left(P, f, \alpha\right) + \epsilon \leqslant \underline{I}\left(f, \alpha\right) + \epsilon$$

that is, $\overline{I}\left(f, \alpha\right) \leqslant \underline{I}\left(f, \alpha\right) + \epsilon$ for every $\epsilon > 0$.

$$\therefore \overline{I}\left(f, \alpha\right) \leqslant \underline{I}\left(f, \alpha\right) + \epsilon \tag{1}$$

But by Theorem 17, we have

$$\underline{I}(f,\alpha) \leqslant \overline{I}(f,\alpha) \tag{2}$$

$\therefore$ from (i) and (ii) we have

$$\underline{I}(f,\alpha) = \overline{I}(f,\alpha)$$

To show that $(iii) \implies (ii)$

Suppose $\underline{I}(f,\alpha) = \overline{I}(f,\alpha) = A$

To show that $\int_a^b f d\alpha$ exists and is equal to A.

Let $\epsilon > 0$.

$\overline{I}(f,\alpha) = \inf_P U(P,f,\alpha)$ and $\underline{I}(f,\alpha) = \inf_P U(P,f,\alpha)$

Therefore By the characterization of infimum, there exists a partition P_1 of $[a,b]$ such that

$$U(P,f,\alpha) < \overline{I}(f,\alpha) + \epsilon \ \ \forall\ P \supseteq P_1$$

Next, by the characterization of Supremum, there exists a partition P_2 of $[a,b]$ such that

$$\underline{I}(f,\alpha) - \epsilon < L(P,f,\alpha) \ \ \forall\ P \supseteq P_\epsilon$$

Let $P_\epsilon = P_1 \cup P_2$. Then for any partition $P \supseteq P_\epsilon$, we have

$$\underline{I}(f,\alpha) - \epsilon < L(P,f,\alpha) \leqslant S(P,f,\alpha) \leqslant U(P,f,\alpha) < \overline{I}(f,\alpha) + \epsilon$$

$$A - \epsilon < S(P,f,\alpha) < A + \epsilon$$

$$|S(P,f,\alpha) - A| < \epsilon$$

Since ϵ is arbitrary, $\int\limits_a^b f d\alpha$ exists and is equal to A. □

3.10 Comparison Theorems

Theorem 14. *Assume that α is monotonically increasing on $[a,b]$. If $f \in R_\alpha[a,b]$ and $g \in R_\alpha[a,b]$ and if $f(x) \leqslant g(x)$ for all $x \in [a,b]$; then we have*

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$$

Proof. For every partition P of $[a,b]$, we have

$$S(P,f,\alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k \leqslant \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k = S(P,g,\alpha)$$

$$\Longrightarrow S(P,f,\alpha) \leqslant S(P,g,\alpha)$$

$$U(P,f,\alpha) \leqslant U(P,g,\alpha)$$

$$\Longrightarrow \inf_P U(P,f,\alpha) \leqslant \inf_P U(P,g,\alpha)$$

$$\Longrightarrow \int\limits_a^b f d\alpha \leqslant \int\limits_a^b g d\alpha$$

□

Note:

If $g(x) \geqslant 0$ and α is monotonic increasing on $[a,b]$ then $\int\limits_a^b d\alpha(x) \geqslant 0$

Theorem 15. *Assume that α is monotonically increasing on $[a,b]$. If $f \in R_\alpha[a,b]$, then $|f| \in R_\alpha[a,b]$ and we have the inequality*

$$\left|\int_a^b f(x)d\alpha(x)\right| \leqslant \int_a^b |f(x)|\, d\alpha(x)$$

Proof. Let $\epsilon > 0$. By Riemann,s condition, there exists a partition P_ϵ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \ \ \forall\ P \supseteq P_\epsilon$$

$$M_k(f) - m_k(f) = \sup\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$$

$$|f(x) - f(y)| \leqslant |f(x) - f(y)| \ \ \forall\ x, y \in [x_{k-1}, x_k]$$

$$\implies M_k(|f|) - m_k(|f|) \leqslant M_k(f) - m_k(f)$$

$$\implies [M_k(|f|) - m_k(|f|)]\,\Delta\alpha_k \leqslant [M_k(f) - m_k(f)]$$

$$\implies \sum_{k=1}^{n} [M_k(|f|) - m_k(|f|)]\,\Delta\alpha_k \leqslant \sum_{k=1}^{n} [M_k(f) - m_k(f)]$$

$$U(P,|f|,\alpha) - L(P,|f|,\alpha) \leqslant U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$|f| \in R_\alpha[a,b]$$

□

The converse of the above theorem need not be true.

Example:

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ -1 & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Theorem 16. *Assume that α is monotonically increasing on $[a, b]$. If $f \in R_\alpha [a, b]$, then $f^2 \in R_\alpha [a, b]$.*

Proof. Let $M = \sup\limits_{x \in [x_{k-1}, x_k]} |f(x)|$

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.
Let $M_k(f) = \sup\limits_{x \in [x_{k-1}, x_k]} |f(x)|$ and $m_k(f) = \inf\limits_{x \in [x_{k-1}, x_k]} |f(x)|$

$$\begin{aligned} \therefore M_k\left(f^2\right) - m_k\left(f^2\right) &= \left[M_k(|f|)\right]^2 - \left[m_k(|f|)\right]^2 \\ &= \left[M_k(|f|) - m_k(|f|)\right]\left[M_k(|f|) + m_k(|f|)\right] \\ &= \left[M + M\right]\left[M_k(|f|) + m_k(|f|)\right] \\ &= 2M\left[M_k(|f|) + m_k(|f|)\right] \end{aligned}$$

$|f| \in R_\alpha [a, b] \implies \exists$ partition P_ϵ of $[a, b]$ such that

$$U(P, |f|, \alpha) - L(P, |f|, \alpha) < \frac{\epsilon}{2M} \forall P \supseteq P_\epsilon$$

$$\begin{aligned} \therefore U\left(P, f^2, \alpha\right) - L\left(P, f^2, \alpha\right) &\leqslant 2M\left[U(P, |f|, \alpha) - L(P, |f|, \alpha)\right] \\ &< 2M \frac{\epsilon}{2M} \quad \text{for} P \supseteq P_\epsilon \end{aligned}$$

$$\therefore f^2 \in R_\alpha [a, b]$$

□

Theorem 17. *Assume that* α *is monotonically increasing on* $[a, b]$. *If* $f \in R_\alpha [a, b]$ *and* $g \in R_\alpha [a, b]$ *then the product* $fg \in R_\alpha [a, b]$.

Proof. $[f(x) + g(x)]^2 = [f(x)]^2 + [g(x)]^2 + 2f(x)g(x) \, \forall x \in [a, b]$

$$f(x)g(x) = \frac{1}{2}\{[f(x) + g(x)]^2 - [f(x)]^2 - [g(x)]^2\}$$

$$f, g \in R_\alpha [a, b] \implies f^2, g^2 \in R_\alpha [a, b]$$

$$f, g \in R_\alpha [a, b] \implies f + g \in R_\alpha [a, b]$$

$$\implies (f + g)^2 \in R_\alpha [a, b]$$

$$\therefore \{(f + g)^2 - f^2 - g^2 \in R_\alpha [a, b]\}$$

$$fg \in R_\alpha [a, b]$$

□

3.11 Integrators of bounded variation

Next, we look at another class of integrator functions, namely, functions of bounded variation, a class of functions closely related to the monotonically increasing functions.

Definition 22 (Variation)**.** Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \to \mathbb{R}$ be a function. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, the sum

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is called the Variation of f in $[a, b]$ over the partition P.

Definition 23 (Function of bounded variation)**.** If $\sup_P V(f, P) < \infty$, then we say that f is a function of bounded variation on $[a, b]$. The value $V_a^b(f) = \sup_P V(f, P)$ is called the variation of f over the interval $[a, b]$.

Definition 24. Let $f \in BV[a, b]$. We define $v : [a, b] \to \mathbb{R}$ by

$$\begin{aligned} v(x) &= 0 \qquad\qquad for \;\; x = a \\ &= V_a^x(f) \qquad for \;\; a < x \leqslant b. \end{aligned}$$

v is called the total variation function for f on $[a, b]$.

Theorem 18. *Both v and $v - f$ are monotonic increasing functions on $[a, b]$.*

Proof. Let $a \leqslant x < y \leqslant b$

Then

$$\begin{aligned} V(y) - V(x) &= V_a^y(f) - V_a^x(f) \\ &= V_x^y(f) \\ &\geqslant |f(y) - f(x)| \\ &\geqslant 0 \end{aligned} \tag{*}$$

$\Longrightarrow$ v is monotonically increasing.

Once again, from $(*)$,

$$\begin{aligned} V(y) - V(x) &\geqslant f(y) - f(x) \\ \Longrightarrow V(y) - f(y) &\geqslant V(x) - f(x) \\ \Longrightarrow (V - f)(y) &\geqslant (V - f)(x) \end{aligned}$$

$\Longrightarrow$ $v - f$ is also monotonically increasing. □

Theorem 19 (Jordan's theorem)**.** $f \in BV[a,b] \iff$ *f can be written as the difference of two monotonic increasing functions.*

Proof. Suppose $f \in BV[a,b]$.

Then both v and $v - f$ are monotonic increasing where

$$v(x) = \begin{cases} 0 & ; x = a \\ V_x^x(f) & a < x \leqslant b \end{cases}$$

by the theorem.

Clearly then $f = v - (v - f)$.

Clearly then $f = v - (v - f)$.

Coversely, suppose $f = g - h$. where both g and h are monotonic increasing.

$\Longrightarrow$ $g,\ h \in BV[a,b]$ ($\because$ every monotonic function on $[a,b]$ is a function of bounded variation.)

$\Longrightarrow$ $g - h \in BV[a,b]$ ($\because$ $BV[a,b]$ is a vector space.)

$$\therefore f \in BV[a,b]$$

□

Jordan's theorem says that every function of bounded variation on $[a,b]$ can be expressed as the difference of two monotonically increasing functiona, say α_1 and α_2, that is, $\alpha = \alpha_1 - \alpha_2$. If $f \in R_{\alpha_1}[a,b]$ and $f \in R_{\alpha_2}[a,b]$, then the linearity of Riemann-Stieltjes integral will imply $f \in R_{\alpha_1-\alpha_2}$, that is, $f \in R_\alpha[a,b]$. However, the converse is not always true. If $f \in R_\alpha[a,b]$, it is quite possible to choose increasing functions α_1 and α_2 such that $\alpha = \alpha_1 - \alpha_2$, but such that neither integrals $\int_a^b f d\alpha_1, \int_a^b f d\alpha_2$ exists. The difficulty is because of the fact that the decomposition $\alpha = \alpha_1 - \alpha_2$ of the function α need not be unique. However, we can prove that there is atleast one decomposition for which the converse us true, namely, when α_1 is the total variation of α and $\alpha_2 = \alpha_1 - \alpha$.

Theorem 20. *Assume that $\alpha \in BV[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ if $a < x \leqslant b$ and let $V(a) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R_\alpha[a,b]$ then $f \in R_V[a,b]$*

Proof. If $V(b) = 0$, then V is constant and the result is trivial.

Suppose $V(b) > 0$.

Suppose also that $|f(x)| \leqslant M \forall x \in [a,b]$.

Let $\epsilon > 0$.

$f \in f[a,b] \implies \exists$ a partition P_1 of $[a,b]$ such that for any partition P finer than P_1 and any $t_k, t'_k \in [x_{k-1} - x_k]$, we have

$$\left| \sum_{k=1}^{n} [f(t_k) - f(t'_k)] \Delta\alpha_k \right| < \frac{\epsilon}{4} \tag{1}$$

$$v(b) = V_a^b(\alpha) = \sup_P v(\alpha, P).$$

By the characterization of Supremum, there exists a partition P_2 of $[a, b]$ such that

$$v(b) - \frac{\epsilon}{4M} < \sum_{k=1}^{n} |\Delta\alpha_k|$$

$$\implies v(b) - \sum_{k=1}^{n} |\Delta\alpha_k| < \frac{\epsilon}{4M} \tag{2}$$

Let $P_\epsilon = P_1 \cup P_2$

$\therefore$ (1) and (2) will hold true for any partition P finer than P_ϵ

Note that $\Delta v_k - |\Delta\alpha_k| \geqslant 0$

$$\begin{aligned}\sum_{k=1}^{n} [M_k(f) - m_k(f)](\Delta v_k - |\Delta\alpha_k|) &\leqslant 2M \sum_{k=1}^{n} (\Delta v_k - |\Delta\alpha_k|) \\ &= 2M \left(v(b) - \sum_{k=1}^{n} |\Delta\alpha_k| \right) \\ &< 2M \cdot \frac{\epsilon}{4M} \\ &= \frac{\epsilon}{2} \end{aligned} \tag{3}$$

Let $A(P) = \{k : \Delta\alpha_k \geqslant 0\}$, $\qquad B(P) = \{k : \Delta\alpha_k < 0\}$

and let $h = \frac{\epsilon}{4v(b)}$

If $k \in A(P)$ by the characterizarion of Supremum, we can choose t_k and t'_k so that

$f(t_k) - f(t'_k) > M_k(f) - m_k(f) - h$

$(\because M_k(f) - m_k(f) = \sup\{f(t_k) - f(t'_k) : t_k, t'_k \in [x_{k-1}, x_k]\})$

but if $k \in B(P)$, we can choose $t'_k, t_k \in [x_{k-1}, x_k]$

$$\ni f(t'_k) - f(t_k) > M_k(f) - m_k(f) - h.$$

Then

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] |\Delta\alpha_k|!$$
$$< \sum_{k=1}^{n} [f(t_k) - f(t'_k)] |\Delta\alpha_k| + \sum_{k=1}^{n} h |\Delta\alpha_k|$$
$$= \sum_{k\in A(P)} [f(t_k) - f(t'_k)] |\Delta\alpha_k| + \sum_{k\in A(P)} [f(t'_k) - f(t_k)] |\Delta\alpha_k|$$
$$+ h \sum_{k=1}^{n} |\Delta\alpha_k|$$
$$= \sum_{k=1}^{n} [f(t_k) - f(t'_k)] \Delta\alpha_k + h \sum_{k=1}^{n} |\Delta\alpha_k|$$

$$< \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4v(b)} \cdot V(b)$$
$$= \epsilon/2 \qquad \cdots (4)$$

Adding (3) and (4), we get

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta v_k < \epsilon$$

$$\implies U(P, f, v) - L(P, f, v) < \epsilon$$

$$\therefore f \in R_v[a, b]$$

□

This theorem together with theorem 19 enables us to reduce the theory of Riemann-Stieltjes integration for integrators of bounded variation to the case of increasing integrators. Riemann's condition then becomes available and it turns out to be a particularly useful tool in the work. As a first application we shall obtain a result which is closely related to theorem 4.

Theorem 21. *Let* $\alpha \in BV\,[a,b]$ *and assume that* $f \in R_\alpha\,[a,b]$*. Then* $f \in R_\alpha\,[c,d]$ *for any subinterval* $[c,d]$ *of* $[a,b]$*.*

Proof. Let

$$\begin{aligned} v(x) &= V_a^b(f) \qquad ; \qquad a < x \leqslant b \\ &= 0 \qquad\qquad ; \qquad x = a \end{aligned}$$

Then $\alpha = v - (v - \alpha)$
where v and $v - \alpha$ are increasing on $[a,b]$.

Now by the previous theorem, $f \in R_v\,[a,b]$ and $f \in R_\alpha\,[a,b] \implies f \in R_{-\alpha}\,[a,b]$.

$$\therefore f \in R_{v-\alpha}\,[a,b]$$

Therefore if the theorem is true for increasing α then the theorem is true for integrators of bounded variation.
Therefore it is enough to prove the theorem for α monotonic increasing on $[a,b]$
By the additive property of Riemann-Stieltjes integrals it is enough to show that each integral $\int_a^c f d\alpha$ and $\int_a^d f d\alpha$ exist.
Assume that $a < c < b$.
If P is a partition of $[a,x]$,
let $\Delta(P,x) = U(P,f,\alpha) - L(P,f,\alpha)$ denote the difference of the upper and lower sums associated with the interval $[a,x]$.

$\because f \in R_\alpha\,[a,b] \therefore$ Riemann's condition holds.

$\therefore$ for $\epsilon > 0, \exists$ a partition P_1 of $[a, b]$ such that

$$\Delta(P, b) < \epsilon \qquad if \qquad P \supseteq P_1. \tag{1}$$

We can assume that $c \in P_1$.

$\therefore [a, c] \cap P_1$ is a partition of $[a, c]$.

Let $P_2 = [a, c] \cap P_1$.

Now $P_2 \subseteq P_1$.

P_2 contains points of $[a, c]$ but P_1 contains points of $[a, c]$ and $[c, b]$ as well.

Suppose P' is a partition of $[a, c]$ such that $P' \supseteq P_2$, then $P = P' \cup P_1$ is a partition of $[a, b]$.

Then the sum defining $\Delta(P', c)$ contains only part of the terms in the sum defining $\Delta(P, b)$.

$\because \Delta(P', c) \geqslant 0, \Delta(P, b) \geqslant 0$ and since $P \supseteq P_1$, we have

$\Delta(P', c) \leqslant \Delta(P, b) < \epsilon$. (from (1))

That is, $P' \supseteq P_2 \implies \Delta(P', c) < \epsilon$.

$\therefore f$ satisfies Riemann's condition on $[a, c]$ and $\int_a^c f d\alpha$ exists.

The same argument shows that $\int_a^d f d\alpha$ exists.

$\therefore$ by the summability of the Riemann-Stieltjes integrals, the integral $\int_c^d f d\alpha$ exists.

$\square$

The next theorem is an application of Theorems 15, 17 and 21.

Theorem 22. *Assume that $f, g \in R_\alpha[a, b]$, where α is monotonic incresing on $[a, b]$. Define*

$$F(x) = \int_a^x f(t)\, d\alpha(t)$$

and

$$G(x) = \int_a^x g(t)\, d\alpha(t)$$

if $x \in [a, b]$.

Then $f \in R_G[a, b]$, $g \in R_F[a, b]$ *and* $f \cdot g \in R_\alpha[a, b]$, *and we have*

$$\begin{aligned}\int_a^b f(x)\, g(x)\, d\alpha(x) &= \int_a^b f(x)\, dG(x) \\ &= \int_a^b g(x)\, dF(x)\end{aligned}$$

Proof. The integral $\int_a^b f \cdot g d\alpha$ exists (By theorem 15).

For every partition P of $[a, b]$, we have

$$\begin{aligned} S(P, f, G) &= \sum_{k=1}^{n} f(t_k)\, \Delta G_k \\ &= \sum_{k=1}^{n} f(t_k)\, [G(x_k) - G(x_{k-1})] \\ &= \sum_{k=1}^{n} f(t_k)\, \{\int_a^{x_k} g(t)\, d\alpha(t) \\ &\qquad - \int_a^{x_{k-1}} g(t)\, d\alpha(t)\} \\ &= \sum_{k=1}^{n} f(t_k) \int_{x_{k-1}}^{x_k} g(t)\, d\alpha(t) \\ &= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t_k)\, g(t)\, d\alpha(t) \end{aligned}$$

and $\int_a^b f(x)\, g(x)\, d\alpha(x) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)\, g(t)\, d\alpha(t)$

$\therefore$ if $M_g = \sup\{|g(x)| : x \in [a, b]\}$, we have

$\left|S(P, f, G) - \int_a^b f \cdot g d\alpha\right|$

$= \left|\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t_k)\, g(t)\, d\alpha(t) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)\, g(t)\, d\alpha(t)\right|$

$$= \left| \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \{f(t_k) - f(t)\} g(t)\, d\alpha(t) \right|$$

$$\leqslant M_g \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t_k) - f(t)|\, d\alpha(t)$$

$$\leqslant M_g \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [M_k(f) - m_k(f)]\, d\alpha(t)$$

$$= M_g \sum_{k=1}^{n} [M_k(f) - m_k(f)] [\alpha(x_k) - \alpha(x_{k-1})]$$

$= M_g [U(P, f, \alpha) - L(P, f, \alpha)]$

$\because f \in R_\alpha[a, b] \therefore$ for each $\epsilon > 0, \exists$ a partition P_ϵ such that $P \supseteq P_\epsilon \implies U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \implies f \in R_G[a, b]$ and $\int_a^b f \cdot g d\alpha = \int_a^b f \cdot dG$

Similarly, it can be proved that $g \in R_F[a, b]$ and that $\int_a^b f \cdot g d\alpha = \int_a^b g \cdot dF$

□

Note:

Theorem 22 is also valid if α is of bounded variation on $[a, b]$.

3.12 Sufficient conditions for existence of Riemann integrals

In most of the previous theorems we have assumed that certain integrals existed and then studied their properties. It is quite natural to ask: When does the integral exist? Two useful sufficient conditions will be obtained.

Theorem 23. *If f is continuous on $[a, b]$ and if α is of bounded variation, then*

$f \in R_\alpha [a, b]$

Proof. It is enough to prove the theorem for α monotonic increasing.

If $\alpha(a) = \alpha(b)$ then we are done.

We have $M_k(f) - m_k(f) = \sup\{f(x) - f(y) : x, y \in [x_{k-1}, x_k]\}$

Suppose $\alpha(a) < \alpha(b)$.

Let $\epsilon > 0$.

f is continuous on $[a, b]$.

$\implies$ f is uniformly continuous on $[a, b]$.

$\therefore \exists \delta > 0$ such that

$$|x - y| > \delta \implies |f(x) - f(y)| < \frac{\epsilon}{a[\alpha(b) - \alpha(a)]}$$

Suppose P_ϵ is a partition of $[a, b]$ such that $\|P_\epsilon\| < \delta$, then for $P \supseteq P_\epsilon$, we have

$$\begin{aligned} M_k(f) - m_k(f) &\leqslant \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \Delta\alpha_k \\ \implies [M_k(f) - m_k(f)] \Delta\alpha_k &\leqslant \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \Delta\alpha_k \\ \implies \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k &\leqslant \sum_{k=1}^{n} \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \\ &= \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \sum_{k=1}^{n} \Delta\alpha_k \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

$$\therefore U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

and we see that Riemann's condition holds.

$$\therefore f \in R_\alpha [a, b]$$

□

We have the special case of the above theorem:

Theorem 24. *Each of the following conditions is sufficient for the existence of the Riemann integral:*

1. *f is continuous on $[a, b]$*
2. *f is of bounded variation on $[a, b]$*

Proof. 1. TST $f \in R[a, b]$

$\alpha(x) = x$ is continuous on $[a, b]$ and $\alpha(x)$ can be expressed as the difference of the two increasing functions x and 0.

$\therefore \alpha(x)$ is of bounded variation on $[a, b]$.

$$\therefore f \in R[a, b]$$

2. f is of bounded variation on $[a, b]$.

$\alpha(x) = x$ is continuous on $[a, b]$

$\implies \alpha \in R_f [a, b]$

$\implies f \in R_\alpha [a, b]$ i.e. $f \in R[a, b]$

□

3.13 Necessary conditions for existence of Riemann-Stieltjes integrals

When $\alpha \in BV[a,b]$, continuity of f s sufficient for the existence of $\int_a^b f d\alpha$. However, the continuity of f throughout $[a,b]$ is not a necessary condition. For example, in Theorem 8, if α is a step function, then f can be defined quite arbitrarily in $[a,b]$ provided only that the continuity of f compensates for the discontinuity of α. The next theorem tells us that if both, α and f have a common discontinuity from the left (or from the right) at some point then the integral $\int_a^b f d\alpha$ cannot exist.

Theorem 25. *Assume that α is monotonic increasing on $[a,b]$ and let $a < c < b$. Assume further that both α and f are discontinuous from the right at $x = c$; that is,assume that there exists an $\epsilon > 0$ such that for every $\delta > 0, \exists x, y \in (c, c+\delta)$ such that*

$|f(x) - f(c)| \geqslant \epsilon$ and $|\alpha(y) - \alpha(c)| \geqslant \epsilon$

Then the integral $\int_a^b f(x)\, d\alpha(x)$ cannot exist.

The integral also fails to exist if α and f are discontinuous from the left at c.

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Suppose $c \in P$ and $c = x_l$

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{k=1}^{n} [M_k(f) - m_k(f)]\, \Delta\alpha_k \\ &\geqslant [M_{l+1}(f) - m_{l+1}(f)]\, \Delta\alpha_{l+1} \\ &= [M_{l+1}(f) - m_{l+1}(f)]\, [\alpha(x_{l+1}) - \alpha(x_l)] \\ &= [M_{l+1}(f) - m_{l+1}(f)]\, [\alpha(x_{l+1}) - \alpha(c)] \\ &\geqslant [M_{l+1}(f) - m_{l+1}(f)] \cdot \epsilon_1 \end{aligned}$$

$\{\because \exists$ an $\epsilon_1 > 0$ such that $\forall x, y \in (c, c+\delta)$, $|\alpha(x) - \alpha(y)| \geqslant \epsilon_1\}$

Also, the hypothesis of the theorem also implies that $\exists \epsilon_2 > 0$ such that $\forall \delta > 0$ and $\forall x, y \in (c, c+\delta)$, $|f(x) - f(y)| \geqslant \epsilon_2$

$$\implies |M_{l+1}(f) - m_{l+1}(f)| \geqslant \epsilon_2$$

Let $\epsilon = min\{\epsilon_1, \epsilon_2\}$

$$\therefore U(P, f, \alpha) - L(P, f, \alpha) \geqslant \epsilon^2$$

$\therefore$ Riemann's condition is not satisfied.

□

3.14 Mean Value Theorems for Riemann-Stieltjes integrals

Although integrals occur in a wide variety of problems, there are relatively fewer cases in which the explicit value of the integral can be obtained. However, it is often enough to have an estimate for the integral rather than its exact value. The mean value theorems of this section are especially useful in making such estimates.

Theorem 26. *Assume that* α *is monotonically increasing and let* $f \in [a, b]$. *Let* M *and* m *denote the Sup and inf of the set* $\{f(x) : x \in [a, b]\}$ *respectively. Then* $\exists c \in \mathbb{R}$ *satisfying* $m \leqslant c \leqslant M$ *such that*

$$\int_a^b f(x)\, d\alpha(x) = c \int_a^b d\alpha(x) = c\,[\alpha(b) - \alpha(a)]$$

In particular, if f *is continuous on* $[a, b]$, *then* $c = f(x_0)$ *for some* x_0 *in* $[a, b]$

Proof. If $\alpha(a) = \alpha(b)$ then $\int_a^b f(x)\,d\alpha(x) = 0$ and $c[\alpha(b) - \alpha(a)] = 0$

$\therefore$ The theorem holds trivially when $\alpha(a) = \alpha(b)$

Suppose $\alpha(a) < \alpha(b)$

Now,$m[\alpha(b) - \alpha(a)] \leqslant L(P, f, \alpha) \leqslant U(P, f, \alpha) \leqslant$

$M[\alpha(b) - \alpha(a)]$

for any partition P of $[a, b]$

$\therefore$ the value of the integral $\int_a^b f d\alpha$ must lie between $m[\alpha(b) - \alpha(a)]$ and $M[\alpha(b) - \alpha(a)]$

That is, $m[\alpha(b) - \alpha(a)] \leqslant \int_a^b d\alpha \leqslant M[\alpha(b) - \alpha(a)]$

$\alpha(a) < \alpha(b) \implies \alpha(a) - \alpha(b) > 0$ that is, $\int_a^b d\alpha(x) > 0$

$$m \leqslant \frac{\int_a^b f d\alpha}{\int_a^b d\alpha} \leqslant M$$

Let $c = \frac{\int_a^b f d\alpha}{\int_a^b d\alpha} \implies m \leqslant c \leqslant M$

$\therefore$ we found $c \in \mathbb{R}$ satisfying $m \leqslant c \leqslant M$ such that $\int_a^b f(x)\,d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$

□

Another theorem of this type can be obtained from the First Mean Value Theorem by using integration by parts.

Theorem 27 (Second MVT for Riemann-Stieltjes integrals)**.** *Assume that α is continuous and that f is monotonically increasing on $[a, b]$. Then $\exists$ a point $x_0 \in$*

$[a, b]$ *such that*

$$\int_a^b f(x)\, d\alpha(x) = f(a) \int_a^{x_0} d\alpha(x) + f(b) \int_{x_0}^b d\alpha(x)$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

Proof. We have the theorem,

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x)$$

Now, apply the previous theorem to the integral on the right hand side.

$\exists c \in \mathbb{R}$ satisfying $m \leqslant c \leqslant M$ such that

$$\int_a^b \alpha(x)\, df(x) = c \int_a^b df(x) = c[f(b) - f(a)]$$

Let M and m be the Supremum and infimum of α on $[a, b]$ respectively.

$m \leqslant c \leqslant M$ and α is continuous on $[a, b] \implies \exists x_0 \in [a, b]$ such that $\alpha(x_0) = c$ □

$$\implies \int_a^b \alpha(x)\, df(x) = \alpha(x_0)[f(b) - f(a)]$$

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)[f(b) - f(a)]$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

3.15 The integral as a function of the interval

If $f \in R_\alpha[a,b]$ and if $\alpha \in BV[a,b]$, then by Theorem 21, the integral $\int_a^b$ exists for each $x \in [a,b]$ and can be studied as a function of x. Let us now obtain some properties of this function.

Theorem 28. *Let $\alpha \in BV[a,b]$ and assume that $f \in R_\alpha[a,b]$. Define F by the equation*

$$F(x) = \int_a^x f d\alpha \quad \text{if } x \in [a,b]$$

Then we have

(i) *$F \in BV[a,b]$.*

(ii) *Every point of continuity of α is also a point of continuity of F.*

(iii) *If α is monotonic increasing on $[a,b]$, the derivative $F'(x)$ exists at each point x_0 in (a,b) where $\alpha'(x_0)$ exists and where f is continuous. For such x_0, we have*

$$F'(x_0) = f(x_0)\,\alpha'(x_0)$$

Proof. It is enough to prove all three statements for the case where α is monotonic increasing on $[a,b]$.

(i) Let m and M be the infimum and supremum of f on $[a,b]$.

If $x \neq y$, then Theorem 26 implies $\exists c \in \mathbb{R}$ satisfying $m \leqslant c \leqslant M$ such that

$$\begin{aligned} F(y) - F(x) &= \int_a^y f d\alpha - \int_a^x f d\alpha \\ &= \int_x^y f d\alpha \\ &= c \int_x^y d\alpha(x) \\ &= c\left[\alpha(y) - \alpha(x)\right] \end{aligned}$$

$\alpha \in BV[a,b] \implies$ For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, $\exists S > 0$ such that

$$\sum_{k=1}^{n} |\alpha(x_k - \alpha(x_{k-1})| \leqslant S$$

$$\begin{aligned} \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| &= \sum_{k=1}^{n} |c[\alpha(x_k) - \alpha(x_{k-1})]| \\ &= |c| \sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \\ &= |c| S \end{aligned}$$

$$\implies F \in BV[a,b]$$

(ii) Let $\epsilon > 0$.

Suppose α is continuous at x_0.

$\implies \exists \delta > 0$ such that $|x - x_0| < \delta \implies |\alpha(x) - (x_0)| < \epsilon$

$$
\begin{aligned}
|F(x) - F(x_0)| &= |c[\alpha(x) - \alpha(x_0)]| \\
&= |c| \, |\alpha(x) - \alpha(x_0)| \\
&< |c| \, \epsilon
\end{aligned}
$$

$\implies F$ is continuous at x_0.

(iii) $x \neq y \implies y - x \neq 0$. Let $x_0 \in (a, b)$

$$\frac{F(y) - F(x)}{y - x} = \frac{c[\alpha(y) - \alpha(x)]}{y - x}$$

Suppose $\alpha'(x_0)$ exists and f is continuous at x_0. For any $x \in [a, b]$,

$$
\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} &= \frac{c[\alpha(x) - \alpha(x_0)]}{x - x_0} \\
\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{c[\alpha(x) - \alpha(x_0)]}{x - x_0} \\
\therefore F'(x_0) &= c \lim_{x \to x_0} \frac{\alpha(x) - \alpha(x_0)}{x - x0}
\end{aligned}
$$

$c \lim\limits_{x \to x_0}$ exists because $\alpha'(x_0)$ exists.

$\therefore F'(x_0)$ exists.

Next,

$$\frac{F(x_0+h)-F(x_0)-F(x_0)}{h} = \frac{\int\limits_a^{x_0+h} f d\alpha - \int\limits_a^{x_0} f d\alpha}{h}$$

$$= \frac{\int\limits_{x_0}^{x_0+h} f d\alpha}{h}$$

$$= \frac{c_h\left[\alpha(x_0+h)-\alpha(x_0)\right]}{h}$$

$$= \lim_{h\to 0} c_h \frac{\alpha(x_0+h)-\alpha(x_0)}{h}$$

$$\therefore F'(x_0) = \lim_{h\to 0} c_h \alpha'(x_0)$$

$$\text{But } \; F'(x_0) = c \lim x \to x_0 \frac{\alpha(x)-\alpha(x_0)}{x-x_0}$$

$$\therefore c \lim_{h\to 0} \frac{\alpha(x_0+h)-\alpha(x_0)}{h} = \lim_{h\to 0} c_h \lim \frac{\alpha(x_0+h)-\alpha(x_0)}{h}$$

$$c = \lim_{h\to 0} c_h$$

□

Theorem 29. *If $f, g \in R_\alpha[a,b]$, let $F(x) = \int\limits_a^x f(t)\,dt, G(x) = \int\limits_a^x g(t)\,dt$ if $x \in [a,b]$.*

Then F and G are continuous functions of bounded variation on $[a,b]$.

Also, $f \in R_{R(G)}[a,b]$ and $g \in R_{R(F)}[a,b]$ and we have

$$\int\limits_a^b f(x)\,g(x)\,dx = \int\limits_a^b f(x)\,dG(x) = \int\limits_a^b g(x)\,dF(x)$$

Proof.

$$F(x) = \int_a^x f(t)\,dt, \quad G(x) = \int_a^x g(t)\,dt \text{ if } \quad x \in [a,b]$$

F and G are continuous functions of bounded variation on $[a,b]$. (By Theorem 28 (i),(ii))

By taking $\alpha(x) = x$ in Theorem 22, we get

$$\int_a^b f(x)\,g(x)\,dx = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x)$$

This theorem converts a Riemann-Stieltjes integral of a product fg into a Riemann-Stieltjes integral $\int_a^b f dG$ with a continuous integrator of bounded variation. □

Note:

When $\alpha(x) = x$, part (iii) of Theorem 28 is sometimes called the *The First Fundamental Theorem of Integral Calculus.* It states that $F'(x) = f(x)$ at each point of continuity of f. A companion result, called the Second Fundamental Theorem of Integral Calculus is given in the next section.

3.16 Second Fundamental Theorem of Integral Calculus

The next theorem tells us how to integrate a derivative.

Theorem 30 (Second Fundamental theorem of integral calculus)**.** *Assume that $f \in R[a,b]$. Let g be a function defined on $[a,b]$ such that the derivative g' exists* *in* (a, b) *and has the value* $g'(x) = f(x)$ *for every* x *in* (a, b).

At the endpoints assume that $g(a) - g(a+) = g(b) - g(b-)$

Then we have

$$\int_a^b f(x)\,dx = \int_a^b g'(x)\,dx = g(b) - g(a)$$

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.

We can write

$$g(b) - g(a) = \sum_{k=1}^{n} \left[g(x_k) - g(x_{k-1})\right]$$

g' is differentiable on $(x_{k-1}, x_k) \implies \exists t_k \in (x_{k-1}, x_k)$ such that

$$g'(t_k) = \frac{g(x_k) - g(x_{k-1})}{x_k - x_{k-1}} = \frac{g(x_k) - g(x_{k-1})}{\Delta x_k}$$

$$g(b) - g(a) = \sum_{k=1}^{n} g'(t_k)\,\Delta x_k = \sum_{k=1}^{n} f(t_k)\,\Delta x_k$$

$f \in R[a, b] \therefore \exists$ a partition P_ϵ of $[a, b]$ such that $\forall P \supseteq P_\epsilon$,

$$\left|g(b) - g(a) - \int_a^b f(x)\,dx\right| = \left|\sum_{k=1}^{n} f(t_k)\,\Delta x_k - \int_a^b f(x)\,dx\right| < \epsilon$$

This proves the theorem. □

The second Fundamental theorem of integral calculus can be combined with Theorem 29 to give the following strengthening of Theorem 6.

Theorem 31. *Assume* $f \in [a, b]$. *Let* α *be a function which is continuous on* $[a, b]$ *and whose derivative* α' *is Riemann-integrable on* $[a, b]$. *Then the following* *integrals exist and are equal.*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha'(x)\, dx$$

Proof. $f \in R[a,b]$.

α is defined on $[a,b]$ and it's derivative α' exists in (a,b).

$\therefore$ by the second Fundamental Theorem, for each $x \in [a,b]$, we have

$$\int_a^x \alpha'(t)\, dt = \alpha(x) - \alpha(a)$$

$f, \alpha' \in R[a,b]$.

Let $F(x) = \int_a^x f(t)\, dt, \quad A(x) = \int_a^x \alpha'(t)\, dt$

Applying Theorem 29 on $f(x)$ and $g(x) = \alpha'(x)$ we get

$$f \in R_A[a,b] \text{ and } \alpha' \in R_R[a,b]$$

and we have

$$\int_a^b f(x)\, \alpha'(x)\, dx = \int_a^b f(x)\, d\alpha'(x) \quad \text{By Theorem 29}$$

$$\Longrightarrow \int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha'(x) \quad \text{By Theorem 6}$$

$\square$

Chapter 4

Geometric interpretation of Riemann-Stieltjes integral

For this discussion, let $\alpha(x)$ be differentiable, apart from being monotonic increasing, as in the definition of the Riemann-Stieltjes integral.

To visualize a Riemann-Stieltjes integral, we assign axes that are perpendicular to each other to x and $f(x)$. We graph the function $f(x)$ and the are under the curve $f(x)$ bounded below by the $x-$axis is the Riemann integral.

we interpret the Riemann-Stieltjes integral in a similar way. But the problem over here is that instead of x, over here we need to integrate with respect the function α. So instead of finding the area under f along $x-$axis, we need to find the area under f along the $\alpha-$ axis.

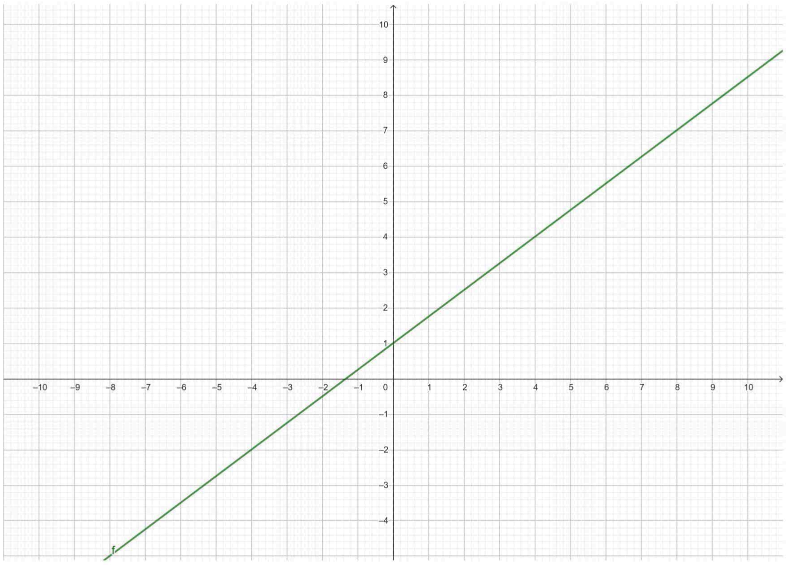


Figure 4.1: $f\left(x\right)=\dfrac{3}{4}x+1$

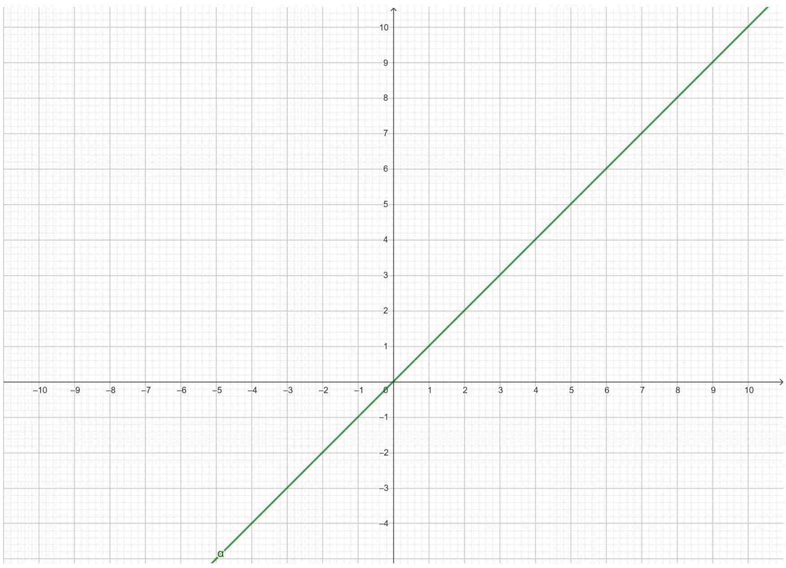


Figure 4.2: $\alpha\left(x\right)=x$

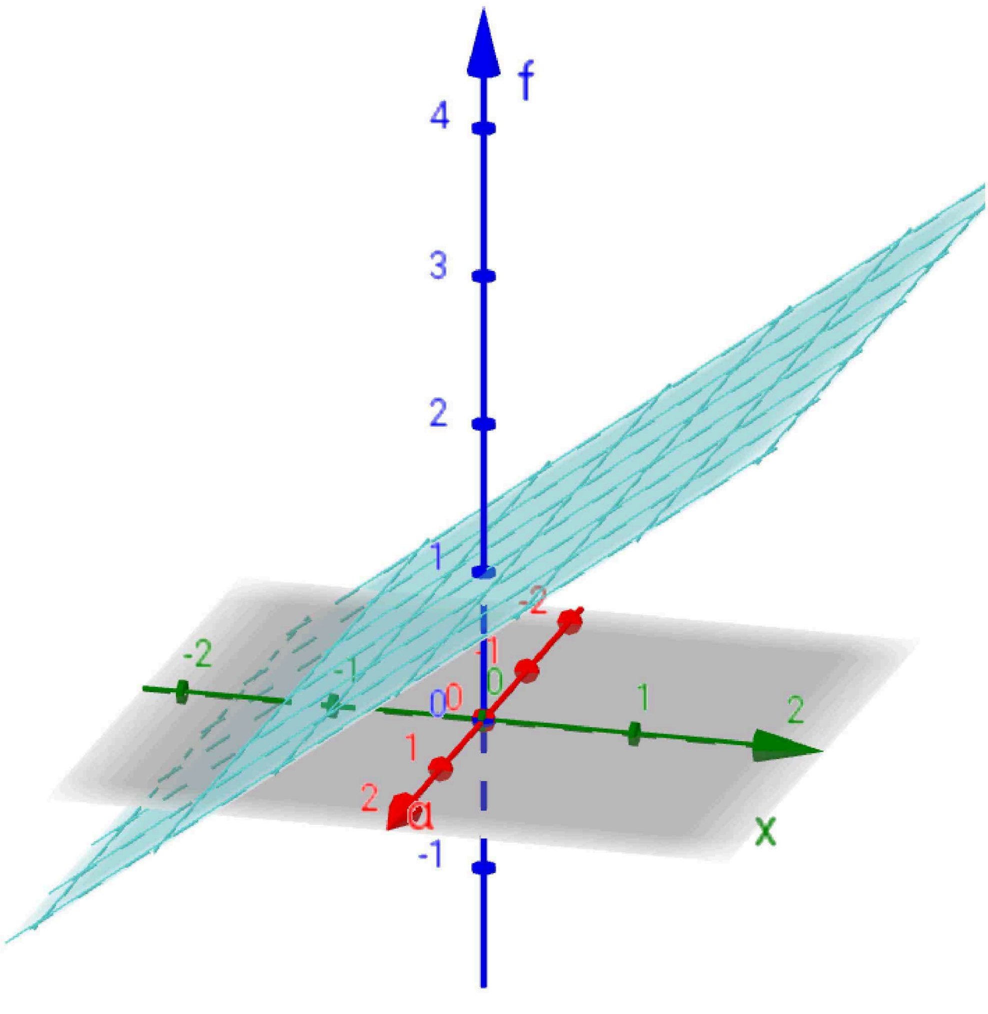


Figure 4.3: $f\left(x\right)=\frac{3}{4}x+1$

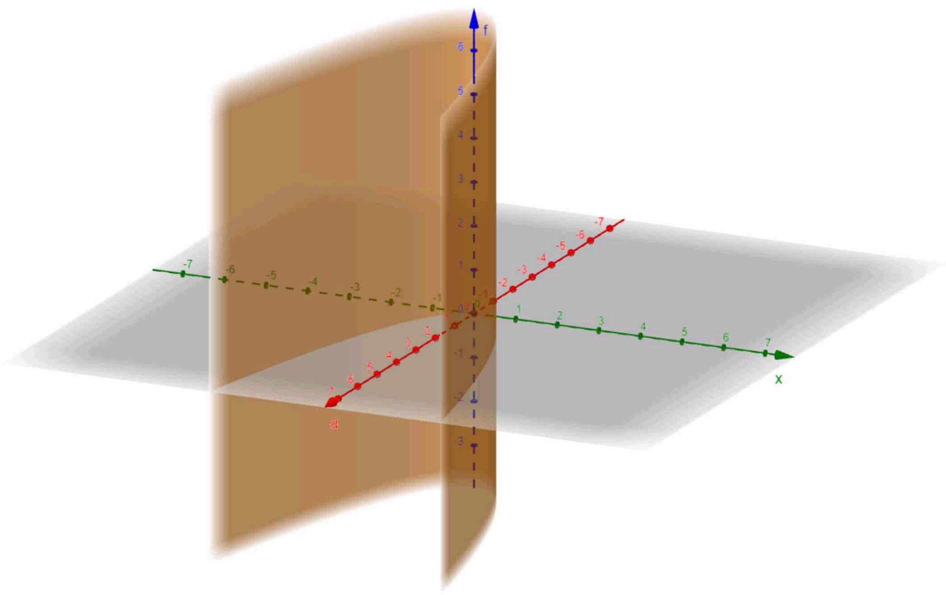


Figure 4.4: $\alpha(x) = x^2$

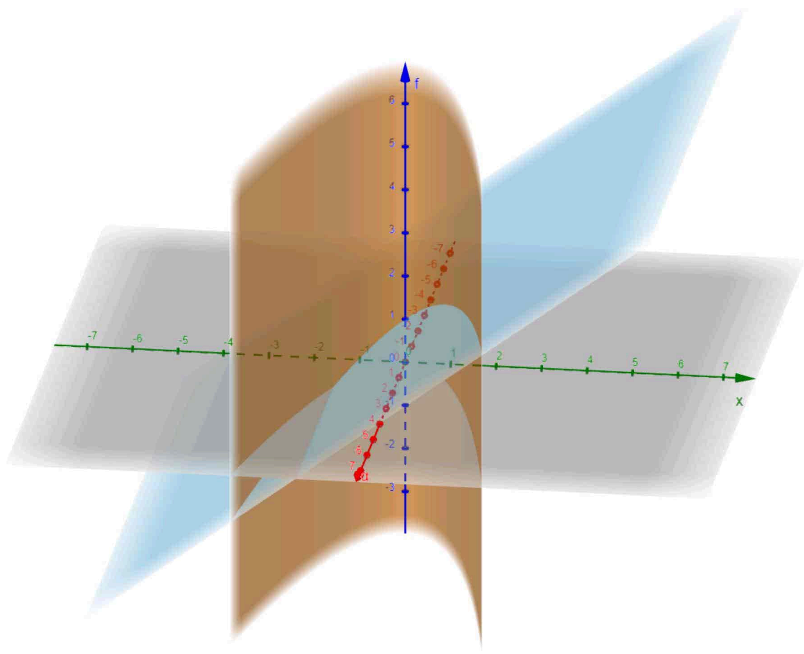


Figure 4.5: Geometric interpretation ofRiemann-Stieltjes integral

While graphing dependent sets , we assign a dimension to each set with direction perpendicular to the other sets. So $f(x)$ and $\alpha(x)$ are graphed as in fig 5.1 and fig 5.2. In order to consider x, $f(x)$ and $\alpha(x)$ simultaneously, each is assigned it's own dimension and direction perpendicular to that of the other two.

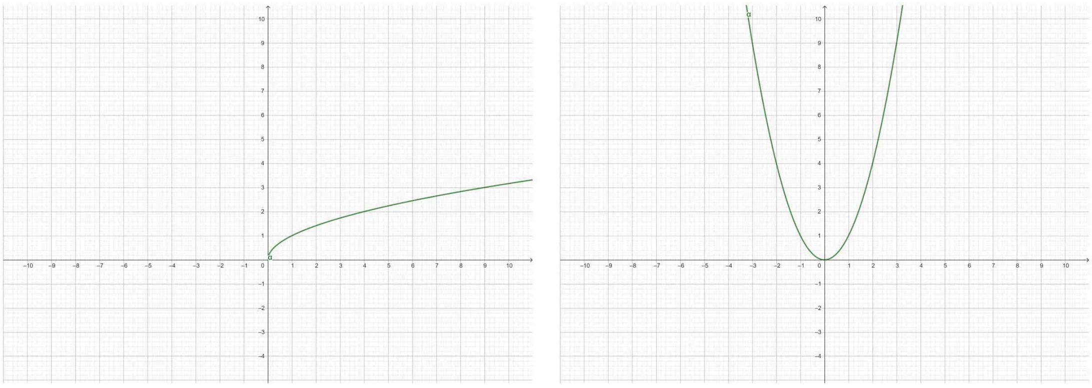
An important thing we need to notice over here is that because $f(x)$ is independent of $\alpha(x)$, therefore for a fixed value of x, $f(x)$ is constant along the $\alpha-$ axis. So $f(x)$ is a cylinder(or more intuitively, a sheet) that is straight in the $\alpha-$ direction. Therefore, if one looks along the sheet in the $x-$ direction, one may see hills and valleys, but one will see only flat terrain in the $\alpha-$ terrain.

Similarly, since $\alpha(x)$ is independent of $f(x)$, therefore the same result will hold for $f(x)$.

If we think of the (α, f) plane as horizontal and the $f-$direction as pointing straight up, then the surface to be considered is like a curved fence. The fence is along the curve $\alpha(x)$ and the height of the fence is given by $f(x)$. So, the fence is actually the section of the $\alpha-$sheet that is bounded between the (α, f) plane and the $f-$sheet.

The Stieltjes integral integrates along this fence. It is the sum of heights and infinitesimal widths. The height is taken as $f(x)$ in each subinterval but for the differential width it only considers $\Delta\alpha_k$, the length of the infinitesimal subinterval in the $\alpha-$direction.

As a result, the area of the integral is actually the area of the projection of the fence onto the (α, f) plane.



((a)) $\alpha(x) = \sqrt{x}$ ((b)) $\alpha(x) = x^2$

Figure 4.6: Graphs of integrators

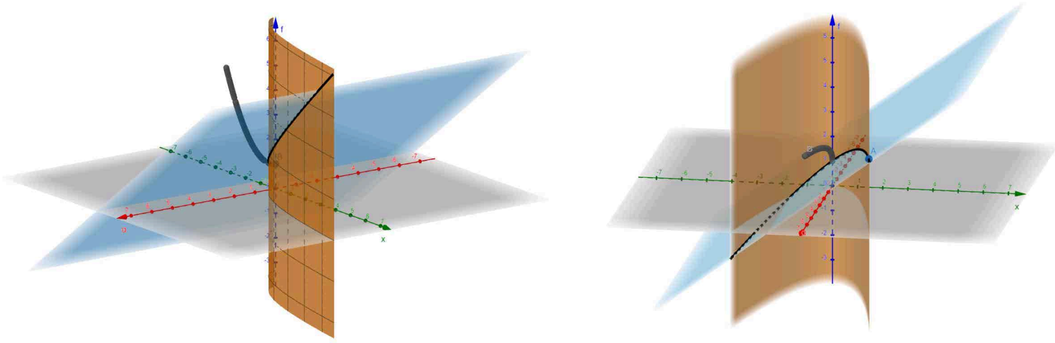
If a spotlight is placed with a beam parallel to the $x-$axis, so as to aim toward the fence, then the area given by the Riemann-Stieltjes integral is the shadow of the fence on the (α, f) plane.

If $\alpha(x) = x$, then the fence will be along a straight line from the origin that makes angle of 45^{deg} with $\alpha-$axis and $x-$axis. Therefore the fence will be symmetric with respect to the (α, f) plane and the (f, x) plane. Hence, the projection of the plane on the (α, f) plane will be symmetric to the projection of the fence on the (f, x) plane and therefore the area of projection on the (f, α) plane is equal to the area of projection on the (f, x) plane. Thus, the Riemann-Stieltjes integral reduces to Riemann integral.

Now let us consider the case where α is not a straight line.

Note that over here, since α is not symmetric with respect to the (α, f) and (f, x) planes, therefore the projection of the fence on (α, f) plane will not be equal to the projection on the (f, x) plane.

Define f as in figure 5.3 and let α be given by figure 5.6(a) and α_2 by figure 5.6(b).



((a)) $\alpha(x) = \sqrt{x}$ ((b)) $\alpha(x) = x^2$

Figure 4.7: Geometric interpretation of Riemann-Stieltjes integral for $f(x) = \frac{3}{4}x + 1$

Let α_1 and α_2 be differentiable. The corresponding figures show the projections and shadows on the (f, α) plane. If we observe the shadows carefully then we can see that the function α weighs the area of the shadow. The values of x for which $\alpha(x)$ has the steepest slope corresponds to regions of the fence that cast the most shadow and thereby carry the most value in the integral.

We know for a fact that the Riemann-Stieltjes integral is equal to the Riemann integral of $f(x)\alpha'(x)$. As we know, $\alpha'(x)$ is the slope of α at x. So we can observe that the regions of the fence in which $\alpha(x)$ has slope 0 cast no shadow at all , that is,

$$\int f d\alpha = \int f(x)\alpha'(x)\,dx$$

where $\alpha'(x) = \frac{d\alpha}{dx}$ is weighing function.

Chapter 5

Applications of Riemann-Stieltjes Integral

5.1 Application in Probability Theory

Definition 25 (Probability Space)**.** A probability space is a triple $(\omega, \mathcal{F}, \mathbb{P})$ where ω is a sample space, F is a $\sigma-$algebra of events and $\mathbb{P}$ is a probability measure on F.

Definition 26 (Sample Space)**.** The set of all possible outcomes for the random experiment we want to model.

Definition 27 (Event space)**.** A collection of all subsets of ω satisfying the following conditions ; making it into a $\sigma-$algebra: undefined.¨See the enumitem package documentation for explanation.˙Type␣␣H␣<return>␣␣for␣immediate␣help

$\phi \in \mathcal{F}$

$A \in \mathcal{F} \implies A^c \in \mathcal{F}$

$A_i \in \mathcal{F}$ for $i = 1, 2, \cdots \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Definition 28 (The function P)**.** The function P is a probability measure defined on $\mathcal{F}$ that is, on the function $P : \mathcal{F} \to [0, 1]$ such that : undefined.¨See the enumitem package documentation for explanation.˙Type␣␣H␣<return>␣␣for␣immediate␣help

$P(\phi) = 0$

$P(A^c) = 1 - P(A)$ where $A \in \mathcal{F}$

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \phi$ when $i \neq j$ where $A_i \in \mathcal{F}$

Example :

Flipping of a fair coin (Outcome is Heads (H) or (T)).

Sample space : $\omega = \{H, T\}$.

Event space : 4 events : $\{H\}, \{T\}, \{\}, \{H, T\}$

$\mathcal{F} = \{\{\}, \{H\}, \{T\}, \{H, T\}\}$

Probability measure : 50% chance of H and 50% chance of T.

$P(\{\}) = 0,\ P(\{H\}) = \frac{1}{2},\ P(\{T\}),\ P(\{H, t\}) = 1$

$\implies (\omega, \mathcal{F}, P)$ is a Probability space.

Definition 29 (Random variable)**.** A random variable X is a function $X : \omega \to \mathbb{R}$ such that $\{\omega : X(\omega) \leqslant x\} \in \mathcal{F}$ for every $x \in \mathbb{R}$. This condition is called measurability.

It ensures that one can define the distribution function F of X, which is defined as

$$F(x) = P(\omega : X(\omega) \leqslant x)$$

$\because$ the set $\{\omega : X(\omega) \leqslant x\}$ is in $\mathcal{F}$ $\therefore$ we can apply $\mathbb{P}$ on it.

Definition 30. Assume that X is a random variable with distribution function$F$. The expectation $E(X)$ of X is defined, if it exists, as

$$E(X) = \int x dF(x)$$

Note: The interpretation of the expectation $E(X)$ is clear from the Riemann-Stieltjes approximation

$$E(X) \approx \sum_{i=1}^{n} \xi_i \left(F(x_i) - F(x_{i-1})\right)$$

with $\xi_i \in (x_{i-1}, x_i]$.

Lemma 1:

Assume F is an increasing step function on I so that

$$F(t) = \sum_{i=1}^{N} a_i,$$

with $t_o = min(I) < t_1 < t_2 < \cdots < t_N = max(I)$ and $a_i \geqslant 0$

Then, if g is continuous,

$$\int_I g(x)\, dF(x) = \sum_{i=1}^{N} a_i$$

Proof. This Lemma is a corollary to Theorem 8 □

Theorem 32. *If X is a random variable and g as above then the random variable $Y = g(x)$ has expectation*

$$E(Y) = \int_{-\infty}^{\infty} g(x)\, dF(x)$$

Proof. A Riemann-Stieltjes sum for the LHS is

$$\begin{aligned}\sum_i \eta_i \left(F_Y\left(y_i\right) - F\left(y_{i-1}\right)\right) &= \sum_i \eta_i P\left(y \in \left(y_{i-1}, yi\right]\right) \\ &= \sum_i \eta_i P\left(g\left(X\right) \in \left(y_{i-1}, yi\right]\right) \\ &= \sum_i \eta_i P\left(X \in g^{-1}\{(y_{i-1}, yi]\}\right), \eta_i \in (y_{i-1}, yi]\end{aligned}$$

Note that

$$\begin{aligned}\eta_i \in (y_{i-1}, yi] &\iff \xi_i := g^{-1}\left(\eta_i\right) \in g^{-1}\{(y_{i-1}, yi]\} \\ &\iff g\left(\xi_i\right) \in (y_{i-1}, yi]\end{aligned}$$

$\therefore$ above is equal to $\sum_i g\left(\xi\right) P\left(X \in g^{-1}\{(y_{i-1}, yi]\}\right)$ with $\xi \in g^{-1}\{(y_{i-1}, yi]\}$

Note that if the intervals $(y_{i-1}, yi]$ form a partition (are disjoint and their union is the whole interval I)

$\therefore$ The above can be written as

$$\sum_i g\left(\xi_i\right) P\left(x \in (x_{i-1}, x_i]\right),$$

with $\xi_i \in (x_{i-1}, x_i]$, which is a Riemann-Stieltjes sum for the RHS.

Hence we are done.

□

Special case:

$g(X) = 1$ if $X \in A, A \subseteq \mathbb{R}$ This is a Bernoulli random variable, it's distribution function is a step function with jumps at 0 and 1, so that

$$\begin{aligned} E(g(X)) &= 0(F(0) - F(0-)) + 1(F(1) - F(1-)) \\ &= F(1) - F(1-) \\ &= P(g(X) = 1) \\ &= P(X \in A) \end{aligned}$$

Also, note that the LHS of this is

$$\begin{aligned} E(g(X)) &= \int_{-\infty}^{\infty} g(x)\, dF(x) \\ &= \int_A dF(x) \end{aligned}$$

and thus we get the formula

$$P(X \in A) = \int_A dF(x)$$

In particular, $1 = P(X \in \mathbb{R}) = \int_R dF(x)$

Theorem 33. *Let X be a random variable with distribution function F. Then*

$$E(aX + b) = aE(X) + b$$

Proof. We have

$$
\begin{aligned}
E(aX+b) &= \int_{-\infty}^{\infty} (aX+b)\, dF(X) \\
&= a\int_{-\infty}^{\infty} x dF(x) + b\int_{-\infty}^{\infty} dF(x) \\
&= aX(X) + b
\end{aligned}
$$

□

Definition 31 (Variance of a random variable)**.** The variance of a random variable X with distributive function is defined (if it exists) as

$$Var(X) = E\left(X^2\right) - E(X)^2$$

Proof.

$$
\begin{aligned}
Var(X) &= E((X-\mu)) \\
&= E\left(X^2 - 2X\mu + \mu^2\right) \\
&= E\left(X^2\right) - 2\mu E(X) + \mu^2 \\
&= E\left(X^2\right) - \mu^2 \\
&= E\left(X^2\right) - E(X)^2
\end{aligned}
$$

□

5.2 Application in Number Theory

5.2.1 Euler's Summation formula

The Euler's summation formula relates the integral of a function over an interval $[a, b]$ with the sum of the function values at the integers in $[a, b]$. Sometimes it can be used to approximate integrals by sums or, Conversely, to estimate the values of certain sums by means of integrals.

Theorem 34. *Euler's Summation formula If f has a continuous derivative f' on $[a, b]$, then we have*

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\,dx\,((x)) + f(a)\,((a)) - f(b)\,((b))$$

where $\sum\limits_{a<n\leqslant b}$ means sum from $n = [a]+1$ to $n = [b]$
When a and b are integers, this becomes

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\,dx\left(x - [x] - \frac{1}{2}\right)$$

Proof. Theorem 6 says that if $f \in R_\alpha[a, b]$ then

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

Let $\alpha(x) = x - [x]$.

$$\int\limits_a^b f(x)\,d(x-[x]) + \int\limits_a^b (x-[x]) = f(b)\,d(b-[b]) - f(a)\,d(a-[a]) \tag{1}$$

Now, since Riemann-Stieltjes integral is linear w.r.t. the integrator

$$\int_a^b f(x)\, d(x - [x]) = \int_a^b f(x)\, dx - \int_a^b f(x)\, d[x] \tag{2}$$

The greatest integer function has unit jumps at the integers $[a] + 1, [a] + 2, \cdots, [b]$.

$\therefore$ using Theorem 11, we get

$$\int_a^b f(x)\, d[x] = \sum_{a<n\leqslant b} f(n) \tag{3}$$

f has a continuous derivative f' on $[a, b]$. $\therefore$ By applying Theorem 7.8, we get

$$\int_a^b (x - [x])\, df(x) = \int_a^b (x - [x])\, f'(x)\, dx$$

$$\therefore \int_a^b (x - [x])\, df(x) = \int_a^b f'(x)\, ((x))\, dx \tag{4}$$

Substituting (2) and (4) in (1), we get

$$\int_a^b f(x)\, dx - \int_a^b f(x)\, d[x] + \int_a^b f'(x)\, d((x)) = f(b)\, ((b)) - f(a)\, ((a)) \tag{5}$$

Next, substituting (3) in (5),

$$\int_a^b f(x)\, dx - \sum_{a<n\leqslant b} f(n) + \int_a^b f'(x)\, d((x)) = f(b)\, ((b)) - f(a)\, ((a))$$

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(x)\, dx + \int_a^b f'(x)\, d((x)) + f(b)\, ((b)) - f(a)\, ((a))$$

□

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s \int_1^n \frac{[x]}{x^{s+1}} dx \text{if} s \neq 1 \tag{5.1}$$

Proof. Let $f(x) = \frac{1}{x^s}$, $s \neq 1$. Recall Theorem 7.11,

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k)\, \alpha_k. \tag{5.2}$$

□

$$\sum_{k=1}^{n} \frac{1}{k} = \log n - \int_1^n \frac{x - [x]}{x^2} dx + 1$$

Proof. Let $f(x) = \frac{1}{x}$

$$\int_1^n \frac{1}{x} d\left[x\right] = \sum_{k=2}^{n} \frac{1}{n}$$

$$\implies \int_1^n \frac{1}{x} + 1 = \sum_{k=1}^{n} \frac{1}{n}$$

$$\int_1^n \frac{1}{x} d\left[x\right] + \int_1^n \left[x\right] d\left(\frac{1}{x}\right) = \frac{1}{n} - \frac{1}{1}\left[1\right]$$

$$\implies \sum_{k=1}^{n} \frac{1}{k} = -\int_1^n \left[x\right] d\left(\frac{1}{x}\right)$$

$$= \int_1^n \frac{1}{x} dx - \int_1^n \frac{1}{x} dx - \int_1^n \left[x\right]\left(\frac{1}{x^2}\right) dx$$

$$= \log n - \int_1^n \frac{x}{x^2} dx + \int_1^n \frac{\left[x\right]}{x^2} dx$$

$$= \log n - \int_1^n \frac{x - \left[x\right]}{x^2} + 1$$

□

If f is continuous on $[1, 2n]$ then

$$\sum_{k=1}^{2n} (-1)^k f\left(k\right) = \int_1^{2n} f'\left(x\right)\left(\left[x\right] - 2\left[\frac{x}{2}\right]\right)$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{2n}(-1)^k f(k) &= -\sum_{k=1}^{2n} f(k) + 2\sum_{k=1}^{2k} \\
&= -\left(\int_1^{2n} f(x)\, d[x] + f(1)\right) + 2\left(\int_1^{2n} f(x)\, d\left[\frac{x}{2}\right]\right)
\end{aligned}$$

$$\begin{aligned}
&= \int_1^{2n} f'(x)[x]\, dx - 2nf(2n) - \int_1^{2n} f'(x)\left[\frac{x}{2}\right] dx + 2nf(2n) \\
&= \int_1^{2n} f'(x)\left([x] - 2\left[\frac{x}{2}\right]\right) dx
\end{aligned}$$

□

Chapter 6

Lebesgue-Stieltjes Integral

Definition 32 (Outer measure)**.** Let E be an arbitrary subset of $\mathbb{R}$. We define the outer measure of E to be the quantity

$$m^*(E) = \{\sum_{n=1}^{\infty} l(I_n) : (I_n) \text{ is a sequence, finite or infinite, such that } E \subseteq \bigcup_{n=1}^{\infty} I_n\}$$

Definition 33 (Measurable set)**.** A subset E of $\mathbb{R}$ is called a measurable set if given $\epsilon > 0$, there exists a closed set F and open set U such that $F \subset E \subset U$ and $m^*(U \backslash F) < \epsilon$.

Definition 34 (Measurable function)**.** Let E be a subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be measurable if $f^{-1}((\alpha, \infty)) = \{x \in E : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

Definition 35 (Borel sets)**.** The collection $\mathcal{B}$ of Borel sets is the smallest $\sigma-$ algebra which contains all of the open sets.

Definition 36 (Baire measure)**.** Let X be the set of real numbers and $\mathcal{B}$ the class of all Borel sets. A measure μ defined on $\mathcal{B}$ and finite for bounded sets is called a

Baire measure (on the real line.

To each finite measure we associate a function F by setting

$$F(x) = \mu(-\infty, x].$$

The function F is called the cumulative distribution function of μ and I real valued and monotonic increasing. We have

$$\mu(a, b] = F(b) - F(a)$$

Since $(a, b]$ is the intersection of the sets $\left(a, b + \frac{1}{n}\right]$, Proposition 11.2 implies that

$$\mu(a, b] = \lim_{n\to\infty} \mu\left(a, b + c\frac{1}{n}\right]$$

and so,

$$F(a, b] = \lim_{n\to\infty} F\left(b + \frac{1}{n}\right] = F(b+).$$

Thus a cumulative distribuion function is continuous on the right.

Similarly,

$$\begin{aligned}
\mu\{b\} &= \lim_{n\to\infty} \mu\left(a, b + \frac{1}{n}\right] \\
&= \lim_{n\to\infty} \{F(b) - F\left(b - \frac{1}{n}\right)\} \\
&= F(b) - F(b-)
\end{aligned}$$

Hence F is continuous at b if and only if the set $\{b\}$ consisting of b alone has measure 0.

Since $\phi = \bigcap_{n=1}^{\infty} (-\infty, -n]$, we have

$$\lim_{n \to -\infty} F(n) = 0$$

and hence

$$\lim_{x \to -\infty} F(x) = 0$$

because of the monotonicity of F. We summarize the properties in the following lemma:

Lemma 1. *If μ is a finite Baire measure on the real line, then it's cumulative distribution function F is a monotonic increasing bonded function, which is continuous on the right.*
Morever, $\lim_{x \to -\infty} F(x) = 0$.

Lemma 2. *Let F be a monotone increasing fuunction continuous on the right. If $[a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$, then*

$$F(b) - F(a) \leqslant \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

Proof:

$$(a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$$

$$\Longrightarrow \mu(a, b] = \mu\{\bigcup_{i=1}^{\infty} (a_i, b_i]\}$$

$$\Longrightarrow F(b) - F(a) \leqslant \sum \mu(a_i, b_i]$$

$$F(b) - F(a) = \sum_{i=1}^{\infty}\{F(b_i) - F(a_i)\}.$$

Theorem 35. *Let μ be a measure on an algebra $\mathcal{A}$ and μ^* the outer measure induced by μ. Then the restriction $\overline{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a σ-algebra containing $\mathcal{A}$. If $\mu$is finite (or $\sigma-$)-finite, so is $\overline{\mu}$. If μ is σ-finite, then $\overline{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{A}$ which is an extension of μ.*

Proposition 1. *Let $\mathcal{C}$ be a semialgebra of sets and μ a non negative set function defined on $\mathcal{C}$ with $\mu(\phi) = 0$ (if $\phi \in \mathcal{C}$). Then μhas a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{C}$ if the following conditions are satisfied:*

(i) If a set C in $\mathcal{C}$ is the union of a finite disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$, then $\mu(C) = \sum_{i=1}^{\infty} \mu(C_i)$

(ii) If a set C is the union of a countable disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$, then $\mu(C) \leqslant \sum_{i=1}^{\infty} \mu(C_i)$

Proposition 2. *Let F be a monotone increasing function which is continuous on the right, then there is a unique Baire measure μ such that for all a and b we have $\mu(a, b] = F(b) - F(a)$.*

Proof. □

Let $\mathcal{C}$ be a semi-algebra consisting of all intervals of the form $(a, b]$ or (a, ∞) and set $\mu(a, b] = F(b) - F(a)$,

then μ satisfies condition (i) of Proposition 1. Also Lemma 2 is precisely the condition (ii) of Proposition 1

therefore by Proposition 1, μ admits a unique extension to a measure on the

$\sigma-$algebra generated by $\mathcal{B}$.

By Theorem 35, this μ can be extended to a $\sigma-$algebra containing $\mathcal{C}$.

Since the class $\mathcal{B}$ of Borel sets is the smallest $\sigma-$algebra conatining $\mathcal{C}$, we have an extension of μ to a Baire measure. The measure μ is $\sigma-$finite, since X is the union of the intervals $(n, n+1]$ and each has finite measure. Thus, the extension of μ to $\mathcal{B}$ is unique.

Corollary 1. *Each bounded monotone function which is continuous on the right is the cumulative distribution function of a unique finite Baire measure provided* $F(-\infty) = 0$.

Definition 37 (Lebesgue-Stieltjes integral)**.** If ϕ is a non-negative Borel measurable function and F is a monotone increasing function which is continuous on the right, we define the Lebesgue-Stieltjes integral of ϕ wit respect to F to be

$$\int \phi dF = \int \phi d\mu$$

where μ is the Baire measure having F as its cumulative distribution function. If ϕ is both positive and negative, we say that it is integrable with respect to F if it is integrable wit respect to μ.

If F is any monotone increasing function, then there is a unique function F^* which is monotone increasing, continuous on the right, and agrees with F wherever F is continuous on the right, and we define the Lebesgue-Stieltjes integral of ϕ with respect to F by

$$\int \phi dF = \int \phi dF^*.$$

If F is a monotone function, continuous on the right, then $\int_a^b \phi dF$ agrees with the Riemann-Stieltjes integral whenever the latter is defined. The Lebesgue-Stieltjes integral is only defined when F is monotone(or more generally, of bounded variation) while the Riemann-Stieltjes integral can exist when F is not of bounded variation, say when F is continuous and ϕ is of bounded variation.

Chapter 7

Conclusion

Chapter 3: The Riemann-Stieltjes integral has basic features such as linearity with respect to integrand and integrator, as well as additivity with respect to the interval of integration. If the intergral exists, it can be simplified to the Riemann integral whenever the integrator has a continuous derivative. The integral can occur at a point where the integrad is continuous from left (or right), as long as the discontinuity is compensated for by the integrator's continuity from left(or right). For monotonic increasing integrators, the presence of a Riemann-Stieltjes integral implies that the integrand satisfies Riemann's condition with respect to the integrator. The existence of the Riemann-Stieltjes integral is guaranteed by the continuity of the integrand and the bounded variation of the integrator. When both the integrand and the integrator are discontinuous at the same point from the same side, the Riemann-Stieltjes integral cannot exist. The mean value theorems for the Riemann-Stieltjes integral can be used to estimate its value. Integrator functions that are not of bounded variation can be explored.

Chapter 4: Geometric interpretations of various functions have been done with the GeoGebra software. Other software packages, such as Scilab, can be used to graph more functions.

Chapter 5: The Riemann-Stieltjes integral is utilized in probability theory. The Riemann-Stieltjes integral makes it simple to prove theorems in number theory. The Riemann-Stieltjes integral is very useful in functional analysis. Applications of the Riemann-Stieltjes integral in Complex analysis can be investigated.

Chapter 6: The Lebesgue-Stiltjes integral has been defined. Properties of the Riemann-Stieltjes/Lebesgue integral that also apply to the Lebesgue-Stieltjes integral can be investigated. Various functions can be used to compare the Riemann-Stieltjes and Lebesgue-Stieltjes integrals.

Bibliography

[1] T.M. Apostol. *Mathematical Analysis*. Addison-Wesley series in mathematics. Addison-Wesley, 1974. ISBN: 9780201002881.

[2] Gregory L Bullock. "A geometric interpretation of the Riemann-Stieltjes integral". In: *The American Mathematical Monthly* 95.5 (1988), pp. 448–455.

[3] Klara Leffler. *The Riemann-Stieltjes integral: and some applications in complex analysis and probability theory*. 2014.

[4] Halsey Royden and Patrick Michael Fitzpatrick. *Real analysis*. China Machine Press, 2010.