# Cantor Sets : Generalizations & Interesting Properties

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# PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " Cantor Sets : Generalizations & Interesting Properties." This survey is divided into five chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

This report is mainly focsed on the standard Cantor set. The brief history,constrction and various properties will be discussed in detail.

In chapter 1, all the basic definations and results will be stated and proved and also a brief history about the origin of the Cantor set is given throughout the report.The Cantor set that is the Cantor 1/3 set is denoted by $\Delta$ and various generalizations are denoted by $\mathcal{C}$. Ternary representation of real numbers is discussed in detail.

In chapter 2, the Cantor Lebesgue function is introduced and its construction and various properties will be explored. At many places, two different proofs for one particular result will be given which are taken from various references.

Chapter 4 will be about various generalizations of the standard cantor set and their properties analogous to $\delta$. A special type of "Fat cantor set" known as SVC set will be discssed.

In chapter 5, the concept of fractral dimension will be discussed very briefly and fractral dimensions of few sets will be computed.

Overall in this report the Cantor set and all its related properties will be discussed which can be used for further applications of Cantor sets in Analysis and Topology.

# ACKNOWLEDGEMENTS

I would like to acknowledge and give my warmest thanks to my guide sir Dr. M Kunhanandan , Department of Mathematics, School of Physical and Applied Sciences , Goa University, who continuously helped me in completion of this dissertation. With his considerable experience and vast knowledge in almost all fields of Mathematics, he tackled all my queries whenever I approached him. Also his valuable inputs and insights helped us in better understanding of understand the concept. I would like to express my gratitude to him for suggesting correct references whenever I needed. His standards and his dedication towards Mathematics have definately inspired me to do better in future.

I would also like to thank my peers and friends during this Master's course with whom I could discuss the difficulties in this process.

Last but not the least I would like to thank the entire Mathematics department for giving us this opportunity and providing us with resources.

# ABSTRACT

The ternary Cantor set , constructed by George Cantor in 1883, is the best known example of a perfect nowhere-dense set in the real line. The present article we study the basic properties of the Cantor set and also study in detail the ternary expansion characterization . We then consider the Cantor-Lebesgue function defined on the Cantor set, prove its basic properties and study its continuous extension to $[0, 1]$. Cantor set as a mathematical object displays various intriguing properties. Although it looks sparse it is an uncountable set. The cantor set is used as a very good example in many theorems and proofs of and topology and measure theory. The Cantor function has various applications in Analysis(ref. [8]) Any compact metric space is a continuous image of the Cantor set is another such result. Thus in this particular report I have tried to study all the basic properties of this very special type of set in detail.

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# Chapter 1

# INTRODUCTION

The Cantor set is an unusual and intriguing subset of $\mathbb{R}$ , also referred to as "Cantor-ternary set" or "Cantor-middle third set". It shows various surprising properties and thus serves as an important counter example in topology , measure theory , and analysis.

Here we shall discuss all the properties of the cantor set in detail and construct the corresponding Cantor function. Later with the help of the standard Cantor set , generalizations of the Cantor set and also various different forms of the Cantor set has been discussed.

## 1.1 Brief History about of origin of Cantor set.

George Cantor(1845 - 1918) was the originator of modern set theory. Cantor first mentioned the Cantor set in connection with the concept of "perfect" sets. But the set itself was not discovered by Cantor. H.J.S. Smith in 1875 introduced the Cantor set in connection with the construction for nowhere dense sets.

The first construction of a perfect, nowhere dense set was by the British mathematician Henry J. S. Smith (1826-1883) in 1875. Smith, who taught at Balliol College in Oxford and was appointed Savilian professor of geometry in 1860, is known primarily for his work in number theory. Not many mathematicians were aware of Smith's construction, a fate that was shared by some of his other ground breaking work. Most of the exciting mathematics was happening in Germany and France, and that is where attention was focused. In 1881, Vito Volterra showed how to construct such a set, but Volterra was still a graduate student, and he published in an Italian journal that was not widely read. Again, little notice was paid. Finally, in 1883, Cantor rediscovered this construction for himself, and suddenly everyone knew about it. Cantor's example is known as the Cantor ternary set.

## 1.2 Preliminaries and Definitions

First let's recall some basic concepts from Analysis on $\mathbb{R}^n$

Let $(\mathbb{R}^n, d)$ be a metric space .

**Definition 1.1.** (Open Ball)

Let $\overline{x} \in \mathbb{R}^n$ and $r > 0$. We define the set $B(\overline{x}, r) = \{\overline{y} \in \mathbb{R}^n | d(\overline{x}, \overline{y}) < r\}$. Then $B(\overline{x}, r)$ is called as the open ball with center $\overline{x}$ and radius $r$.

**Definition 1.2.** (Open set)

A subset $U$ of $\mathbb{R}^n$ is called an open set if for each $\bar{x} \in U$ , $\exists\ r > 0$ s.t. $B(\overline{x}, r) \subset U$

**Definition 1.3.** (Interior point)

$S \subset \mathbb{R}^n$. A point $x \in S$ is said to be an interior point of S if $\exists\, r > 0$ s.t. the open ball $B(x, r) \subset S$.

Interior of a set is an open set.

**Definition 1.4.** (Closed Set)

A subset $E$ of $\mathbb{R}^n$ is said to be closed if its complement , $E^c$ is open.

**Definition 1.5.** (Limit point of a set)

Let $A \subseteq \mathbb{R}^n$. A point $\overline{x_0} \in \mathbb{R}^n$ is said to be a limit point of $A$ if

$\forall\, r > 0, (B(\overline{x}, r) \cap A) \setminus \{x_0\} \neq \emptyset$

**Theorem 1.6.** *Let $A \subset \mathbb{R}^n$, a point $x \in A$ is a limit point of $A$ iff $\exists\,(x_n)$ , a sequence in $A$ s.t. $x_n \longrightarrow x$.*

**Definition 1.7.** (Derived set)

Let E be a subset of $\mathbb{R}^n$. The set of all limit points of $E$ is called as derived set of $E$ and denoted by $E'$ .

**Definition 1.8.** (Nowhere dense set)

A subset A of a metric space M is nowhere dense if its closure has an empty interior. That is $int(A) = \phi$.

**Definition 1.9.** (Perfect set)

A set is perfect if it is equal to its derived set. In other words, $S$ is perfect if and only if every point of $S$ is an accumulation point of $S$, and all accumulation points of $S$ are in $S$.

**Definition 1.10.** (Outer Measure)

For any subset of $\mathbb{R}$, say $E \subset \mathbb{R}$, the Lebesgue outer measure $m^*(E)$ is defined as an infimum, which is represented by

$$m^*(E) = \inf \left\{ \sum l(I_n) \mid (I_n) \text{ is a sequence of intervals s.t. } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

where $l(I_n) =$ length of the interval $I_n$

* If $I_n = (a, b)$ or $[a, b]$ then length of $I_n = l(I_n) = b - a$

**Proposition 1.11.** *Properties of outer measure*

- *The outer measure $m^*$ is a real-valued set function defined on a space $X$ with the properties:*
  - *It is a non-negative set function defined for all subsets of X*
  - $m{*}(O) = 0$*, i.e. the outer measure of the empty set is zero.*
  - *It is monotone increasing, i.e. if $A \subseteq B$then $m^*(A) \leq m^*(B)$*

- *It is countably subadditive, i.e.* $m^*(\bigcup A_j) \leq \sum m^*(A_j)$
- *It is countably additiveb for any pairwise disjoint sets* $A_j$*. That is,* $m^*(\bigcup A_j) = \sum m^*(A_j)$ *if* $A_j$*'s are pairwise disjoint.*

**Definition 1.12.** (Separated sets)

Two subsets $A, B$ of $\mathbb{R}^n$ are said to be separated if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$

**Definition 1.13.** (Connected set)

A subset $E$ of $\mathbb{R}^n$ is said to be connected if $E$ cannot be written as the union of two non-empty seperated sets.

**Definition 1.14.** (Totally disconnected set)

If E contains no interval then E is totally disconnected.

A set S is called totally disconnected if for each distinct $x, y \in S$ there exist disjoint open set $U$ and $V$ such that $x \in U, y \in V$ and $(U \cap S) \cup (V \cap S) = S$.

A totally disconnected space is a topological space that has no non-trivial (singletons and empty set) connected subsets. In other words, the only connected components in any totally disconnected space X are the one-point sets.

**Definition 1.15.** (Self Similar sets)

A set S is self similar if it can be divided into N congruent subsets, each of which when magnified yields the entire set S.

**Definition 1.16.** (Fractal dimension)

Let $S$ be a compact set and $N(S, r)$ be the minimum number of balls of radius $r$ needed to cover $S$. Then the fractal dimension of $S$ is defined as

$$dim\, S = \lim_{r \to 0} \frac{logN(S, r)}{log(1/r)}$$

## Fractals

**Definition 1.17.** (Fractals)

A fractal is an object or quantity that displays self-similarity. Fractal is a pattern that repeats forever, and every part of the Fractal, regardless of how zoomed in, or zoomed out you are, it looks very similar to the whole image.

- It is a curve or geometrical figure, each part of which has the same statistical character as the whole.
- A fractal is a never-ending pattern. Fractals are infinitely complex patterns that are self-similar across different scales.
- A Fractal is a type of mathematical shape that is infinitely complex. In essence, a Fractal is a pattern that repeats forever, and every part of the Fractal, regardless of how zoomed in, or zoomed out you are, it looks very similar to the whole image.

**Definition 1.18.** (Absolutely Continuous functions)

A real-valued function $f$ on a closed, bounded interval $[a, b]$ is said to be absolutely continuous on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $(a_k, b_k)_{k=1}^{n}$ of open intervals in $(a, b)$, if

$$\sum_{k=1}^{n} |b_k - a_k| < \delta, \text{ then } \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$$

**Definition 1.19.** (Dyadic rational) In mathematics, a dyadic rational or binary rational is a number that can be expressed as a fraction whose denominator is a power of two. For example, 1/2, 3/2, and 3/8 are dyadic rationals, but 1/3 is not.

A rational number p/q in simplest terms is a dyadic rational when q is a power of two. Another equivalent way of defining the dyadic rationals is that they are the real numbers that have a terminating binary representation.

The dyadic rationals are precisely those numbers possessing finite binary expansions. Their binary expansions are not unique; there is one finite and one infinite representation of each dyadic rational other than 0 (ignoring terminal 0s). For example, 0.112 = 0.10111...2, giving two different representations for 3/4. The dyadic rationals are the only numbers whose binary expansions are not unique.

Analogous to dyadic rationals we have ternary rationals the only difference being denominator is a power of 3 instead of 2. That is a rational number $\frac{p}{q}$ in simplest terms is a ternary (tri-adic) rational when q is a power of three.

# Chapter 2

# The standard Cantor set

The standard Cantor set or the Cantor ternary set, denoted by $\Delta$ is an unusual subset of $[0, 1]$ , which is uncountable, perfect , totally disconnected and nowhere dense. This is an example of an Uncountable set which has Lebesgue measure zero. we will prove all the above stated properties in the subsequent section, before that let us look at the construction of the Cantor set.

## 2.1 The Cantor ternary set

### 2.1.1 Construction of the Cantor ternary set

The Cantor set $\Delta$ is produced by the iterated process of removing the middle third from the previous segments. Begin with the closed real interval $[0, 1]$ and divide it into three equal open sub intervals. Remove the central open interval $(\frac{1}{3}, \frac{2}{3})$. Removing the middle

third, leaving us with the union of two closed intervals of length $\frac{1}{3}$ each.
Now we remove the middle third from each of these intervals, leaving us with the union of four closed intervals of length $\frac{1}{9}$ each.

We continue this process inductively, then for each n=1,2 3,.. we get a set $C_n$ which is the union of $2^n$ closed intervals of length $\frac{1}{3^n}$. This iterative construction is illustrated in the following figure, for the first four steps:

Consider closed interval $[0, 1]$. Let $C_0 = [0, 1]$

$C_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3})$

$= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = ([0, \frac{1}{3}] \setminus (\frac{1}{3^2}, \frac{2}{3^2})) \bigcup([\frac{2}{3}, 1] \setminus (\frac{7}{3^2}, \frac{8}{3^2}))$

$= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$C_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup \cdots \cup [\frac{3^n-3}{3^n}, \frac{3^n-2}{3^n}] \cup [\frac{3^n-1}{3^n}, 1]$

Continuing this way we obtain a sequence of sets such that

1. $C_1 \supset C_2 \supset C_3 \cdots$
2. $C_n$ is the union of $2^n$ intervals, each of length $\frac{1}{3^n}$
3. The set $\Delta = \bigcap_{n=1}^{\infty} C_n$ is called the Cantor set.

### 2.1.2 Illustration of the Construction

The following is the illustration of the first few iterations of the cantor set.

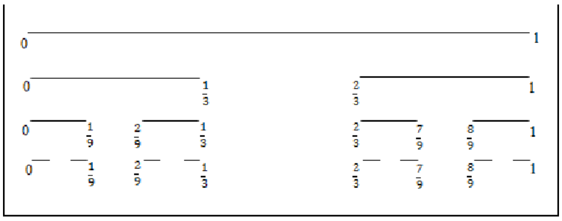


Figure 2.1: Cantor ternary set

### 2.1.3 Basic results

**Observation 2.1.** *The Cantor set is non empty.*

Note that in process of construction we never removed the endpoints. for e.g. $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9} \cdots \in \Delta$. So $\Delta \neq \phi$.

Also note that $(C_n)$ is a decreasing sequence of non empty closed and compact sets. Therefore by the Cantor intersection theorem , $\Delta$ is non-empty.

**Theorem 2.2.** $\Delta$ *is uncountable.*

*Proof.* This is quite a surprising result as though The Cantor set looks very discrete and scarce, we can actually prove that it is uncountable.

The proof that $\Delta$ is uncountable is based on characterization of $\Delta$ in terms of *ternary* (base 3) decimals. This we shall discuss in the next section. □

**Proposition 2.3.** $\Delta$ *is closed.*

$\Delta$ is a closed set, being the countable intersection of closed sets, and trivially bounded, since it is a subset of [0,1]. Therefore, by the Heine-Borel theorem $\Delta$ is a compact set.

**Proposition 2.4.** *The cantor set has Lebesgue measure zero.*

*Proof.* The Cantor set $\Delta$ is an intersection of union of closed intervals. Thus complement of $\Delta$ in $[0,1]$ is a union of disjoint open intervals. Therefore , from additivity of outer measure, (ref : 1.11 ) length of the removed intervals at each stage that is length of $[0,1] \setminus C_n$ for each $n$ will add up.

$$\begin{aligned}
\text{Total length removed } &= \frac{1}{3} + 2(\frac{1}{3})^2 - 2^2(\frac{1}{3})^3 + \cdots \\
&= \frac{1}{3}\Big(1 + \frac{2}{3} + (\frac{2}{3})^2 + \cdots\Big) \\
&= \frac{1}{3}\Big(\frac{1}{1-\frac{2}{3}}\Big) \\
&= \frac{1}{3}(\frac{1}{\frac{1}{3}}) \\
&= 1
\end{aligned}$$

Thus the total length of the removed intervals is 1

Implies,the remaining length that is the length of the intervals in $\Delta$

= length of $[0,1] - 1$

$= 1 - 1 = 0$

Thus $\Delta$ has outer measure zero. $\square$

## 2.2 Ternary Expansion and the Cantor set

There is an alternative characterization of C, the ternary expansion characterization. Consider the ternary representation for $x \in [0, 1]$.We shall describe this in detail in the following sections.

For that let us briefly discuss binary, ternary and decimal expansion of real numbers.

Let $p \in \mathbb{N}$ be greater than or equal to 2. Then we want to show that there exists p-expansion for any real number $x$ in the following sense: There exists an integer $x_0$ and numbers $a_k$ lying in $\{0, 1, ..., p - 1\}$ such that $x = x_0 + \displaystyle\sum_k \frac{a_k}{p^k}$

It will suffice to consider real numbers between 0 and 1, since the representations for other real numbers can then be obtained by adding a positive or negative integer. Any real number $x$ can be written in the form $x = x_0 + a$, where $x_0 \in \mathbb{Z}$ and $a \in [0, 1)$. This suggests that it suffices to consider only $x \in [0, 1]$. The most widely used are the cases when p=2,3,4 ; and we shall explore these three cases only. When $p = 2, 3, 10$, the expansions are respectively called **binary, ternary and decimal**. We shall discuss this in detail in the following sections.

### 2.2.1 The decimal Expansion of real numbers

**Definition 2.5.** (Positive decimal)

An expression of the form $a_0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \cdots$

where $a_0$ is a positive integer and $a_k \in \{0, 1, 2, \cdots 9\}$ for each $n \in \mathbb{N}$

This series represents the number $a_0.a_1a_2 \ldots a_n \cdots$

If $a_0 \in \mathbb{Z}^+$ then $a_0 = z_m 10^m + z_{m-1} 10^{m-1} + \cdots + z_0 10^0$ for some $m \in \mathbb{N}$

Then the above series representation can also be written as

$$a_0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \cdots$$
$$= z_m 10^m + z_{m-1} 10^{m-1} + \cdots + z_0 10^0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \cdots.$$

The following are some of the well known results on the decimal expansion of real numbers. They are just stated here without giving proofs.

**Proposition 2.6.**
- *Every infinite decimal is a real number.*
- *Every (positive) real number has an infinite decimal representation.*
- *A real number is rational if and only if it has periodic (repeating) decimal representation.*
- *This representation is unique except for rational numbers of the form $\frac{m}{10^n}$ for some $m, n \in \mathbb{N}$ in its simplest form.*

### 2.2.2 Binary Expansion of numbers

Similar to the properties of decimal expansion we have binary expansion of real numbers. This we shall formally construct and prove.

If $x \in [0, 1]$, we will use a repeated bisection procedure to associate a sequence $(\eta_n)$ of 0's and 1's as follows.

If $x \neq \frac{1}{2}$ belongs to the left sub interval $[0, \frac{1}{2}]$ we take $\eta_1 := 0$, while if x belongs to the right sub interval $[\frac{1}{2}, 1]$ we take $\eta_1 := 1$. If $x = \frac{1}{2}$, then we may take $\eta_1$ to be either 0 or 1. In any case, we have

$$\frac{\eta_1}{2} \leq x \leq \frac{\eta_1 + 1}{2}$$

. We now bisect the interval $[\frac{1}{2}\eta_1, \frac{1}{2}(\eta_1 + 1)]$. If $x$ is not the bisection point and belongs to the left sub interval we take $\eta_2 := 0$, and if $x$ belongs to the right sub interval we take $\eta_2 := 1$. If $x = \frac{1}{4}$ or $x = \frac{3}{4}$, we can take $\eta_2$ to be either 0 or 1. In any case, we have

$$\frac{\eta_1}{2} + \frac{\eta_2}{2^2} \leq x \leq \frac{\eta_1}{2} + \frac{\eta_2 + 1}{2^2}$$

We continue this bisection procedure, assigning at the $n^{th}$ stage the value $\eta_n := 0$ if $x$ is not the bisection point and lies in the left subinterval, and assigning the value $\eta_n := 1$ if $x$ lies in the right subinterval. In this way we obtain a sequence $(\eta_n)$ of 0's or 1's that correspond to a nested sequence of intervals containing the point $x$. For each n, we have the inequality

$$\frac{\eta_1}{2} + \frac{\eta_2}{2^2} + \cdots \frac{\eta_{n-1}}{2^{n-1}} + \frac{\eta_n}{2^n} \leq x \leq \frac{\eta_1}{2} + \frac{\eta_2}{2^2} + \cdots + \frac{\eta_{n-1}}{2^{n-1}} + \frac{\eta_n + 1}{2^n} \tag{2.1}$$

If $x$ is the bisection point at the $n^th$ stage, then $x = \frac{m}{2n}$ with m odd. In this case, we may choose either the left or the right subinterval; however, once this subinterval is chosen, then all subsequent subintervals by the bisection procedure are determined. For instance, if we choose the left subinterval so that $\eta_n = 0$, then x is the right endpoint of all subsequent subintervals, and hence $\eta_k = 1$ for all $k \geq n+1$. On the other hand, if we choose the right subinterval so that $\eta_n = 1$, then $x$ is the left endpoint of all subsequent subintervals, and hence $\eta_k = 0$ for all $k \geq n + 1$. For example, if $x = \frac{3}{4}$, then the two possible expansions for $x$ are $0.10111\cdots_{\text{(base 2)}}$ and $0.11000\cdots_{\text{(base 2)}}$

To summarize: If $x \in [0,1]$, then there exists a sequence $(\eta_n)$ of 0's and 1's such that inequality above holds $\forall n \in \mathbb{N}$. In this case we write

$$x = 0.\eta_1\,\eta_2\,\eta_3\,\cdots\eta_n\cdots_{\text{(base 2)}}$$

where $\eta_i \in \{0,1\}$and call it binary representation of $x$.This representation is unique except when $x = \frac{m}{2n}$ with m odd, in which case $x$ has two representations

$$\begin{aligned} x &= 0.\eta_1\,\eta_2\,\eta_3\,\cdots\eta_{n-1}1000\cdots \text{(base 2)} \\ &= 0.\eta_1\,\eta_2\,\eta_3\,\cdots\eta_{n-1}0111\cdots \text{(base 2)} \end{aligned}$$

one ending in 0's and the other ending in 1's. Conversely, each sequence of 0's and 1's is the binary representation of a unique real number in $[0,1]$. The inequality (2.1) determines a closed interval with length $\frac{1}{2^n}$ and the sequence of these intervals is nested. Therefore, the nested interval property implies that there exists a unique real number $x$ satisfying the inequality for every $n \in \mathbb{N}$. Consequently, $x$ has the binary representation

$$x = 0.\eta_1\,\eta_2\,\eta_3\,\cdots\eta_n\cdots_{\text{(base 2)}}$$

Thus any real number $x \in [0,1]$ can be represented as the sum of series

$$x = \frac{\eta_1}{2^1} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + \frac{\eta_4}{2^4} + \cdots \tag{2.2}$$

where , $\eta_n$ is an integer s.t. $0 \le \eta_n \le 1$ for each $n \in \mathbb{N}$, which represents the number $x = 0.\eta_1\,\eta_2\,\eta_3\,\cdots\eta_n\cdots_{\text{(base 2)}}$

This is called as binary representation of the number.

**Examples**

1. $x = \frac{1}{2} = 0.1 = 0.0111\cdots_{\text{(base 2)}}$

   - If $[0,1]$ is bisected then $\frac{1}{2}$ is the midpoint and it is a dyadic rational number. Thus there as discussed earlier it has 2 binary expansions.

2. $x = \frac{1}{4} = 0.01 = 0.00111\cdots_{\text{(base 2)}}$

3. $x = \frac{3}{2} = 1.1 = 1.0111\cdots_{\text{(base 2)}}$

4. $x = \frac{1}{16} = 0.0001 = 0.0000111\cdots_{\text{(base 2)}}$

5. $x = \frac{5}{9} = 0.\overline{100011}_{\text{(base 2)}}$

6. $x = 1 = 0.111\cdots_{\text{(base 2)}}$

### 2.2.3 Ternary Representation

Analogously, if $x \in [0,1]$, we will use a repeated trisection procedure to associate a sequence $(\varepsilon_n)$ of 0's, 1's and 2's as follows. If $x \neq \frac{1}{3}$ and belongs to the left sub interval $[0, \frac{1}{3}]$ we take $\varepsilon_1 := 0$, if x belongs to the right sub interval $[\frac{1}{3}, \frac{2}{3}]$ we take $\varepsilon_1 := 1$ while if x belongs to the right sub interval $[\frac{2}{3}, 1]$ we take $\varepsilon_1 := 2$. If $x$ is the end points of each of the sub intervals ; that is if $x = \frac{1}{3}$, then we may take $\varepsilon_1$ to be either 0 or 1, and if $x = \frac{2}{3}$, then we may take $\varepsilon_1$ to be either 1 or 2. In any case, we have

$$\frac{\varepsilon_1}{3} \leq x \leq \frac{\varepsilon_1 + 1}{3}$$

We now trisect the interval $[\frac{1}{3}\varepsilon_1, \frac{1}{3}(\varepsilon_1+1)]$ If x is not the trisection point and belongs to the left sub interval we take $\varepsilon_2 := 0$, and if $x$ belongs to the right sub interval we take $\varepsilon_2 := 1$. If $x = \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}$ or $\frac{8}{9}$, we can take $\varepsilon_2$ to be either 0, 1 or 2. In any case we have

$$\frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3^2} \leq x \leq \frac{\varepsilon_1}{3} + \frac{\varepsilon_2 + 1}{3^2}$$

We continue this process , at each $n^{th}$ stage assigning the value $\varepsilon_n := 0$ if $x$ is not the trisection point and lies in the left sub interval, assigning the value $\varepsilon_n := 1$ if $x$ lies in the middle sub interval and assigning the value $\varepsilon_n := 2$ if $x$ lies in the right sub interval. In this way we obtain a sequence $(\varepsilon_n)$ of 0's, 1's or 2's that correspond to a nested sequence of intervals containing the point $x$. For each $n$, we have the inequality,

$$\frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3^2} + \cdots \frac{\varepsilon_{n-1}}{3^{n-1}} + \frac{\varepsilon_n}{3^n} \leq x \leq \frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3^2} + \cdots + \frac{\varepsilon_{n-1}}{3^{n-1}} + \frac{\varepsilon_n + 1}{3^n} \tag{2.3}$$

If $x$ is the trisection point at the $n^{th}$ stage, then $x = \frac{m}{3^n}$. In this case, we may choose either the left or the right sub interval; however, once this sub interval is chosen, then all subsequent sub intervals in the trisection procedure are determined. (For instance, if we choose the left sub interval so that $\varepsilon_n = 0$, then $x$ is the right endpoint of all subsequent sub intervals, and hence $\varepsilon_k = 1$ for all $k \geq n+1$. On the other hand, if we choose the right sub interval so that $\varepsilon_n = 2$, then $x$ is the left endpoint of all subsequent subintervals, and hence $\varepsilon_k = 0$ for all $k \geq n+1$.

For example, if $x = \frac{2}{3}$, then the two possible sequences are $2, 0, 0, 0, \cdots$ and $1, 2, 2, 2, \cdots$

That is, if $x = \frac{2}{3}$, then the two possible representations of $x$ are

$$
\begin{aligned}
x = \frac{2}{3} &= 0.2000\cdots_{\text{(base 3)}} \\
&= 0.1222\cdots_{\text{(base 3)}}
\end{aligned}
$$

Thus to summarize:

- If $x \in [0, 1]$, then there exists a sequence $(\varepsilon_n)$ of 0's, 1's and 2's such that inequality above holds for all $n \in \mathbb{N}$. In this case we write

  $$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_n\cdots_{\text{(base 3)}}$$

  and call it as ***ternary representation*** of $x$.

  (Also called representation of x in ***base 3***)

- This representation is unique except when $x = \frac{m}{3^n}$ for some $m$, in which case $x$ has the two representations one ending in 0's and the other ending in 2's

  $$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}1000\cdots_{\text{(base 3)}}$$

  $$= 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}0222\cdots_{\text{(base 3)}}$$

- The inequality in (2.3) determines a closed interval with length $\frac{1}{3^n}$ and the sequence of these intervals is nested. Therefore,the nested interval property implies that there exists a unique real number x satisfying the inequality for every $n \in \mathbb{N}$.

Consequently, $x$ has the ternary representation

$$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3 \cdots \varepsilon_n \cdots_{\text{(base 3)}}$$

.

- Any $x \in [0, 1]$ can be written in the form

$$x = \frac{\varepsilon_1}{3^1} + \frac{\varepsilon_2}{3^2} + \frac{\varepsilon_3}{3^3} + \frac{\varepsilon_4}{3^4} + \cdots \tag{2.4}$$

where , $\varepsilon_n$ is an integer s.t. $0 \leq x_n \leq 2$ for each $n \in \mathbb{N}$ and this represents the real number $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3 \cdots \varepsilon_n \cdots_{\text{(base 3)}}$ and this is called as ternary representation of $x$

- For any ternary(tri-adic) rational number $\frac{m}{3^n}$ there are two possible ternary expansions, because

$$\begin{aligned}\frac{m}{3^n} &= \frac{m-1}{3^n} + \frac{1}{3^n} \\ &= \frac{m-1}{3^n} + \sum_{k=n+1}^{\infty} \frac{2}{3^k}\end{aligned}$$

Depending on where the rational number $\frac{m-1}{3^n}$ lies among the three sub intervals created at each stage in the trisection process, we will have the ternary representation of $\frac{m-1}{3^n}$ as we have discussed earlier.

- Therefore the decimal,binary (dyadic) and the ternary representations are unique except for the decimal fraction or dyadic or ternary rational numbers,() in that case we will take the infinite expansions representations for the dyadic and ternary rational numbers.

**Some examples for the illustration.**

$$
\begin{aligned}
(0.10101)_{\text{(base 3)}} &= 1/3 + 0/3^2 + 1/3^4 + 1/3^5 \\
&= 91/243 \\
(0.22222....)_{\text{(base 3)}} &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \\
&= \frac{1}{2}\Big(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\Big) \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) \\
&= 1
\end{aligned}
$$

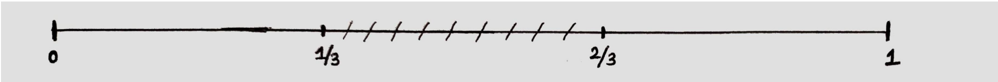
$$
\begin{aligned}
\frac{1}{3} &= 0.1_{\text{(base 3)}}\, 0.22222...._{\text{(base 3)}} \\
\frac{2}{3} &= 0.2_{\text{(base 3)}} = 0.1222\cdots_{\text{(base 3)}} \\
\frac{1}{4} &= 0.010101\cdots_{\text{(base 3)}} \\
\frac{7}{9} &= (0.21)_3 = 0.02222\cdots_{\text{(base 3)}}
\end{aligned}
$$

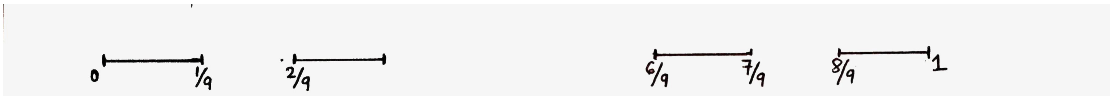
Observe that the rational numbers of the form $\frac{m}{3^n}, n \in \mathbb{N}$ have 2 different expansions, also note that in the construction of the Cantor set the end points of the sub intervals created at each $n^{th}$ stage are of the form $\frac{m}{3^n}$.

Now we characterize the elements in the Cantor set.

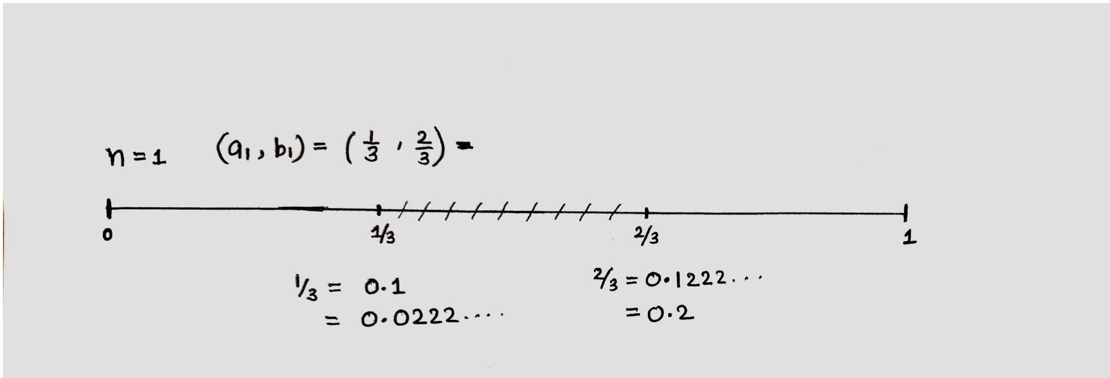
**Theorem 2.7.** *$x \in \Delta$ if and only if $x$ can be written as $\displaystyle\sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k}$ , where $a_n = 0$ or $2$.*

*Proof.* Recall in the Construction of cantor set we divide the unit interval into three parts and then remove the central open interval.



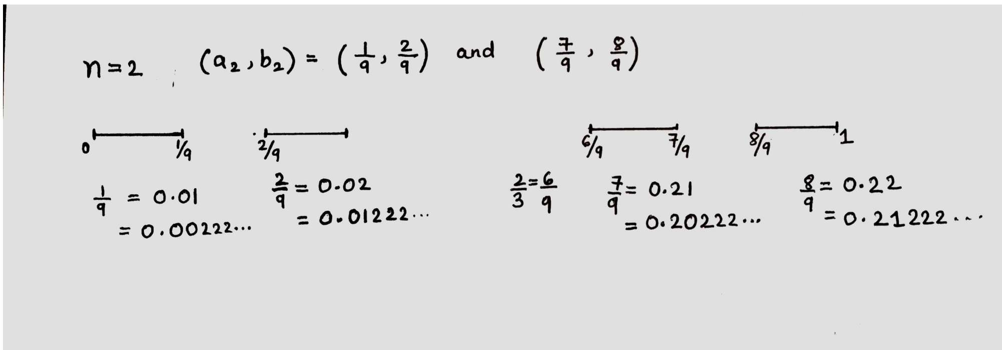

We have seen that, for $n = 1$, an interval $(a_n, b_n) = (a_1, b_1)$ is removed if $a_1 = 0.1_{\text{(base 3)}} = 0.0\bar{2}_{\text{(base 3)}}$ and $b_1 = 0.2_{\text{(base 3)}}$.




For $n = 1$, we remove points between $\frac{1}{3}$ and $\frac{2}{3}$.
That is between $0.0\bar{2}_{(base3)} = 0.0222\cdots_{\text{(base 3)}}$ and $0.2_{\text{ (base 3)}}$

For $n = 2$, it is removed if $a_2 = 0.01_{\text{(base 3)}} = 0.00\bar{2}, b_2 = 0.02_{\text{(base 3)}}$ or $a_2 = 0.21_{\text{(base 3)}} = 0.20\bar{2}_{\text{(base 3)}}, b_2 = 0.22_{\text{(base 3)}}$.

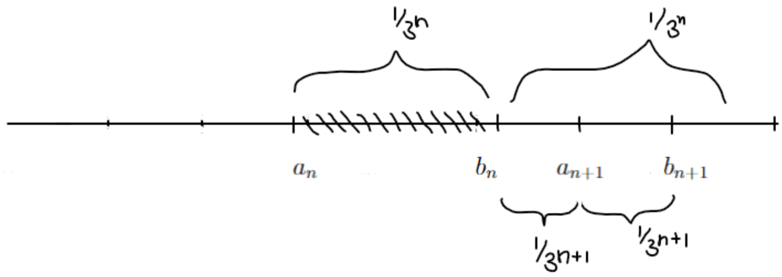
In general, any interval $(a_n, b_n)$ is removed if and only if $a_n, b_n$ can be written in the




form

$$
\begin{aligned}
a_n &= 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}1_{\text{(base 3)}}\\
&= 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}0222\cdots_{\text{(base 3)}}\\
&= \sum_{k=1}^{n-1}\frac{\varepsilon_k}{3^k} + \sum_{k=n+1}^{\infty}\frac{1}{3^k}\\
b_n &= \sum_{k=1}^{n-1}\frac{\varepsilon_k}{3^k} + \sum_{k=n+1}^{\infty}\frac{2}{3^k}\\
&= 0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_{n-1}2_{\text{(base 3)}}
\end{aligned}
$$

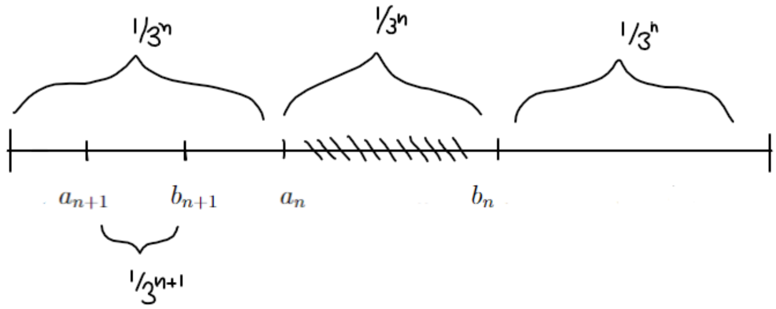
Assume this is true for a given $n$, then for $(n+1)^{th}$ step. At each step we remove $2^n$ open intervals. Suppose we remove right most interval from one particlar segment formed at the n-th stage then we have,



$$
\begin{aligned}
a_{n+1} &= b_n + \frac{1}{3^{n+1}} \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}2 + \frac{1}{3^{n+1}} \qquad = 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}21 \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}20222\cdots \\
b_{n+1} &= b_n + \frac{2}{3^{n+1}} \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}22
\end{aligned}
$$

# OR

If we remove the open interval $(a_n, b_n)$ from the left most segment then we have




$$
\begin{aligned}
a_{n+1} &= a_n - \frac{2}{3^{n+1}} \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}1 - \frac{1}{3^{n+1}} \qquad = 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}01 \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}00222\cdots \\
b_{n+1} &= a_n - \frac{1}{3^{n+1}} \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}1 - \frac{1}{3^{n+1}} \\
&= 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}02
\end{aligned}
$$

Therefore by the induction principle we conclude that

$$x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k}, \text{ where } \epsilon_k = 0 \text{ or } 2$$

Observe that this implies, in particular, that the left end points of the disjoint intervals constituting $C_n$ have a finite triadic representation with ending digit 2; more precisely, they can be written as
$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where $a_n \in \{0,2\}$ for $0 < k \leq n$,
$a_k = 2$ and $a_k = 0$ for $k > n$ , and the right end points of the disjoint intervals constituting $C_n$ have a infinite periodic triadic representation (endpoints are of the form $\frac{m}{2^n}$) with period 2 (means 2 is recurring). Moreover, observe that $C_n$ consists of exactly $2^n$ disjoint intervals, which is exactly the number of points in $[0, 1]$ of the form
$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ , where $a_k \in \{0,2\}$ for $0 < k \leq n$ .
and $a_k = 0$ for $k > n$ □

Therefore we can conclude that if $x \in \Delta$ then

$$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3 \cdots \varepsilon_n \cdots_{\text{(base 3)}} \quad \text{where } \varepsilon_k \in \{0,2\}$$

**Theorem 2.8.** *Every point in the Cantor ternary set $\Delta$ is a limit point of $\Delta$.*

*Proof.* Recall the definition of a limit point: Let $A \subset \mathbb{R}$, a point $x \in A$ is a limit point of A iff $\exists\,(x_n)$ , a sequence in A s.t. $x_n \to x$. We will prove that for every $x \in \Delta$, $\exists$ a sequence in $\Delta$ converging to it.

Case 1 : Suppose $x$ has only finite number of digits, say n.

Approach $x$ through the sequence $x + \dfrac{2}{3^{n+1}},\ x + \dfrac{2}{3^{n+2}},\ x + \dfrac{2}{3^{n+3}}, \cdots$ where all the terms are also elements in the standard cantor set. Therefore $x$ is a limit point of $\Delta$

Case 2 : Ternary expansion of x has infinite digits.

$x = x_1x_2x_3x_4...._{\text{(base 3)}}$ where each $x_k$ is either 0 or 2. Now approach x through the sequence $0.x_1$ , $0.x_1x_2$ , $0.x_1x_2x_3$ , $0.x_1x_2x_3x_4$ , ...

($x_k = k^{th}$ digit)

All the terms in the sequence that converge to x , belong to $\in \Delta$, since $x_k$ is either 0 or 2. As a result, x is a limit point of the Cantor ternary set.

□

**Corollary 2.9.** *The Cantor set is perfect.*

*Proof.* Recall that a set $S$ is said to be perfect if $S$ is closed and $S = S'$.
Since a closed set contains all its limit points , it is equivalent to prove that every point of a closed set is a limit point. The Cantor set $\Delta$ is closed and $\forall x \in \Delta, x$ is a limit point of $\Delta$. □

**Proposition 2.10.** *$\Delta$ is totally disconnected*

*Proof.* We will show that any two elements of the Cantor set are separated by at least one point not in $\Delta$.

Let $a, b \in \Delta$. Then we know that written in base 3, both a and b are composed of only 0 and 2.

Now, starting from left decimal digit locate the first digit that differ in $a$ and $b$ and replace it with a 1.

w.l.o.g. , if $a = 0.20202...$ and $b = 0.20222...$ in base 3, then let $c = 0.2021...$ in base 3, where the remaining digits don't really matter. Since the first 3 digits of a, b and c are the same, we are able to argue that $a < c < b$ and that c is not in the ternary Cantor set since it contains the digit 1.

If this is the case, no two distinct points can be part of the same connected component, so the set is totally disconnected. □

**Proposition 2.11.** $\Delta$ *is nowhere dense.*

*Proof.* The Cantor set contains no interval of non-zero length.

For, suppose $\Delta$ contains some interval $(a, b)$ . Then $(a, b) \subset C_n\ \forall n$. But each $C_n$ consists of $2^n$ closed disjoint intervals. So $(a, b)$ is contained in exactly one of the $2^n$ closed intervals, say $J_n$ .

But length of each $J_n$ is $1/3^n$.

$\implies |b - a| < \frac{1}{3^n}\ \forall n.$

Thus as $n \to \infty, \quad 1/3^n \to 0.$

$\implies a = b$ and hence $(a, b) = \phi$ So $int(\Delta) = \phi$.

□

So summing up this chapter , some of the key points in this chapter are noted down here.

- For any point in the Cantor set and any arbitrarily small neighborhood of the point, there is some other number with a ternary numeral of only 0s and 2s, as well as numbers whose ternary numerals contain 1s. Hence, every point in the Cantor set is an accumulation point (also called a cluster point or limit point) of the Cantor set, but none is an interior point. A closed set in which every point is an accumulation point is also called a perfect set in topology, while a closed subset of the interval with no interior points is nowhere dense in the interval.

- It is worth emphasizing that numbers like 1, $1/3 = 0.1_3$ and $7/9 = 0.21_3$ are in the Cantor set, as they have ternary numerals consisting entirely of 0s and 2s: 1 = 0.222...3 = $0.\overline{2}_3$, 1/3 = 0.0222...3 = $0.0\overline{2}_3$ and 7/9 = 0.20222...3 = $0.20\overline{2}_3$. All the latter numbers are "endpoints", and these examples are right limit points of $\Delta$. The same is true for the left limit points of $\Delta$, e.g. $2/3 = 0.1222..._3 = 0.1\overline{2}_3 = 0.2\overline{0}_3$ and $8/9 = 0.21222..._3 = 0.21\overline{2}_3 = 0.22\overline{0}_3$. All these endpoints are proper ternary fractions (elements of $\mathbb{Z} \cdot 3^{-\mathbb{N}_0}$) of the form p/q, where denominator q is a power of 3 when the fraction is in its irreducible form.[10] The ternary representation of these fractions terminates (i.e., is finite) or — recall from above that proper ternary fractions each have 2 representations — is infinite and "ends" in either infinitely many recurring 0s or infinitely many recurring 2s. Such a fraction is a left limit point of $\Delta$ if its ternary representation contains no 1's and "ends" in infinitely many recurring 0s. Similarly, a proper ternary fraction is a right limit point of $\Delta$ if it again its ternary expansion contains no 1's and "ends" in infinitely many recurring 2s.

- The Cantor set contains as many points as the interval from which it is taken, yet itself contains no interval of nonzero length. The irrational numbers have the same property, but the Cantor set has the additional property of being closed, so it is not even dense in any interval, unlike the irrational numbers which are dense in every interval.

- This set of endpoints is dense in $\Delta$ (but not dense in [0,1]) and makes up a countably infinite set. The numbers in $\Delta$ which are not endpoints also have only 0s and 2s in their ternary representation, but they cannot end in an infinite repetition of the digit 0, nor of the digit 2, because then it would be an endpoint. Thus there are as many points in the Cantor set as there are in the interval [0, 1] (which has the uncountable cardinality $\mathfrak{c} = 2^{\aleph_0}$). However, the set of endpoints of the removed intervals is countable, so there must be uncountably many numbers in the Cantor set which are not interval endpoints. As noted above, one example of such a number is 1/4, which can be written as $0.020202\cdots_3 = 0.\bar{02}$ in ternary notation.

Thus we have seen that the Cantor ternary set enjoys various interesting properties:

- $\Delta$ is uncountable.
- $\Delta$ is closed and compact.
- $\Delta$ is totally disconnected perfect set.
- $\Delta$ is nowhere dense set, having Lebesgue measure zero.

Having discussed the ternary representation let us now explore the corresponding Cantor function and ten discuss some properties of the Cantor set.

# Chapter 3

# The Cantor-Lebesgue function

Now let us discss the Cantor Lebesge function associated with the Cantor middle-thhird set.

The Cantor Lebesgue function or simply the "the Cantor function" is defined on $\Delta$, is constructed using the Cantor set.

We know that every element $x \in \Delta$, Cantor set, x can be written as $\sum_{i=1}^{\infty} \frac{2a_i}{3^i}$ where $a_i \in \{0, 1\}$

$f : \Delta \to [0, 1],$ is defined by

$$f\Big(\sum_{i=1}^{\infty} \frac{2a_i}{3^i}\Big) = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \text{ where } a_i \in \{0, 1\} \tag{3.1}$$

Basically, $x \in \Delta$ can be written as:

$$x = 0.a_1a_2a_3 \cdots_{\text{(base 3)}}$$

where each $a_n = 0$ or $2$

Observe that the action of the function $f$ can be written as

$$f(x) = f(0.a_1a_2a_3\cdots) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\cdots_{\text{(base2)}}$$

- f is well defined since from the discussion in the previous chapter, for almost all points of $[0,1]$ (except for the binary and ternary rationals, $\frac{m}{2^n}, \frac{m}{3^n}$) have a unique representation. For the case of the binary and tri-adic rationals they have two representations, in order to apply the definition of $f(x)$, we need the infinite representation with period 2

$$x = \frac{m}{3^n} = 0.a_1a_2a_3\cdots a_{n-1}1000\cdots = 0.a_1a_2a_3\cdots a_{n-1}02222\cdots_{\text{(base 3)}}$$

$$\begin{aligned} F(x) &= 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\cdots a_{n-1}01111\cdots_{\text{(base 2)}} \\ &= 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\cdots a_{n-1}10000\cdots_{\text{(base 2)}} \end{aligned}$$

- f is not one-one.

  Observe that $0.0\bar{2}_{\text{(base3)}}$ and $0.2_{\text{(base3)}}$ are mapped to the same element ;

  $0.0\bar{1}_{\text{(base2)}} = 0.1_{\text{(base 3)}}$ in base 2 expansion.

  $f(1/3) = f(0.0222..._3) = (0.0111...)_{\text{(base 2)}} = (0.1)_2 = 1/2$

  $f(2/3) = f(0.2_3) = (0.1)_2 = 1/2$

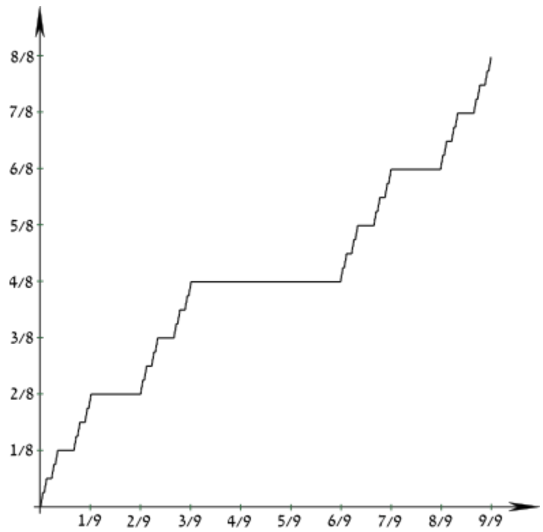



Figure 3.1: The Cantor Function

**Proposition 3.1.** *The cantor function is surjective.*

*Proof.* Given any $y$ in $[0, 1]$, we can find its pre-image in the Cantor set.

Let $y \in [0, 1]$ with binary expansion of the form $\sum_{k=1}^{n=\infty} \dfrac{\eta_k}{2^k}$ with $\eta_k \in \{0, 1\}$.

Then consider $x = \sum_{k=1}^{n=\infty} \dfrac{2\eta_k}{3^k}$ with $\eta_k \in \{0, 1\}$.

It clear that $x \in \Delta$ and by the definition of $f$ , $f(x) = y$.

This proves that $f$ is surjective. $\square$

**Theorem 3.2.** *The cantor function $f : \Delta \to \Delta$ is continuous.*

*Proof.* Let us now prove the continuity of f on $\Delta$. Suppose that $x_0 \in \Delta$ , is fixed, and let $x \in \Delta$, such that $|x - x_0| < \dfrac{1}{3^{2n}}$ . Then, $x_0$ and $x$ cannot differ in the first $2n$ ternary places and we have

$x_0 = 0.\epsilon_1\epsilon_2 \cdots \epsilon_{2n-1}\epsilon_{2n}\epsilon_{2n+1} \cdots$( base 3)

$x = 0.\epsilon_1\epsilon_2 \cdots \epsilon_{2n-1}\tau_{2n}\tau_{2n+1} \cdots$( base 3)

Suppose, to the contrary, that $\epsilon_{2n} \neq \tau_{2n}$. Then, $|\epsilon_{2n} - \tau_{2n}| = 2$ and

$$x - x_0 = \frac{(\tau_{2n} - \epsilon_{2n})}{3^{2n}} + \frac{(\tau_{2n+1} - \epsilon_{2n+1})}{3^{2n+1}} + \cdots$$

and, hence,

$$\begin{aligned} |x - x_0| &\geq \frac{2}{3^{2n}} - \frac{2}{3^{2n+1}}(1 + \frac{1}{3} + \frac{1}{9} + \cdots) \\ &= \frac{2}{3^{2n}} - \frac{1}{3^{2n}} = \frac{1}{3^{2n}}, \\ &\text{instead of } < \frac{1}{3^{2n}}. \end{aligned}$$

Therefore, given $\epsilon > 0$ there exists N such that $\frac{1}{2^{2N}} < \epsilon$, and taking $\delta = \frac{1}{3^{2N}}$ we have that if $|x - x_0| < \frac{1}{3^{2N}}$

then

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \frac{\frac{\tau_{2N+1}}{2} - \frac{\epsilon_{2N+1}}{2}}{2^{2N+1}} + \frac{(\frac{\tau_{2N+2}}{2} - \frac{\epsilon_{2N+2}}{2})}{2^{2N+2}} + \cdots \right| \\ &= \left| \frac{(\tau_{2N+1} - \epsilon_{2N+1})}{2^{2N+2}} + \frac{(\tau_{2N+2} - \epsilon_{2N+2})}{2^{2N+3}} + \cdots \right| \\ &\leq \frac{2}{2^{2N+2}}(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \\ &= \frac{1}{2^{2N}} \\ &< \epsilon \end{aligned}$$

Hence the continuity of $f$ at $x_0$ follows. We have actually proved that $f$ is uniformly continuous on $\Delta$, since $\delta$ is independent of $x_0$, since given $\epsilon > 0$, taking $\frac{1}{2^{2N}} < \epsilon$, and $\delta = \frac{1}{3^{2N}}$ we have for all $x, y \in \Delta$ such that $|x - y| < \delta$

$\implies |f(x) - f(y)| \leq \epsilon$ $\square$

Note that $f(0) = 0$ and $f(1) = 1$. Because,

$$0 = \sum_{k=1}^{\infty} \frac{0}{3^k} \qquad \text{and } 1 = \sum_{k=1}^{\infty} \frac{2}{3^k} = 0.222\cdots_{\text{(base 3)}}$$

$$f(0) = \sum_{k=1}^{\infty} \frac{0}{2^k} = 0 \qquad \text{and } f(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Or $1 = 0.222\cdots_{\text{(base 3)}}$

$f(1) = 0.111\cdots_{\text{(base 2)}} = 1$

We can extend the Cantor function defined on $\Delta$ to $[0, 1]$.

$$F : [0,1] \longrightarrow [0,1] \text{ s.t } .F\Big|_{\Delta} = f$$

$$F(x) = f(x) \text{ for } x \in \Delta.$$

$$F(x) = sup\{f(y) : y \in \Delta, y \leq x\}$$

First, we will prove that if $(a_n, b_n)$ is an open interval of the complement of $\Delta$, $\Delta^c$ then $F(a_n) = F(b_n)$. Hence we may define F to have the constant value $F(a_n)$ in that interval. Then, the definition of F may be extended into all of $[0, 1]$ by defining it on $\Delta$ as follows:

If $x \in \Delta^c$, then $x \in (a_n, b_n)$ where the open interval $(a_n, b_n)$ is one of those that has been removed from $[0, 1]$ in the construction of the Cantor set. Then, necessarily

$a_n = 0.\epsilon_1\epsilon_2\epsilon_3...\epsilon_{n-1}1_{\text{(base 3)}} = 0.\epsilon_1\epsilon_2\epsilon_3...\epsilon_{n-1}0\bar{2}_{\text{(base 3)}}$

$b_n = 0.\epsilon_1\epsilon_2\epsilon_3...\epsilon_{n-1}2_{\text{(base 3)}}$

and therefore,

$$
\begin{aligned}
F(a_n) &= 0.\frac{\epsilon_1}{2}\frac{\epsilon_2}{2}\frac{\epsilon_3}{2}\cdots\frac{\epsilon_{n-1}}{2}\,0\bar{1}_{\text{(base 2)}} \\
&= 0.\frac{\epsilon_1}{2}\frac{\epsilon_2}{2}\frac{\epsilon_3}{2}\cdots\frac{\epsilon_{n-1}}{2}\,1_{\text{(base 2)}} = F(b_n)
\end{aligned}
$$

Thus $F$ is constant on each of the subintervals removed at each $n-th$ stage.

Now, to prove that the extension, also denoted by $F$, is continuous on $[0,1]$, i.e., we need to prove that for any $x_0 \in [0,1]$ given $\epsilon > 0$ there exist $\delta > 0$ such that if $x \in [0,1]$ such that $|x - x_0| < \delta$ then $|F(x) - F(x_0)| < \epsilon$.

- If $x_0 \in \Delta^c$ , then $x_0 \in (a_n, b_n)$ where the open interval $(a_n, b_n)$ is one of those that has been removed from $[0,1]$ in the construction of the Cantor set and then as we have proved above $F$ is constant on $(a_n, b_n)$ so $F$ is trivially continuous at $x_0$.
- If $x_0 \in \Delta$, then, as $\Delta$ is perfect, it is either a left accumulation point, a right accumulation point or an accumulation point. We will only consider the case where $x_0$ is a left accumulation point, which makes that it a right end point of an interval that has been removed in the construction of $\Delta$.
  From above we know that namely $x_0 = 0.\epsilon_1\epsilon_2\epsilon_3\cdots\epsilon_{n-1}2_{\text{(base3)}}$. Then we know that the restriction of $F$ to $\Delta$, is continuous on it and, since $\Delta$ is compact, it is uniformly continuous on $\Delta$, i.e.,
  for any $\epsilon > 0$, there is $\delta > 0$ such that

  $$|F(x_1) - F(x_2)| < \delta,$$

  for all $x_1, x_2 \in \Delta$ for which $|x_1 - x_2| < \delta$. Let us now examine $|F(x) - F(x_0)|$.

There are two possibilities:

- $x \in \Delta$: then, we already know $|F(x) - F(x_0)| < \epsilon$ for $|x - x_0| < \delta$,
- $x \in \Delta^c$: then, $x \in (a, b)$, where $(a, b)$ is one of the intervals that have been removed in the construction of $\Delta$.

  Then $a, b \in \Delta$ and $(a, b) \subset \Delta^c$.

  But again, as $F$ is constant on $(a, b)$, then $F(t) = F(a) = F(b) \quad \forall t \in (a, b)$.

  Now, let $(a, b)$ be any such interval with $x_0 < a < b < x_0 + \delta$,

  $|F(t) - F(x_0)| = |F(a) - F(x_0)| < \epsilon$, for all $t \in (a, b) \subset \Delta^c$, as $a \in \Delta$.

  This, together with the fact that to the left of $x_0$ ($x_0$ being the right endpoint of a removed interval) which implies the continuity of $F$ at $x_0 \in \Delta$. An analogous proof applies to a right accumulation point of $\Delta$, and both proofs together take care of a two-sided accumulation point of $\Delta$.

***Another proof that the extended Cantor function $F$ defined from $[0, 1]$ to $[0, 1]$ is continuous.***

To prove that the $F : [0, 1] \to [0, 1]$ is continuous let us look at the following two results:

**Theorem 3.3.** *If $f : (a, b) \to \mathbb{R}$ is monotone , then $f$ has at most countably many points of discontinuity in (a,b), all of which are jump discontinuities.*

*Proof.* Let $f : I = (a, b) \to J$ be monotonic increasing. $E = \{x \in I \mid f$ is discontinuous at $x\}$.
TPT: $E$ is at most countable.
Recall that for monotonic increasing function $f(x^-) \le f(x) \le f(x^+)$.

$f$ is increasing; and $f$ is discontinuous at $x \iff f(x^-) < f(x^+)$. By density of rationals in $\mathbb{R}$, $\exists r_x$ a rational in $I$ such that $f(x^-) < r_x < f(x^+)$.

Suppose $x, y \in E$ and $x < y$ then we have

$$f(x^-) < f(x^+) < f(y^-) < f(y^+)$$

and thus $r_x \neq r_y$ if $x \neq y$.

Thus for the distinct points $x \in E$ we can assign distinct rational numbers $r_x$.

i.e we can have a function , $g : E \to \mathbb{Q}$ such that $g$ is injective, and $g(x) = r_x$.

Thus $card(E) \leq card(\mathbb{Q})$. $\therefore E$ is at most countable.

□

**Corollary 3.4.** *If $f : [a,b] \to [c,d]$ is both monotone and onto , then $f$ is continuous.*

*Proof.* Let $f : [a,b] \to [c,d]$ be monotonic increasing and onto.

Let $x_0 \in [a,b]$ & $\varepsilon > 0$

Since $f$ is increasing , $f(x_0) - \varepsilon \leq f(x_0) \leq f(x_0) + \varepsilon$

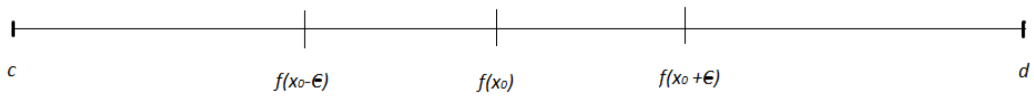
Consider rationals $q_j$ , $q_k$ s.t.

$$q_j \in (f(x_0) - \varepsilon, f(x_0))$$

$$q_k \in (f(x_0), f(x_0) + \varepsilon)$$

Then $q_j \leq q_k$.

□




Since f is onto, $\exists\ x_j, x_k$ in $[a, b]$ s.t. $f(x_j) = q_j$ and $f(x_k) = q_k$.


Let $\delta = \min\{x_0 - x_j, x_k - x_0\}$

Without loss of generality assume $\delta = x_k - x_0$.

If $x \in [a, b]\ s.t\ |x - x_0| < \delta$ then

$$|x - x_0| < x_k - x_0$$

$$\implies x_0 - x_k < x - x_0 < x_k - x_0$$

$$x_j - x_0 < x_0 - x_k < x - x_0 < x_k - x_0$$

$$x_j < 2x_0 - x_k < x < x_k$$

$$x_j < x < x_k$$

By monotonicity, $f(x_j) \leq f(x) \leq f(x_k)$

Thus $q_j \leq f(x) \leq q_k \implies f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon$

That is, $|f(x) - f(x_0)| \leq \varepsilon$

Thus we have proved that $f$ is continuous.

**Theorem 3.5.** *The Cantor function is uniformly continuous.*

*Proof.* given $\epsilon > 0$ let N be the smallest integer such that, $\frac{1}{2^N} < \epsilon$ let $\delta = \frac{1}{3^N}$

Let $x_0 \in [0, 1]$. For any $x \in (x_0 - \delta, x_0 + \delta)$,

i.e. $|x - x_0| < \delta = \frac{1}{3^N} \implies x$ and $x_0$ cannot differ in first N places.

for, suppose

$$x = 0.a_1a_2a_2a_4....a_{N-1}a_Na_{N+1}\dots$$

$$x_0 = 0.a_1a_2a_2a_4....a_{N-1}b_Nb_{N+1}\dots$$

and if we have $a_N \neq b_N$ , then $|a_N - b_N| = 2$ and

$$\begin{aligned}
x - x_0 &= \frac{a_N - b_N}{3^N} + \frac{a_{N+1} - b_{N+1}}{3^{N+1}} + \frac{a_{N+2} - b_{N+2}}{3^{N+2}} + \cdots \\
&= \frac{2}{3^N} + \frac{a_{N+1} - b_{N+1}}{3^{N+1}} + \frac{a_{N+2} - b_{N+2}}{3^{N+2}} + \cdots \\
&\geq \frac{2}{3^N} \\
&> \delta \\
&= \frac{1}{3^N}
\end{aligned}$$

Thus if $|x - x_0| < \delta = \frac{1}{3^N}$ then $x$ and $x_0$ cannot differ in first N places.

Therefore, given $\epsilon > 0, \; \exists \; N$ such that $\dfrac{1}{2^N} < \epsilon$, and taking $\delta = \dfrac{1}{3^N}$ we have that ,

if $|x - x_0| < \dfrac{1}{3^N}$ then,

$$
\begin{aligned}
F(x)-F(x_0) &= \frac{(a_{N+1}-b_{N+1})/2}{2^{N+1}} + \frac{(a_{N+2}-b_{N+2})/2}{2^{N+2}} + \frac{(a_{N+3}-b_{N+3})/2}{2^{N+3}} + \cdots \\
&= \frac{(a_{N+1}-b_{N+1})/2}{2^{N+1}} + \frac{(a_{N+2}-b_{N+2})/2}{2^{N+2}} + \frac{(a_{N+3}-b_{N+3})/2}{2^{N+3}} + \cdots \\
&= \frac{(a_{N+1}-b_{N+1})}{2.2^{N+1}} + \frac{(a_{N+2}-b_{N+2})}{2.2^{N+2}} + \frac{(a_{N+3}-b_{N+3})}{2.2^{N+3}} + \cdots \\
&\leq \frac{1}{2}\frac{1}{2^{N+1}}\big(2 + \frac{2}{2^N} + \frac{2}{2^2}\big) + \cdots \\
&\leq \frac{1}{2}\frac{1}{2^{N+1}}2\big(1 + \frac{1}{2^N} + \frac{1}{2^2} + \cdots\big) \\
&\leq \frac{1}{2^N} < \epsilon
\end{aligned}
$$

□

Recall how we defined the absolutely continuous functions.

A real-valued function $f$ on a closed, bounded interval $[a, b]$ is said to be absolutely continuous on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $(a_k, b_k)_{k=1}^{n}$ of open intervals in $(a, b)$, if

$$\sum_{k=1}^{n} |b_k - a_k| < \delta, \; then \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$$

**Proposition 3.6.** *The Cantor function is uniformly continuous but not absolutely continuous.*

*Proof.* The Cantor-Lebesgue function $\phi$ is increasing and continuous on $[0, 1]$, but it is not absolutely continuous.

To see that $\phi$ is not absolutely continuous, let n be a natural number. At the $n^{th}$ stage of the construction of the Cantor set, a disjoint collection $\{[c_k, d_k]\}_{1<k<2^n}$ of $2^n$ sub intervals of $[0, 1]$ is constructed that cover the Cantor set. Each interval at the n-th stage which constitutes $C_n$ has length $(1/3)^n$.

The Cantor function is constant on each of the intervals in the complement of this collection of intervals (constant on each interval in the complement of the Cantor set). Therefore, since $\phi$ is increasing and $\phi(1) - \phi(0) = 1$,

$$\sum_{1\leq k\leq 2^n} |d_k - c_k| = \left(\frac{2}{3}\right)^n \ while \sum_{1\leq k\leq 2^n} |\phi(d_k) - \phi(c_k)| = 1.$$

□

**Proposition 3.7.** *The Cantor function has zero derivative on $\Delta^c$ (the complement of the cantor set on the interval (0,1) and is not differentiable on the ternary Cantor set.*

*Proof.* The first half of the theorem is trivial, since the Cantor function is constant on each of the open interval in the $\Delta^c$ . Therefore the derivative is zero on $\Delta^c$ .

The second half can be proved as follows.

Given $x \in \Delta$,

**Case 1:** $x$ has a finite number of digits in base 3.

Denote that number as $n$. Let $h = \frac{2}{3^k}\ where\ k > n.$

Then $\lim_{k\to\infty} h = 0.$

So $\lim_{h\to 0} \dfrac{f(h+x) - f(x)}{h}$

$= \lim_{k\to\infty} \dfrac{f(h+x) - f(x)}{h}$

$= \frac{1}{2^k} / \frac{2}{3^k}$

$= \lim_{k\to\infty} \frac{3^k}{2^{k+1}} = \infty.$ Similarly, it can be proved that the limit does not exist when h is negative.

**Case 2:** x has an infinite number of digits in base 3. Let y be the first k digits of x, then

y is also in $\Delta$.

so $\lim_{x\to y} \dfrac{(f(x)-f(y))}{x-y}$

$= \lim_{k\to\infty} \dfrac{(f(x)-f(y))}{x-y}$

$> \lim_{k\to\infty} \dfrac{(\frac{1}{2^{k+1}})}{(\frac{2}{3^{k+1}})}$

$= \lim_{k\to\infty} \dfrac{3^{k+1}}{2^{k+2}} = \infty$

Thus limit does not exist in any case. therefore F is not differentiable on $\Delta$ $\square$

We will see another proof for the same result.

**Alternate proof**

*Proof.* Let $x, x_n \in \Delta$

$x = 0.\epsilon_1\epsilon_2\cdots\epsilon_{n-1}\epsilon_n\epsilon_{n+1}\epsilon_{n+2}\cdots_{(\text{base } 3)}$

$x_n = 0.\epsilon_1\epsilon_2\cdots\epsilon_{n-1}\epsilon_n\tau_{n+1}\epsilon_{n+2}\cdots_{(\text{base } 3)}$

where $\tau_{n+1} = \epsilon_{n+1} + 1(\text{mod } 2)$.

Then $|x - x_n| = \dfrac{2}{3^{n+1}}$ and it is easy to see that

$$F(x) - F(x_n) = \frac{\epsilon_{n+1} - \tau_{n+1}}{2^{n+1}}$$

Hence,

$$\frac{F(x) - F(x_n)}{x - x_n} = \frac{3}{4}(\frac{9}{2})^n \longrightarrow \infty$$

Thus, $F$ is nowhere differentiable on $\Delta$. Therefore, as $F$ is trivially differentiable on $\Delta^c$, with derivative zero, we can conclude that $F$ is a singular function, i.e., $F'(x)$ is zero $a.\,e.$, for $x \in [0, 1]$ without being constant. $\square$

Let us now go back to some properties of the Cantor set .

**Theorem 3.8.** $\Delta$ *has same cardinality as* $[0, 1]$.

*Proof.* The Cantor function $F : \Delta \to [0, 1]$ is onto.

This implies that card($\Delta$)$\geq$ card($[0, 1]$). But since $\Delta$ is a subset of $[0, 1]$,

card($\Delta$)$\leq$ card($[0, 1]$)

Thus card($\Delta$)$=$ card($[0, 1]$) $= c$

Therefore $\Delta$ is uncountable. $\square$

# Chapter 4

# Generalizations of the Cantor set

In the previous chapters we saw the construction and properties of the standard Cantor set , (Cantor middle third or ternary set) and the corresponding Cantor function. We also proved all the properties associated with the Cantor set and the Cantor function.

In this section various generalizations of The standard Cantor set has been discussed. Similar to Cantor $\frac{1}{3}$ set , Cantor $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ sets and in general the cantor middle $\frac{1}{2m+1}$ set where $1 \leq m < \infty$ has been discussed.

Another Category of Cantor sets called as "fat Cantor sets" has been introduced. In that the special case of Smith-Volterra Cantor sets is also been discussed. [2] "Two ratio Cantor set" is discussed in brief.

Let look at various generalizations one at a time.

## 4.1 Generalization I

### 4.1.1 Cantor middle 1/2m+1 set.

we generalize the Cantor set by removing the central open interval of length $\frac{1}{n}$ from equal parts of length $n$, where $n$ is an odd number say, $n = 2m + 1$ from the unit interval $C_0 = [0, 1]$ . Before discussing this construction for $n = 2m + 1$ let us look at the particular cases of $n = 5, 7, 9$. For $n = 3$ we obtain the standard Cantor ternary set which has already been discussed.

### 4.1.2 Cantor 1/5 set

Let us look at the particular case of Cantor$(\frac{1}{2m+1})^{th}$ set when $m = 2$.
For $m = 1$ it is the Cantor middle$(\frac{1}{3})^{rd}$ set which we have already discussed in detail.

#### Construction

To build this set (denoted by $\mathcal{C}(\frac{1}{5})$) we can follow the same procedure as construction of the middle-third Cantor's set. First we delete the open interval covering its middle fifth from the unit interval $C_0 = [0, 1]$. That is, we remove the open interval $(\frac{2}{5}, \frac{3}{5})$. The set of points that remain after this step will be called $C_1$ That is, $C_1 = [0, \frac{2}{5}] \cup [\frac{3}{5}, 1]$
In the second step, we remove the middle fifth portion of each of the 2 closed intervals of $C_1$ and the remaining set is $C_2$:
$C_2 = [0, \dfrac{4}{25}] \cup [\dfrac{6}{25}, \dfrac{2}{5}] \cup [\dfrac{3}{5}, \dfrac{19}{25}] \cup [\dfrac{21}{25}, 1]$

Note that at the first step we obtained $2^1$ closed intervals , each of length $(\frac{2}{5})$. At the second step ; in $C_2$ we have $2^2 = 4$ intervals of length $\frac{4}{25}$ each.

The following are the steps for first few iterations.

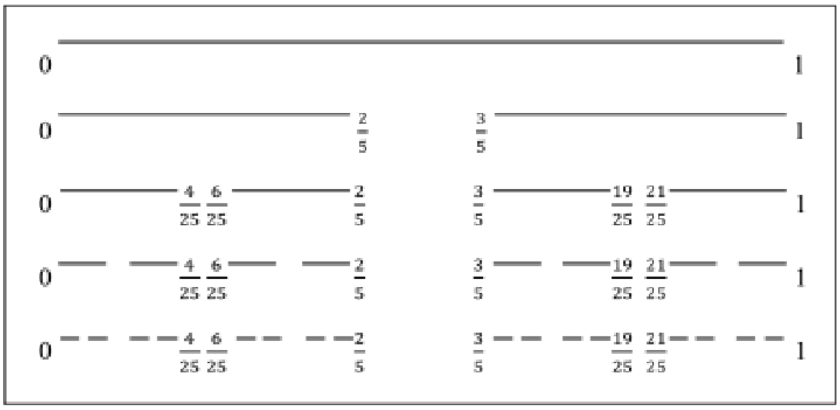



**Figure 3.** Construction of the Cantor middle-*1/5* set.

Repeating this process, we get a limiting set $\mathcal{C}(\frac{1}{5})$,

$$\mathcal{C}(\tfrac{1}{5}) = \bigcap_{n=1}^{\infty} C_n$$

and call it the Cantor middle $\frac{1}{5}$ set.

**Properties of Cantor middle $\frac{1}{5}$ set.**

**Proposition 4.1.** *Outer measure of the Cantor $\frac{1}{5}$ set is zero*

*Proof.*

The length of removed intervals

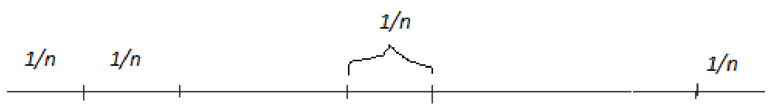
$$
\begin{aligned}
&= \frac{1}{5} + 2(\frac{2}{5^2}) + 2^2(\frac{2^2}{5^2}) + 2^3(\frac{2^3}{5^4}) + \cdots \\
&= \frac{1}{5} + \frac{4}{5^2} + \frac{4^2}{5^3} + \frac{4^3}{5^4} + \cdots \\
&= \frac{1}{5} + \frac{4}{5^2}(1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \cdots \\
&= \frac{1}{5} + \frac{4}{5^2}\left(\frac{1}{1-\frac{4}{5}}\right) \\
&= \frac{1}{5} + \frac{4}{5^2}\left(\frac{1}{\frac{1}{5}}\right) \\
&= \frac{1}{5} + \frac{4}{5} \\
&= 1
\end{aligned}
$$

Therefore the length of remaining set is$= 1 - 1 = 0$ □

**Construction of Cantor middle C($\frac{1}{2m+1}$) set.**

Having discussed few examples previously let us us look at the generalized construction of the set as introduced the subsection 4.1.1. ,i.e $C(\frac{1}{2m+1})$ set. or $C(\frac{1}{2m+1})$ set where $n = 2m+1,\ m = 1, 2, 3, \cdots$

Consider the unit interval $C_0 = [0, 1]$. Divide it into $n$ equal parts and remove remove it the central open interval of length $\frac{1}{n}$.



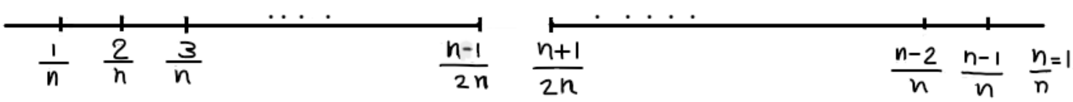
That is, we are removing

$$Z_1 = \{x \in [0,1] : \frac{n-1}{2n} < x < \frac{n+1}{2n}\}$$

We remove from $[0,1]$, $Z_1$ an open interval of total length $= \frac{1}{n}$.
we remove $Z_1 = \left\{x \in [0,1] \;\middle|\; \frac{n-1}{2n} < x < \frac{n+1}{2n}\right\}$ from $F_0 = [0,1]$.
Then we get union of two closed intervals as

$$\begin{aligned} C_1 &= C_0 \setminus Z_1 \\ &= \left\{x \;:\; 0 \leq x \leq \frac{n-1}{2n}\right\} \bigcup \left\{x \;:\; \frac{n+1}{2n} \leq x \leq 1\right\} \\ &= L \cup R \end{aligned}$$

where $L = \left\{x \mid 0 \leq x \leq \frac{n-1}{2n}\right\}$ and $R = \left\{x \;:\; \frac{n+1}{2n} \leq x \leq 1\right\}$
i.e $L = \left[0, \frac{n-1}{2n}\right]$ and $R = \left[\frac{n+1}{2n}, 1\right]$.




Next we divide each of $L$ and $R$ into n equal parts and remove from it the central open interval. . That is, we remove

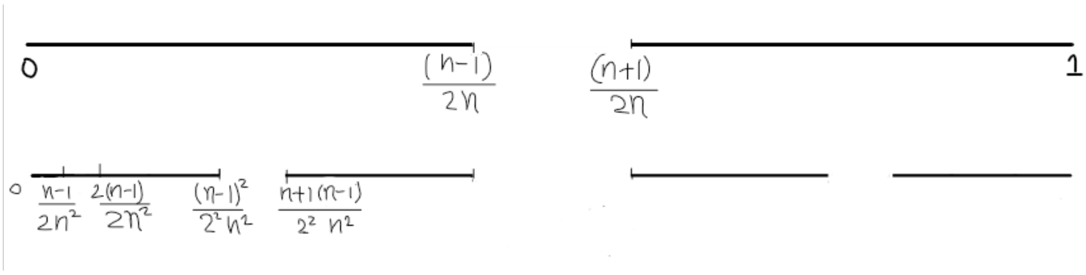
$$Z_2 = \left\{x \;\middle|\; \frac{n^2-2n+1}{4n^2} < x < \frac{n^2-1}{4n^2}\right\} \bigcup \left\{x \;:\; \frac{3n^2+1}{n^2} < x < \frac{3n^2+2n-1}{4n^2}\right\}$$

We remove from $[0,1] \setminus Z_1 = C_2$, a union of $2^1$ open intervals of total length $= \frac{1}{2}\left(\frac{n-1}{n^2}\right)^k$
After this removal, we set the remaining part as $C_2$, i.e., $C_2 = F_0 \setminus Z_1 \cup Z_2$ which is union of four closed intervals, viz. LL, LR, RL, RR, each of length $\left(\frac{n-1}{2n}\right)^2$. Thus we

get $C_2 = LL \cup LR \cup RL \cup RR$, where

$$LL = \left[0, \frac{n^2 - 2n + 1}{4n^2}\right] \qquad \text{and } LR = \left[\frac{n+1}{2n}, \frac{3n^2+1}{4n^2}\right] \tag{4.1}$$
$$RL = \left[\frac{n+1}{2n}, \frac{3n^2+1}{4n^2}\right] \qquad \text{and } RR = \left[\frac{3n^2+2n-1}{4n^2}, 1\right] \tag{4.2}$$
$$\tag{4.3}$$

Proceeding this way, we get a sequence of closed intervals $\{C_k\}$ where $C_k$ is union of




$2^k$ number of closed intervals of length $\left(\frac{n-1}{2n}\right)^k$ each. At each $k-th$ step we remove $2^{k-1}$ intervals of length $\frac{1}{2^{k-1}}\left(\frac{(n-1)^{k-1}}{n^k}\right)$ Hence, in this process of generalization, limiting set formed from this process is called set, i.e. Cantor$(\frac{1}{n})$ the set, and formally defined as

$$C(\tfrac{1}{n}) = [0,1] \setminus \left\{\bigcup_{k=1}^{\infty} R_k\right\} = \bigcap_{k=1}^{\infty} C_k$$

where $n = 2m+1, \; m = 1, 2, 3, \cdots$

**Properties of Cantor middle $\frac{1}{2m+1}$ set.**

Properties of Cantor $\frac{1}{5}, \frac{1}{7}$ sets have been discussed.Let us look at the properties of the generalized form.

**Proposition 4.2.** *The set $C(\frac{1}{(2m+1)})$ is disconnected.*

*Proof.* Let $2m + 1 = n$.

The set $C(\frac{1}{(2m+1)}) = C(\frac{1}{n})$ is totally disconnected since it is constructed so as to contain no intervals other than points. For, suppose it contains an interval of positive length $\epsilon$ then this interval would be contained in each $C_k$, but $C_k$ contains no interval of length greater than $\left(\frac{n-1}{2n}\right)^k$. (For each $n \in \mathbb{N}, C_k$ is an union of $2^k$ closed disjoint intervals each having length $\left(\frac{n-1}{2n}\right)^k$.)

For any $n \in \mathbb{N}$,

$$n < 2n$$

$$n - 1 < 2n$$

$$\implies \frac{n-1}{2n} < 1$$

$$\implies \left(\frac{n-1}{2n}\right)^k < 1 \quad \forall\, k \in \mathbb{N}$$

So using Archimedean property if n is chosen to be large enough so that $\left(\frac{n-1}{2n}\right)^k$ is less than $\epsilon$, then there is no interval of length $\epsilon$ in $F_k$ . □

**Proposition 4.3.** *The set* $C(\frac{1}{(2m+1)})$ *is nowhere dense*

*Proof.* Let $2m + 1 = n$.

The Cantor middle $\frac{1}{2m+1}$ set contains no interval of non-zero length.

For, suppose $\Delta$ contains some interval $(a, b)$ . Then $(a, b) \subset C_k\ \forall n$. But each $C_k$ consists of $2^k$ closed disjoint intervals. So $(a, b)$ is contained in exactly one of the $2^k$ closed intervals, say $J_{i_k}, i = 1, 2, \cdots, k$.

But length of each $J_{i_k}$ is $\left(\frac{n-1}{2n}\right)^k$.

$\implies |b - a| < \left(\frac{n-1}{2n}\right)^k\ \forall k$.

Thus given $n = 2m + 1$ as $k \to \infty$,
$\left(\frac{n-1}{2n}\right)^k = (\frac{1}{2})^k \left(\frac{n-1}{n}\right)^k = (\frac{1}{2})^k \left(1 - \frac{1}{n}\right)^k = (\frac{1}{2})^k \longrightarrow 0.$
$\implies a = b$ and hence $(a, b) = \phi$ $\square$

**Proposition 4.4.** *The generalized Cantor set , $\mathcal{C}(\frac{1}{(2m+1)})$ is measurable and has Lebesgue measure zero.*

*Proof.* Let $2m + 1 = n$
At the first step we remove one open interval of length $\frac{1}{(2m+1)} = \frac{1}{n}$
What remains is the union of two closed intervals, each of length $(\frac{n-1}{2n})$

$$[0, \frac{n-1}{2n}] \cup [\frac{n+1}{2n}, 1]$$

. And at 1st step we remove open interval of length $\frac{1}{n}$. Then at step 2 we divide intervals of length $\frac{n-1}{2n}$ by $n$ and remove 2 , central open intervals of length $\left(\frac{(n-1)^2}{2n^2}\right)$ from the above two disjoint sets
Inductively at $k^{th}$ step we remove $2^{k-1}$ intervals of length $\dfrac{(n-1)^{k-1}}{2^{k-1}n^k}$.

Thus the total length removed =

$$
\begin{aligned}
&= \frac{1}{n} + 2(\frac{n-1}{2n^2}) + 2^2\frac{(n-1)^2}{2^2n^3} + \cdots \\
&= \frac{1}{n} + (\frac{n-1}{n^2}) + \frac{(n-1)^2}{n^3} + \cdots \\
&= \frac{1}{n} + \frac{n-1}{n^2}\Big[1 + \frac{(n-1)}{n} + (\frac{n-1}{n})^2 + \cdots\Big] \\
&= \frac{1}{n} + \frac{n-1}{n^2}\Big[\frac{1}{1-\frac{n-1}{n}}\Big] \\
&= \frac{1}{n} + \frac{n-1}{n^2}\Big[\frac{1}{\frac{1}{n}}\Big] \\
&= \frac{1}{n} + \frac{n-1}{n} \\
&= 1
\end{aligned}
$$

Therefore the outer measure of the set is $1 - 1 = 0$ $\square$

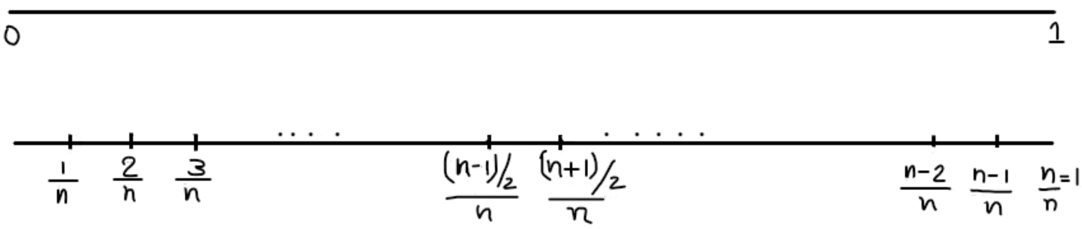
### 4.1.3 Generalization II

## Cantor($\frac{m}{n}$) set

In this generalization, we divide the unit interval $C_0 = [0,1]$ in $n \geq 3$ parts, where $n = 2m+1$, some odd number. And then remove from it the alternate $m$ open intervals,

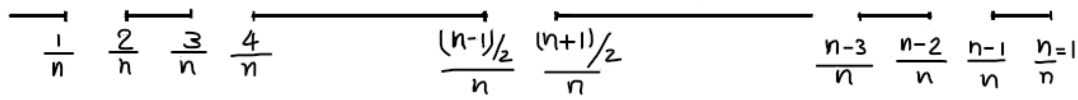
$$(\frac{1}{n}, \frac{2}{n}) \cup (\frac{3}{n}, \frac{4}{n}), (\frac{5}{n}, \frac{6}{n}), ..., (\frac{2m-1}{n}, \frac{2m}{n})$$

The remaining set $C_1 = [0,1] \setminus C_0$ turns out to be the union of $m+1$ intervals, i.e




$$C_1 = \left[0, \frac{1}{n}\right], \left[\frac{2}{n}\frac{3}{n}\right] \cup \left[\frac{4}{n}, \frac{5}{n}\right] \cup \dots \left[\frac{2m}{n}, 1\right]$$

Note that here each interval has length $\frac{1}{n}$.




In the next step, we divide each of the above sub interval into $n = 2m + 1$ equal parts again, and then remove alternate open intervals. What remains is a set $C_2$ a union of $(m + 1)^2$ closed intervals, each of length $\frac{1}{n^2}$

In general, after n iterations we obtain a set

$$C_n = [0, \frac{1}{(2m+1)^n}] \cup [\frac{2}{n}, \frac{3}{(2m+1)^n}] \cup \cdots \cup [\frac{2m-2}{2m+1^n}, \frac{2m-1}{(2m+1)^n}] \cup [\frac{2m}{(2m+1)^n}, 1]$$

Since n is odd $n = 2m+1$ Therefore we construct a decreasing sequence $(C_k)$ of closed sets, that is

$C_{k+1} \subseteq C_k \ \forall k \in \mathbb{N}$,

each $C_k$ consists of $n^k$ intervals of length $\frac{1}{(n)^k}$.

The limiting set obtained in this process is called Cantor$\frac{m}{n}$ set, $\mathcal{C}(\frac{m}{n}) = \bigcap_{n=1}^{\infty} C_k$.

where $n = 2m + 1$

**Construction of C(2/5)**

We start with the closed interval $C_0 = [0, 1]$.

Remove the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. This leaves a new set

$$C_1 = [0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$$

Each iteration removes the open $2^{nd}$ and $4^{th}$ interval from each segment of the previous iteration. Thus the next set would be

$$M_2 = [0, \frac{1}{25}] \cup [\frac{2}{25}, \frac{3}{25}] \cup [\frac{4}{25}, \frac{1}{5}] \cup [\frac{2}{5}, \frac{11}{25}] \cup [\frac{12}{25}, \frac{13}{25}] \cup [\frac{14}{25}, \frac{3}{5}] \cup [\frac{4}{5}, \frac{21}{25}] \cup [\frac{22}{25}, \frac{23}{25}] \cup [\frac{24}{25}, 1]$$

In general, after $n$ iterations we obtain $C_n$ as follows

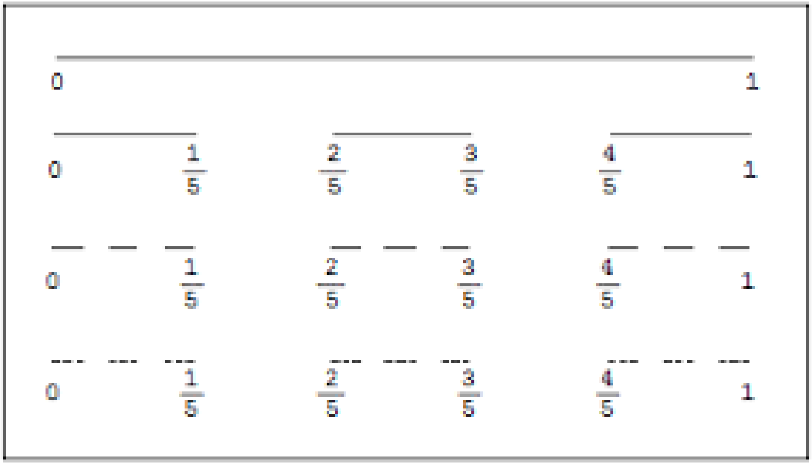
$$C_n = \big[0, \frac{1}{5^n}\big] \cup \left[\frac{2}{5^n}, \frac{3}{5^n}\right] \cup \cdots \cup [\frac{5^n - 3}{5^n}, \frac{5^n - 2}{5^n}] \cup [\frac{5^n - 1}{5^n}, 1]$$

where $n \geq 1$

Therefore, we construct a decreasing sequence $(C_n)$ of closed sets, that is, $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, so that every $C_n$ consists of $3^n$ closed intervals each of them having the same length $\frac{1}{5^n}$

We define $C(\frac{2}{5})$ as $C(\frac{2}{5}) = \bigcap_{n=1}^{\infty} C_n$ and call it the Cantor $\frac{2}{5}$ set.

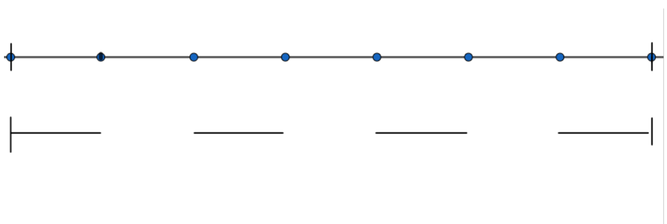
Geometrical representation




**Construction of Cantor $\frac{3}{7}$ set.**

In the construction of the Cantor $\frac{3}{7}$ set we consider the closed unit interval $[0, 1]$ and then divide it into 7 equal parts each part having length one by 7. Then from this remove open

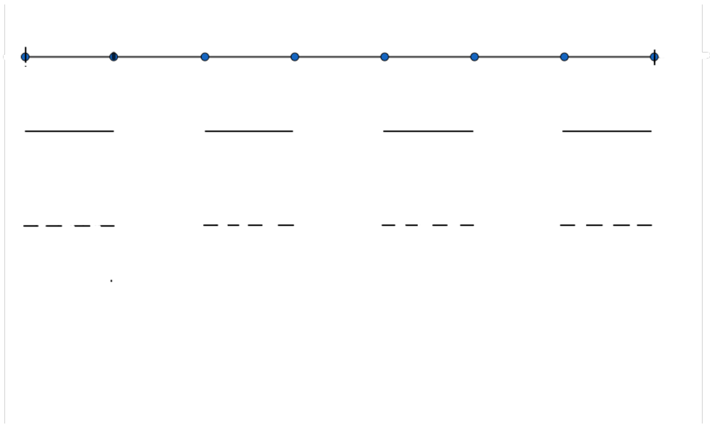
intervals alternatively that is, remove 2-nd, 4-th and 6-th open intervals from the above 7 parts. This leaves us a new set $M_1 = [0, \frac{1}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \cup [\frac{4}{7}, \frac{5}{7}] \cup [\frac{6}{7}, 1]$



Note that $C_1$ is a union of $2^2$ closed intervals of length $\frac{1}{7}$; and we removed $3(4^0) = 3$ open intervals, each of length $\frac{1}{7}$. Thus the total length removed at the step one is $3(\frac{1}{7})$.

In the second step we divide each of the four closed intervals in $C_1$ into 7 equal parts of length $\frac{1}{7^2}$.

Then remove from each segment in the previous iteration the $2nd, 4th$ and $6th$ open interval. Thus the next set will be



$$M_2 = [0, \frac{1}{49}] \cup [\frac{2}{49}, \frac{3}{49}] \cup [\frac{4}{49}, \frac{5}{49}] \cup [\frac{6}{49}, \frac{7}{49}] \cup [\frac{2}{7}, \frac{15}{49}] \cup \cdots \cup [\frac{46}{49}, \frac{47}{49}] \cup [\frac{48}{49}, 1]$$

Note that $C_2$ is a union of $4^2$ closed intervals , each of length $\frac{1}{7}$ and we removed

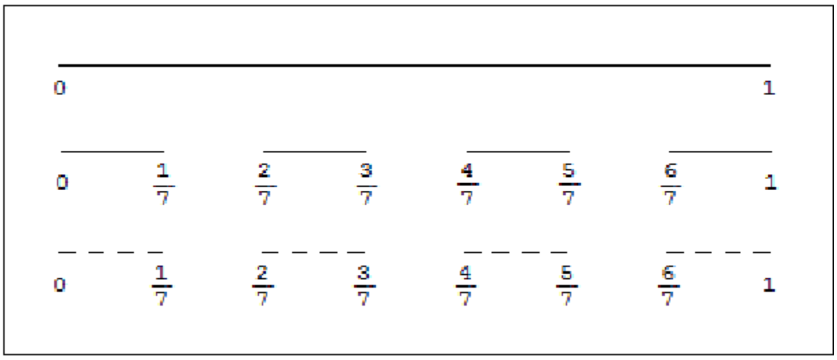



**Figure 6.** Construction of the middle-3/7 Cantor set.

$3(4^1) = 12$ open intervals, each of length $\frac{1}{7}$.

In general, after $n$ iterations we obtain $C_n$ as follows

$$C_n = [0, \frac{1}{7^n}] \cup \left[\frac{2}{7^n}, \frac{3}{7^n}\right] \cup \cdots \cup [\frac{7^n - 3}{7^n}, \frac{7^n - 2}{7^n}] \cup [\frac{7^n - 1}{7^n}, 1]$$

where $n \geq 1$

Therefore, we construct a decreasing sequence $(C_n)$ of closed sets, that is, $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, so that every $C_n$ consists of $4^n$ closed intervals each of them having the same length $\frac{1}{7^n}$ $7 = 2(3) + 1$

We define $C(\frac{3}{7})$ as $C(\frac{3}{7}) = \bigcap_{n=1}^{\infty} C_n$ and call it the Cantor $\frac{3}{7}$ set.

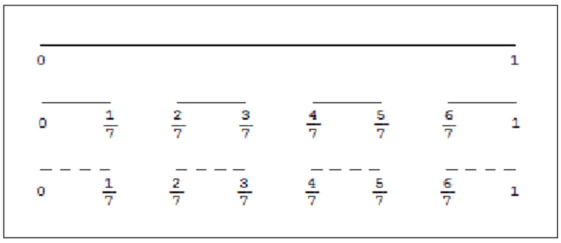
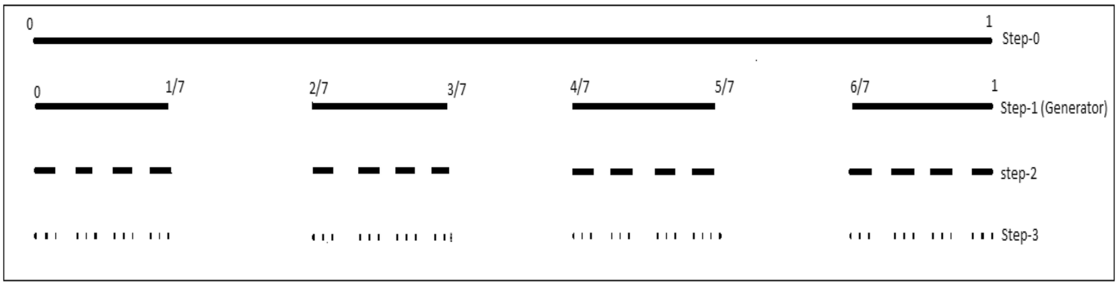



**Figure 6.** Construction of the middle-3/7 Cantor set.




### 4.1.4 The Smith-Volterra-Cantor Sets (SVC sets)

A particular family of SVC sets consists of those formed by, at the k-th iteration, removing an open interval of length $\frac{1}{n^k}$ from the center of each of the remaining closed intervals. We shall denote the resulting set SVC(n), $n \geq 3$. For $n = 3$, i.e. SVC(3) is noting but the standard Cantor ternary set

We shall see the case for $n = 4$ To create the set SVC(4), we remove an open interval of length $\frac{1}{4}$ from the middle of $[0, 1]$

First remove middle-1/4 from the interval $[0, 1]$ then we get two remaining intervals $[0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$ For the second step the remove $(\frac{1}{4})^2$ from both the remaining intervals. Subsequently, remove sub intervals of width $(1/4)^n$ from the middle of each of the remaining intervals Following is the illustration for the same.

**Fat Cantor sets**

Instead of removing a constant portion of the original set in each iteration, fat Cantor sets are created by removing progressively smaller portions of the original set in each step such that the ratio of what is being removed to the interval it is being removed from goes to 0 as n goes to infinity.

Smith Volterra Cantor sets SVC(n) for $n \geq 4$ are examples of Fat cantor sets.
Remove the middle $(\frac{1}{k})^n$ from $C_{n-1},\ k > 3$.
at each stage we remove $2^n$ intervals; each of length $(\frac{1}{k})^n$.
The Lebesgue measure of the removed intervals.
$= \frac{1}{k} + 2(\frac{1}{k})^2 + 4(\frac{1}{k})^3 + \ldots$

$= \frac{1}{k}\left(1 + \frac{2}{k} + (\frac{2}{k})^2 + (\frac{2}{k})^3 + \dots\right)$
$= \frac{1}{k}\left(\dfrac{1}{1-\frac{2}{k}}\right)$
$= \frac{1}{k}\left(\dfrac{k}{k-2}\right) = \frac{1}{k-2}$

Thus the Lebesgue measure of SVC(k) $= 1 - \frac{1}{k-2} = \dfrac{k-2-1}{k-2} = \dfrac{k-3}{k-2}$
Thus since $k > 3$, SVC(k) has non zero Lebesgue measure.

A closed set is nowhere dense if and only if it is equal to its boundary,

A nowhere dense set is not necessarily negligible in every sense. For example, if X is the unit interval [0,1], not only is it possible to have a dense set of Lebesgue measure zero (such as the set of rationals), but it is also possible to have a nowhere dense set with positive measure.

# Chapter 5

# CANTOR SET AS A FRACTAL

A Fractal is a type of mathematical shape that is infinitely complex. In essence, a Fractal is a pattern that repeats forever, and every part of the Fractal, regardless of how zoomed in, or zoomed out you are, it looks very similar to the whole image.

**Definition 5.1.** A set S is self similar if it can be divided into N congruent subsets, each of which when magnified yields the entire set S.

**Definition 5.2.** (Topological Dimension)
It is defined as the number of independent coordinates needed to specify the location of a point in a space.
Basically, the topological dimension of an object is a topological measure of the size of its covering properties. Roughly speaking, it is the number of coordinates needed to specify a point on the object. For example, a rectangle is two-dimensional, while a cube is three-dimensional. It a quantity having integer values.

Fractals are basically, self similar objects.

**Definition 5.3.** (Fractal dimension or Hausdorff Dimension)

Let S be a compact set and $N(S, r)$ be the minimum number of balls of radius r needed to cover S. Then the fractal dimension of S is defined as

$$dimS = lim_{r \to 0} \frac{logN(S, r)}{log1/r}$$

**Definition 5.4.** A fractal set is a set whose fractal dimension exceeds topological dimension.

Fractal dimension (also called Hausdorff dimension) is a fine tuning of the definition of topological dimension that allows notions of objects with dimensions other than integers.

**Cantor set has topological dimension 0**.

Lets find out **fractal dimension** of a Cantor ternary set.

In general, cantor set $\Delta$ consists of $2^n$ intervals, each of length $\frac{1}{3^n}$

Further we know that $\Delta$ contains the end points that lie $\frac{1}{3^n}$ apart.

Therefore, the smallest number of $\frac{1}{3^n}$-balls covering $\Delta$ is

$N(\Delta, 1/3n) = 2^n$

$$dim(\Delta) = lim_{n \to \infty} \frac{logN(\Delta, \frac{1}{3^n})}{log3^n}$$

$$= \frac{n\,log2}{n\,log3} = \quad approx.\ 0.6309$$

Hence, the fractal dimension exceeds the topological dimension.

Let us see one more example to understand the concept of self similarity and fractal dimension.

- The koch curve is constructed very differently start with a closed unit interval. At the 1st stage remove the middle third of the interval and replace it with two line segments of length 1/3 to make a tent. The resulting set consists of 4 line segments of length 1/3. At the next stage, repeat this procedure on all of the existing line segments, resulting in a set that contains 16 line segments of length 1/9. At each stage there are $4^n$ line segments of length $\frac{1}{3^n}$.

  When n $\rightarrow \infty$ the resulting set is called koch curve.

- The set is self-similar,with $4^n$ subsets at nth stage of length $(\frac{1}{3}^n)$ So, the fractal dimension is:

  $$lim_{n\to\infty}\frac{log4^n}{log3^n} = \frac{n\,log4}{n\,log3} = log_3(4) = approx.\ 1.2619$$

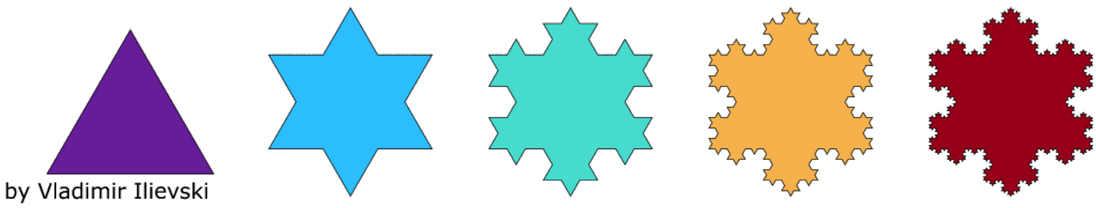



Figure 5.1: koch snowflake

# CONCLUSION

Following the construction of the standard Cantor set (denoted as $\Delta$ throughout this report) in geometrical way we discussed the ternary representation of the real numbers and their relation with the Cantor set. We concluded that any element in $[0, 1] \in \Delta$ iff its ternary representation contains the digit 0 and 2 only. Thus $\Delta$ is an example of set of cardinality same as [0,1] yet it has zero outer measure. The Cantor Lebesgue function which was discussed in chapter 3 is an example of monotonic increasing continuous function; in fact uniformly continuous function which is not absolutely continuous. It is also an example of a singular function, that is a function whose derivative is zero almost everywhere on [0,1]. Chapter 4 was attributed to various generalizations of the Cantor set and their Lebesgue measure. In the last chapter a brief light was drawn on fractals and their relation with the Cantor set.

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