Study of Variations in Lucky labeling of Graphs: A Comprehensive Analysis

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Study of Variations in Lucky labeling of Graphs: Comprehensive Analysis" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Jessica Fernandes e Pereira and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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COMPLETION CERTIFICATE

This is to certify that the dissertation report "Study of Variations in Lucky labeling of Graphs: Comprehensive Analysis" is a bonafide work carried out by Ms.Swejal Shankar Kalangutkar under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences, Goa University.

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PREFACE

This Thesis Report has been created as a part of the MAT-651 Discipline Specific Dissertation for the M.Sc. in Mathematics program during the academic year 2023-2024. It is on the topic "Study of Variations in Lucky labeling of Graphs:A Comprehensive Analysis" and is divided into four chapters, each systematically covering different aspects of the topic.

CHAPTER ONE :

The Introductory stage of this thesis report is based on overview of lucky labeling of different graphs, the definitions of the classes and types of graphs used throughout the thesis and few results based on lucky labeling used to prove theorems on labeling.

CHAPTER TWO:

This chapter is about the lucky Number of triangular Snake graphs, quadrilateral snake graphs, bloom graphs, extended triplicate of a star graph

CHAPTER THREE:

The chapter discusses the Proper Lucky labeling of triangular snake graph, quadrilateral snake graph, bloom graph, extended triplicate of a star graph, family of ladder graphs.

CHAPTER FOUR

It further deals with d-Lucky labeling of butterfly network graph, benes network, mesh network.

CHAPTER FIVE:

In this chapter we propose theorems on lucky numbers of butterfly network, cycles, wheel graph, mycielskian of paths , mycielskian of cycles, Lolipop graph and proper lucky number of butterfly network.

CHAPTER SIX:

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This chapter Concludes the lucky numbers that had been discussed in the preceding chapters.

ACKNOWLEDGEMENTS

Words cannot express my gratitude to my Professor Jessica Fernandes e Periera for her invaluable patience and feedback, I gained all my confidence to execute and complete this work by Her. I also could not have undertaken this journey without her support who generously provided knowledge and expertise. Additionally this endeavor would not have been possible without the generous support from the librarians, and study participants from the university, who impacted and inspired me.

Lastly, I would be remiss in not mentioning my family, especially my parents and siblings. Their belief in me has kept my spirits and motivation high during this process. I would also like to thank My cat Cookie for all the entertainment and emotional support.

ABSTRACT

Let $f:V(G) \to N$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of labels of the neighbors of the vertex v in G. If v is an isolated vertex of G we put $s(v) = 0$. Define $s(v) = \sum_{u \in N(v)} f(u)$, as the sum of neighborhood of vertex v, where $N(v)$ denotes the open neighborhood of $v \in V$. A labeling f is lucky if $s(v) \neq s(u)$ for every pair of adjacent vertices u and v.

After Lucky labeling, new labelling called Proper Lucky labeling, d-lucky labelling were introduced. In this work we will see the lucky, proper lucky and d-lucky numbers of few classes of graphs. We also found the Lucky numbers of butterfly network topology, cycle graph, wheel graph, mycielskian of path and cycle, lolipop graph, banana graph, and finally the proper lucky number of butterfly network topology..

Keywords: Lucky labeling; Lucky Number; Proper lucky labeling; Proper lucky Number; d-lucky labeling; d-lucky lucky number.

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Notations and Abbreviations

$\eta(G)$	Lucky number
$\eta_p(G)$	Proper Lucky number
$\eta_{dl}(G)$	d-Lucky number
$V(G)$	vertex set of G
$E(G)$	edge set of G
$\chi(G)$	chromatic number of G
P_n	path on n vertices
C_n	cycle on n vertices
K_n	complete graph
$\Delta(G)$	maximum degree
P_n	path on n vertices
$\delta(G)$	minimum degree
C_n	cycle on n vertices
K_n	complete graph
$B_{(m,n)}$	Bloom graph
$BF(n)$	Butterfly graph
TS_n	Triangular Snake Graph
DTS_n	Double Triangular Snake Graph
ATS_n	Alternate Triangular Snake Graph
$DATS_n$	Double Alternating Triangular Snake Graph
QS_n	Quadrilateral Snake Graph
DQS_n	Double Quadrilateral Snake Graph
AQS_n	Alternating Quadrilateral Snake Graph
$DAQS_n$	Double Alternating Quadrilateral Snake Graph
L_n	Ladder Graph
OL_n	Open Ladder Graph
TL_n	Triangular Ladder Graph
OTL_n	Open Triangular Ladder Graph
DL_n	Diagonal Ladder Graph
ETG_n	Triplicate Graph of Star Graph
M_{nxn}	Mesh Graph
EX_{nxn}	Extended Mesh
EN_{nxn}	Enhanced Mesh
Q_n	n-Dimensional Hypercube
BB_n	n-Dimensional Benes Network
$N(v)$	open Neighborhood of v
$N[v]$	closed Neighborhood of v
$B(m,n)$	banana tree graph
$\mu(P_n)$	mycielski graph of P_n
$\mu(C_n)$	mycielski graph of C_n
W_n	wheel graph on n vertices
$L_{m,n}$	lollipop graph
$f(u_i;j)$	Label of Vertices u_i in level j

Chapter 1

DEFINITIONS

1.1 Lucky Labeling and its Type

Definition 1.1.0.1. Lucky Labeling of Graphs

Let $f : V(G) \in N$ be a labeling of the vertices of a graph G by positive integers. Let $s(v)$ denote the sum of labels of the neighbors of the vertex v in G. If v is an isolated vertex of G we put $s(v) = 0$. Define $s(v) = \sum_{u \in N(v)} f(u)$, as the sum of neighborhood of vertex v, where $N(v)$ denotes the open neighborhood of $v \in V$. A labeling f is lucky if $s(v) \neq s(u)$ for every pair of adjacent vertices u and v.

Definition 1.1.0.2. Lucky Labeling Number

The lucky number of a graph G denoted by $\eta(G)$ is the least positive integer k such that G has a lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels.

Definition 1.1.0.3. Proper Lucky Labeling of Graphs A lucky labeling is proper lucky labeling if the labeling G is proper as well as lucky i.e. if u and v are adjacent in G then $f(u) \neq f(v)$ and $s(u) \neq s(v)$.

Definition 1.1.0.4. Proper Lucky Labeling Number

The proper lucky number of G is denoted by $\eta_p(G)$ is the least positive integer k such that G has a proper lucky labeling with $\{1,2,...,k\}$ as the set of labels.

Definition 1.1.0.5. d-lucky labeling of Graph

Let $l : V(G) \in N$ be a labeling of the vertices of a graph G by positive integers. Define $c(u) = \sum_{u \in N(v)} l(v) + d(u)$, where $d(u)$ denotes the degree of u and $N(v)$ denotes the open neighborhood of u. A labeling l is said to be d-lucky labeling if $c(u) \neq c(v)$, for every pair of adjacent vertices u and v in G.

Definition 1.1.0.6. d-lucky Labeling Number

the d-Lucky Label of a graph G denoted by $\eta_{dl}(G)$, is the least positive k such that G has d-lucky labeling with $\{1,2,...,k\}$ as the set of labels.

1.2 Graph Classes

1.2.1 Family of Triangular Graphs

Definition 1.2.1.1. Triangular Snake Graph

The triangular snake TS_n is constructed with $2n-1$ nodes where $n > 2$, u_i have n nodes, v_i have $n-1$ nodes and the edge $E(TS_n) = \{u_i u_{i+1} : 1 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_{1+i} v_i : 1 \leq i \leq n\}$.

Definition 1.2.1.2. Double Triangular Snake Graph

A double triangular snake graph DTS_n is constructed with the the $3n-2$ nodes where $n > 2$, u_i have n nodes, w_i have $n-1$ nodes and the edge set $E(DTS_n) = E(TS_n) \cup \{u_i w_i : 1 \leq i \leq n\} \cup \{u_{1+i} w_i : 1 \leq i \leq n\}$.

Definition 1.2.1.3. Alternate Triangular Snake Graph An alternate triangular snake ATS_n is constructed with the $\frac{3n}{2}$ nodes where $n > 4$, u_i have n nodes, v_i have $\frac{n}{2}$ nodes, and the edge set $E(ATS_n) = \{u_iu_{i+1} : 1 \leq i \leq n\} \cup \{u_{2i-1}v_i : 1 \leq i \leq n\} \cup \{u_{2i}v_i : 1 \leq i \leq n\}$.

Definition 1.2.1.4. Double Alternating Triangular snake

A double triangular snake $DATS_n$ is constructed with $2n$ nodes where $n > 2$, u_i have n nodes, v_i have $\frac{n}{2}$ nodes, w_i have $\frac{n}{2}$ nodes and the edge set $E(DATS_n) = E(ATS_n) \cup \{u_{2i-1}w_i : 1 \leq i \leq n\} \cup \{u_{2i}w_i : 1 \leq i \leq n\}$.

1.2.2 Family of Quadrilateral Graphs

Definition 1.2.2.1. Quadrilateral snake graph

A quadrilateral snake graph QS_n is obtained from $3n-2$ vertices where $n > 1$, u_i have n vertices, v_i have $2n-2$ vertices and the edge set $E(QS_n) = \{u_iu_{i+1} : 1 \leq 1 \leq n\} \cup \{v_{2i-1}v_{2i} : 1 \leq 1 \leq n\} \cup \{u_iv_{2i-1} : 1 \leq 1 \leq n\} \cup \{u_{i+1}v_{2i} : 1 \leq 1 \leq n\}$.

Definition 1.2.2.2. Alternating Quadrilateral Snake Graph

An alternating quadrilateral snake graph AQS_n consists of $2n$ vertices $n > 1$, u_i have n vertices, v_i have n vertices and the edge set $E(AQS_n) = \{u_iu_{i+1} : 1 \leq 1 \leq n\} \cup \{v_{2i-1}v_{2i} : 1 \leq 1 \leq n\} \cup \{u_{2i}v_{2i} : 1 \leq 1 \leq n\} \cup \{u_{2i-1}v_{2i-1} : 1 \leq 1 \leq n\}$.

Definition 1.2.2.3. A double quadrilateral snake graph DQS_n consist of a $5n-2$ vertices where $n > 1$, u_i have n vertices, v_i have $2n-2$ vertices and the edge set $E(DQS_n) = E(QS_n) \cup \{w_{2i-1}w_{2i-1} : 1 \leq 1 \leq n\} \cup \{u_{i+1}w_{2i} : 1 \leq 1 \leq n\}$.

Definition 1.2.2.4. Double Alternate Quadrilateral Snake Graph

An alternate double quadrilateral snake Graph DAQ_n is consist of a $3n$ vertices where $n > 1$, u_i have n vertices, v_i have n vertices and the edge set $E(DAQS_n) = E(AQS_n) \cup \{w_{2i-1}w_{2i} : 1 \leq 1 \leq n\} \cup \{u_{2i}w_{2i} : 1 \leq 1 \leq n\} \cup \{u_{2i-1}v_{2i-1} : 1 \leq 1 \leq n\}$.

1.2.3 Family Of Ladder Graphs

Definition 1.2.3.1. Ladder Graph

The ladder graph has vertices u_{i} and v_{j} are the two paths in the graph. $V(G) = \{u_{\mathrm{i}}v_{\mathrm{j}} : i = j, 1 < i \leq n\}$ and the edge set is given by $E(G) = \{u_i u_{i+1}, v_i vi + 1 : i = j, 1 < i \leq n$ $cup\{u_j v_j : i = j, 1 < i \leq n\}$ It is denoted by L_n .

Definition 1.2.3.2. Open Ladder Graph

An open ladder Graph generated from a ladder graph with $n > 2$ by excluding the Edges $u_{\mathrm{i}}\ v_{\mathrm{j}}$, for $i = 1$ and $n, j = 1$ and n . It is denoted by OL_{n}.

Definition 1.2.3.3. Slanting Ladder Graph

A Slanting ladder is the graph obtained from two paths u_{i} and v_{j} by joining each u_{j} with $v_{\mathrm{j+1}}$, $1 \leq i \leq n-1$, $1 \leq j \leq n-1$. It is denoted by SL_{n}.

Definition 1.2.3.4. Triangular Ladder Graph

A Triangular ladder Graph is obtained from l_n with $n \geq 2$ by adding the edges $E(G) = \{u_{i+1}v_j : i = j, 1 \leq n-1, 1 \leq j \leq n-1\}$. It is denoted by TL_{n}.

Definition 1.2.3.5. Open Triangular Graph

An open Triangular Ladder Graph is generated from a Triangular ladder graph with $n > 2$ by removing the edges $u_{\mathrm{i}}v_{\mathrm{j}}$, for $i = 1$ and $n, j = 1$. It is denoted by OTL_{n}.

Definition 1.2.3.6. Diagonal Ladder Graph

A Diagonal ladder is a graph obtained from L_{n} by adding the edges $E(G) = \{u_{\mathrm{i}}v_{\mathrm{j+1}} : i = j, 1 \leq n-1, 1 \leq j \leq n-1\} \cup \{u_{\mathrm{i+1}}v_{\mathrm{j}} : i = j, 1leqi \leq n-1, 1 \leq æ \leq n-1\}$. For all $n \geq 2$, it is denoted by DL_{n}.

Definition 1.2.3.7. Open Ladder Graph

An open Diagonal Ladder graph is generated from a diagonal Ladder graph by excluding the edges $u_{\mathrm{i}}\ v_{\mathrm{j}}$, for $i = 1$ an d $n, j = 1$. It is denoted by ODL_{n}.

1.2.4 Other Graphs

Definition 1.2.4.1. The Bloom graph

$B_{m,n}$, $m,n > 2$ is defined as follows:

$V(B_{m,n}) = \{(x,y) : 0 \leq m-1, 0 \leq y \leq y \leq n-1\}$ two distinct vertices (x_1, y_1) and (x_1, y_2) being adjacent $\Longleftrightarrow$

(i) $x_2 = x_1 + 1$ and $y_1 = y_2$

(ii) $x_1 = x_2$ and $y_1 = y_1(modn)$

(iii) $x_1 = x_2$ and $y_1 + 1 = y_2(modn)$

(iv) $x_2 = x_1 + 1$ and $y_1 = y_2(modn)$

Definition 1.2.4.2. Triplicate Graph of Star Graph

Let G be a Star graph $K_{1,n}$. The triplicate graph for star graph with vertex set $\delta(G)$ and edge set $\zeta(G)$ is given by $\delta(G) = \{bc_i{}' \cup b'c_i \cup b'c_i{}'' \cup b''c_i{}'/1 \leq i \leq n\}$ and $\zeta(G) = \{bc'_i \cup b'c_i \cup b'c_i{}'' \cup b"c'_i/1 \leq i \leq n\}$. Clearly, triplicate graph of star graph $TG(K_{1,n})$ with this vertex set and edge set is disconnected. To make this a connected graph, include a new edge bc_1 to the edge set $\zeta(G)$. Thus, we get an extended triplicate graph of star with vertex set δ ' = $\zeta(G)$ and edge set $\zeta'(G) = \zeta \cup bc_1$, denoted by $ETG(K_{1,n})$. clearly, $ETG(K_{1,n})$. has $3(n+1)$ vertices and $4n+1$ edges.

Definition 1.2.4.3. *n*-dimensional Hypercube

The vertex set of Q_n consists of all binary sequence of length n on the set $0,1$ i.e,$V = \{x_1, x_2, .., x_n \in \{0,1\}, i = 1,2,..,n\}$.Two vertices are linked by an edge $\Longleftrightarrow$ x and y differ in exactly one coordinate. Q_n is defined recursively as follows, $Q_1 = k_2$, $Q_n = Q_{n-1}XQ_1 = K_2XK_2...K_2$

Definition 1.2.4.4. *n*-dimensional Benes Network

The $n-dimensional$ benes network consists of back to back butterfly ,denoted by

$BB(n)$.The $BB(n)$ has $2n+1$ levels, each with 2^n vertices.The first and last $n+1$ levels of $BB(n)$ form two $BF(n)$ respectively. While the middle level in $BB(n)$ is shared by these butterfly networks.The $n-dimensional$ Benes network has $(n+1)2^{n+1}$ vertices and $n2^{n+2}$ edges.It has only 2-degree and 4-degree vertices, and thus eulerian.

Definition 1.2.4.5. X-Tree

An X-Tree XT_{n} is obtained from complete binary tree on $2^{n+1}-1$ vertices of length 2^t-1 and adding paths P_{i} left to right through all the vertices at level i ; $1 \leq i \leq n$.

Definition 1.2.4.6. Butterfly network topology

A butterfly network topology consists of $(k+1)2^k$ nodes arranged in k+1 ranks, each row or rank containing $n = 2^k$ nodes. k is called the order of the network. For $P(I,J)$ the j^{th} node to i^{th} rank where $(1 < i < k)$ and $0 \leq j \leq n$ .Then construct the two nodes $p(i-1,m)$ $p(i-1,m)$ on rank $i,i-1$, where m is the integer found by inverting the i^{th} bit in the binary representation of j.

Definition 1.2.4.7. Banana Graph

An (n,k)-banana tree, as defined by Chen et al. (1997), is a graph obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex that is distinct from all the stars

Definition 1.2.4.8. Lollipop Graph

The (m,n)-lollipop graph is the graph obtained by joining a complete graph K_{m} to a path graph P_{n} with a bridge.

Definition 1.2.4.9. Mycielski Graph

The Mycielski Graph of a graph G is obtained with $V(G) = \{v_1, v_2, ..., v_{\mathrm{n}}\}$ is the graph which is obtained by applying the following steps:

1. Corresponding to each vertex v_{i} in $V(G)$, introduce a new vertex u_{i} and let $U = \{u_{\mathrm{i}} : 1 \leq i \leq n\}$. Add edges from each vertex u_{i} of U to the vertex v_{j} if $v_{\mathrm{i}}v_{\mathrm{j}} \in E(G)$.

2. Take another vertex w and add edges from w to all the vertices in U.

The new graph thus obtained is called the Mycielski Graph and is denoted by $\mu(G)$.

1.3 Theorems used

Theorem 1.3.0.1. *The chromatic number of complete graph K_n is $\chi(K_n) = n$.*

Theorem 1.3.0.2. *The chromatic number of any graph G is greater than or equal to the clique number i.e. $\omega \leq \chi$*

Theorem 1.3.0.3. *For any connected graph G, the chromatic number is less than or equal to proper lucky number i.e. $\chi \leq \eta_p$*

Theorem 1.3.0.4. *For any connected graph G, the chromatic number is less than or equal to proper lucky number i.e. $\omega \leq \eta_p$*

Theorem 1.3.0.5. *For any connected Graph G, $\eta(G) \leq \eta_p(G)$*

Chapter 2

LUCKY LABELING OF GRAPHS

Definition 2.0.0.1. Let $f : V(G) \in N$ be a labeling of the vertices of a graph G by positive integers. Let $s(v)$ denote the sum of labels of the neighbors of the vertex v in G. If v is an isolated vertex of G we put $s(v) = 0$. Define s(v)=$\sum_{u \in N(v)} f(u)$, as the sum of neighborhood of vertex v, where $N(v)$ denotes the open neighborhood of $v \in V$. A labeling f is lucky if $s(v) \neq s(u)$ for every pair of adjacent vertices u and v.

Theorem 2.0.0.2. *The Triangular Snake TS_n with $n > 2$ is Lucky Graph with $\eta(TS_n) = \frac{\Delta(TS_n)}{2}$=2*

Proof. Let $f : V(TS_\text{n}) \to \{1,2\}$ be defined by following 2 cases

case(i) $n =$ even

Labeling part:

$$f(u_\text{i}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

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$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i > 1 \\ 2, & i = 1 \end{cases}$$

Sum of the Labels of the adjacent vertices:

$$s(v_{\mathrm{i}}) = 3$$

$$s(u_{\mathrm{i}}) = \begin{cases} 2, & i = n \\ 4, & i = 1, i > 2, i > n, \text{ } i \text{ even} \\ 5, & i = 2 \\ 6, & i > 1, \text{ } i \text{ odd} \end{cases}$$

Therefore TS_{n} with $n > 2$ is Lucky Graph with $\eta(TS_{\mathrm{n}}) = 2$ for even n.

case(ii) n= odd

Labeling Part:

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i > n, \text{ } i \text{ odd} \\ 2, & i = n, \text{ } i \text{ even} \end{cases}$$

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i > 1 \\ 2, & i = n \end{cases}$$

Sum of the Labels of the adjacent vertices

$$s(u_{\mathrm{i}}) = \begin{cases} 3, & i = n \\ 4, & i > 2, \text{ } i < n-1, \text{ } i = 1, \text{ } i \text{ even} \\ 5, & i = 2, n-1 \\ 6, & 1 < i < n, \text{ } i \text{ odd} \end{cases}$$

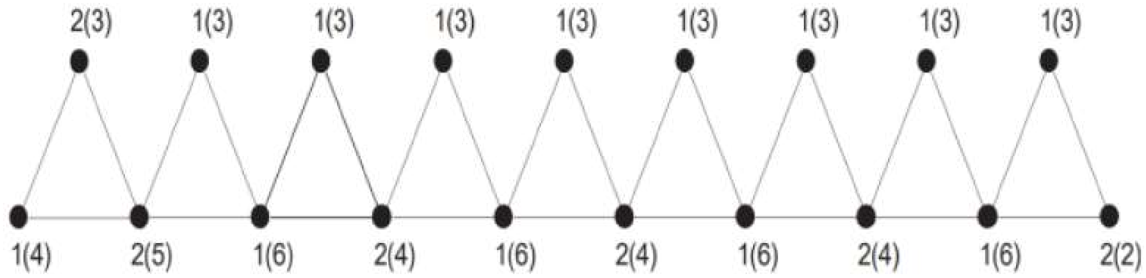

Figure 2.1: Lucky triangular Snake Graph TS_{10}

$$s(v_i) = \begin{cases} 3, & i > n \\ 4, & i = n \end{cases}$$

Therefore TS_n with $n > 2$ is Lucky graph with $\eta(TS_n) = 2$ for odd n.

Hence TS_n with $n > 2$ is Lucky graph with

$\eta(TS_{\mathbf{n}}) = \frac{\Delta(TS_{\mathbf{n}})}{2}$=2 □

Theorem 2.0.0.3. *The double triangular snake DTS_n with $n > 2$ with $\eta(DTS_n) = \Delta(\frac{DTS_n}{3}) = 2$*

Proof. Let f:V(TS_n)$\rightarrow \{1,2\}$

case(i) $n =$ even

Labeling Part:

$$f(u_i) = \begin{cases} 1, & i = 2, i = n, i \text{ odd} \\ 2, & 2 < i < n, i \text{ even} \end{cases}$$

$$f(v_i) = 1$$
$$f(w_i) = 1$$

Sum of the Labels of the adjacent vertices

$$s(u_{\mathrm{i}}) = \begin{cases} 3 & i = 1 \\ 4 & i = n \\ 6, & i < n, i \text{ even} \\ 7, & i = 3, n-1 \\ 8, & i > 3, i < n-1, i \text{ odd} \end{cases}$$

$$s(v_{\mathrm{i}}) = \begin{cases} 2, & i = 1,2,n \\ 3, & 2 < i < n \end{cases} s(w_{\mathrm{i}}) = \begin{cases} 2, & i = 1,2,n \\ 3, & 2 < i < n \end{cases}$$

Therefore for $n > 2$, $\eta(DTS_{\mathrm{n}}) = 2$ for even n

case(ii) $n = \text{odd}$

Labeling Part:

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i = 2, i \text{ odd} \\ 2, & i > 2, i \text{ even} \end{cases} f(v_{\mathrm{i}}) = 1$$

$$f(w_{\mathrm{i}}) = 1$$

Sum of the Labels of the adjacent vertices:

$$s(u_{\mathrm{i}}) = \begin{cases} 6, & i \text{ even} \\ 7, & i = 3 \\ 8, & i > 3, i < n-1, i \text{ odd} \end{cases}$$

$$s(v_{\mathrm{i}}) = \begin{cases} 2, & i = 1,2 \\ 3, & i > 2 \end{cases}$$

$$s(w_{\mathrm{i}}) = \begin{cases} 2, & i = 1,2 \\ 3, & i > 2 \end{cases}$$

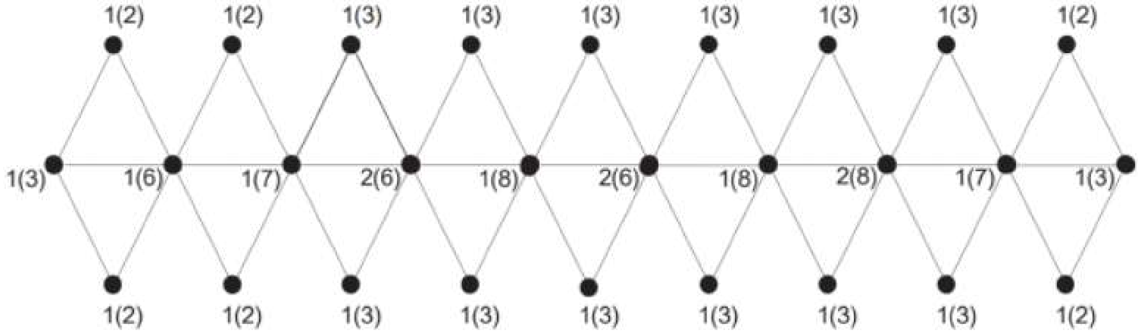

Figure 2.2: Lucky Double triangular snake graph DTS_{10}

Therefore the sum of the labels of the adjacent vertices are distinct

Therefore for DTS_{n} with $n > 2$, $\eta(DTS_{\text{n}}) = 2$ for odd n.

Hence for DTS_{n} with $n > 2$, $\eta(DTS_{\text{n}}) = \frac{DTS_{\text{n}}}{3}$=2 □

Theorem 2.0.0.4. *The alternate triangular snake graph ATS_n with $n > 4$ is Lucky graph with $\eta(ATS_n) = \Delta(ATS_n) - 1 = 2$*

Proof. Let $f : V(ATS_{\text{n}}) \rightarrow \{1,2\}$ be defined by

Labeling Part:

$$f(u_{\text{i}}) = \begin{cases} 1, & i \neq 3, i = n, \, i \text{ odd} \\ 2, & i = 3, i \neq n, \, i \text{ even} \end{cases}$$

$$f(v_{\text{i}}) = 2$$

Sum of the Labels of the adjacent vertices:

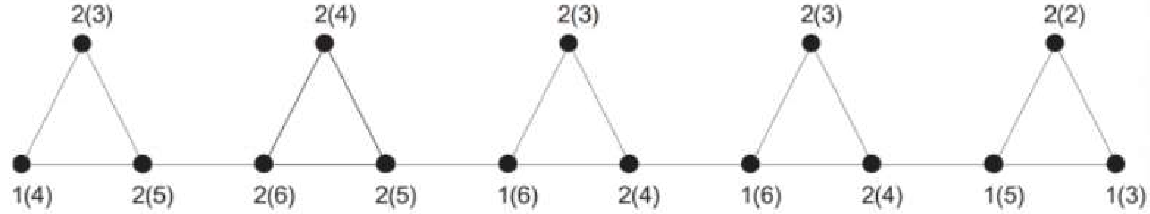

Figure 2.3: Lucky Alternate triangular snake graph ATS_{10}

$$s(u_i) = \begin{cases} 3, & i = n \\ 4, & i > 4, i \neq n, i = 1, i \text{ even} \\ 5, & i \leq 4, i = n-1, i \text{ even} \\ 6, & i > 1, i < n-1, i \text{ odd} \end{cases}$$

$$s(v_i) = \begin{cases} 2, & i = \frac{n}{2} \\ 3, & i \neq \frac{n}{2}, i \neq 2 \\ 4, & i = 2 \end{cases}$$

Therefore for ATS_n with $n > 4$, $\eta(ATS_n) = \Delta(ATS_n) - 1$

Therefore the sum of the labels of the adjacent vertices are distinct

□

Theorem 2.0.0.5. *The Double alternate triangular snake graph $DATS_n$ with $n > 2$ is Lucky graph with $\eta(DATS_n) = \frac{\Delta(DATS_n)}{2} = 2$*

Proof. Let $f : V(DATS_n) \to \{1, 2\}$ be defined by

Labeling Part:

$$f(u_i) = \begin{cases} 1, & i = 2, i \text{ odd} \\ 2, & i > 2, i \text{ even} \end{cases}$$

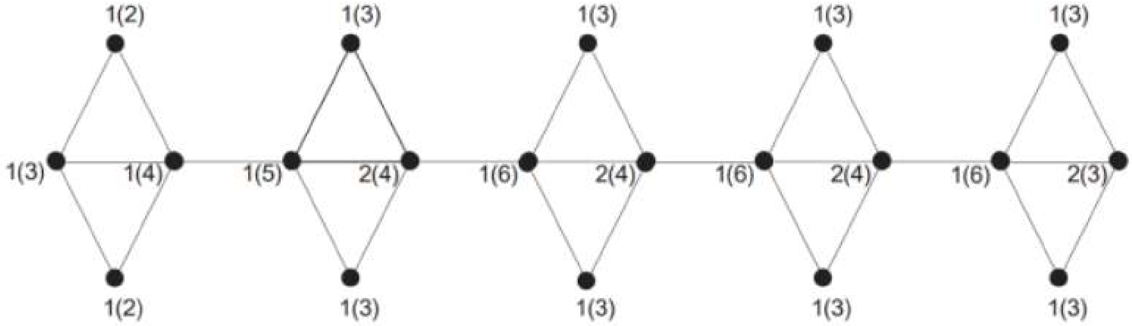

Figure 2.4: Lucky double alternate triangular snake graph $DATS_{10}$

$$f(v_{\mathrm{i}}) = 1$$
$$f(w_{\mathrm{i}}) = 1$$

Sum of the Labels of the adjacent vertices:

$$s(u_{\mathrm{i}}) = \begin{cases} 3, & i = 1, n \\ 4, & i < n, i \text{ even} \\ 5, & \text{even } i = 3 \\ 6, & i > 3, i \text{ odd} \end{cases} \quad s(v_i) = \begin{cases} 2, & i = 1 \\ 3, & i > 1 \end{cases}$$

$$s(w_i) = \begin{cases} 2, & i = 1 \\ 3, & i > 1 \end{cases}$$

Therefore the sum of the labels of the adjacent vertices are distinct

Therefore for $DATS_{\mathrm{n}}$ with $n > 2$, $\eta(DATS_{\mathrm{n}}) = \frac{\Delta(DATS_{\mathrm{n}})}{2} = 2$

□

Theorem 2.0.0.6. *The quadrilateral snake QS_n with $n > 1$ admits Lucky labeling with lucky number $\eta(QS_n) = 2$*

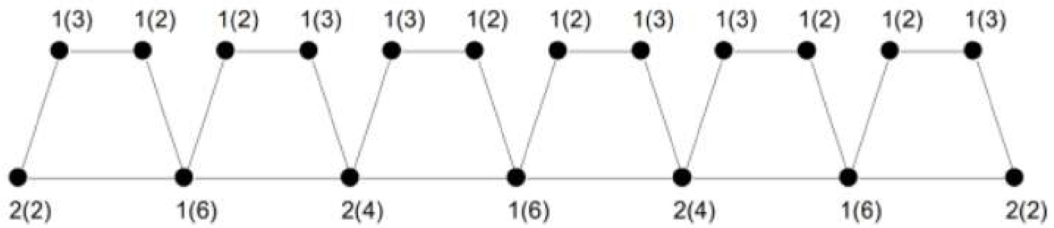

Figure 2.5: Lucky quadrilateral Snake Graph QS_7

Proof. let $f : V(QS_n) \to \{1,2\}$ be defined by

Labeling Part:

$$f(u_i) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$
$$f(v_i) = 1$$

Sum of the Labels of the adjacent vertices:

$$s(u_i) = \begin{cases} 2, & i = 1, i = \text{odd } n \\ 3, & i = \text{even } n \\ 4, & 1 < i < n, i \text{ odd} \\ 6, & i < n, i \text{ even} \end{cases}$$

for $k \in \mathbf{N}$

$$s(v_i) = \begin{cases} 2, & i = 4k-1, 4k-2 \\ 3, & i = 4k, 4k-3 \end{cases}$$

Therefore sum of adjacent vertices are not same.

Therefore for QS_n with $n > 1$, $\eta(QS_n) = 2$ □

Theorem 2.0.0.7. *The double quadrilateral snake graph DQS_n with $n > 1$ admits Lucky labeling with lucky labeling Number $\eta(DQS_n) = 2$*

Proof. Let $f : V(DQS_n) \to \{1,2\}$ be defined by

case(i) n = even

Labeling Part:

$$f(u_i) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i \neq 2 \\ 2, & i = 2 \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i \neq 2 \\ 2, & i = 2 \end{cases}$$

Sum of the labels of the adjacent vertices:

$$s(u_1) = 3$$

$$s(u_n) = 4$$

$$s(u_i) = \begin{cases} 6, & i > 1, i \text{ odd} \\ 8, & n > i > 2, i \text{ even} \\ 10, & i = 2 \end{cases}$$

for $k \in \mathbf{N}$

$$s(v_i) = \begin{cases} 2, & i = 4k-1, 4k-2 \\ 3, & i = 4k, 4k-3, i \neq 1 \\ 4, & i = 1 \end{cases}$$

$$s(w_{\mathrm{i}}) = \begin{cases} 2, & i = 4k-1, 4k-2 \\ 3, & i = 4k, 4k-3, i \neq 1 \\ 4, & i = 1 \end{cases}$$

The DQS_{n} with $n > 1$ for n=even has $\eta(DQS_{\mathrm{n}}) = 2$.

case(ii) $n =$ odd Let $f : V(DQS_{\mathrm{n}}) \rightarrow \{1, 2\}$ be defined by

Labeling Part:

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i \neq 2, n-1 \\ 2, & i = 2, n-1 \end{cases}$$

$$f(w_{\mathrm{i}}) = \begin{cases} 1, & i \neq 2, n-1 \\ 2, & i = 2, n-1 \end{cases}$$

Sum of the labels of the adjacent vertices:

$$s(u_1) = 3$$

$$s(u_{\mathrm{n}}) = 3$$

$$s(u_{\mathrm{i}}) = \begin{cases} 6, & \text{odd } n > i > 1, \\ 8, & \text{even } 2 < i < n-1 \\ 10, & i = 2, n-1 \end{cases}$$

for $k \in \mathbf{N}$

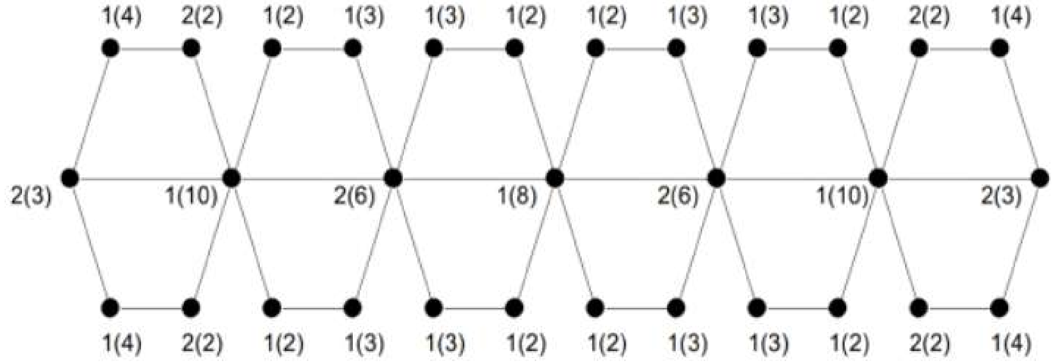

Figure 2.6: Lucky Double quadrilateral Snake Graph DQS_7

$$s(v_{\rm i}) = \begin{cases} 2, & i = 4k-1, 4k-2, i \neq n \\ 3, & i = 4k, 4k-3, i \neq 1, n \\ 4, & i = 1, n \end{cases}$$

$$s(w_{\rm i}) = \begin{cases} 2, & i = 4k-1, 4k-2, i \neq n \\ 3, & i = 4k, 4k-3, i \neq 1, n \\ 4, & i = n, 1 \end{cases}$$

The $DQS_{\rm n}$ with $n > 1$ for n=odd has $\eta(DQS_{\rm n}) = 2$.

Hence for the double quadrilateral snake $DQS_{\rm n}$ with $n > 2$, $\eta(DQS_{\rm n}) = 2$ □

Theorem 2.0.0.8. *The alternate quadrilateral snake graph AQS_n with $n > 1$ admits Lucky Labeling with lucky Number $\eta(AQS_n)$=2*

Proof. Let $f : V(AQS_{\rm n}) \to \{1, 2\}$ defined by

Labeling Part:

$$f(u_{\rm i}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

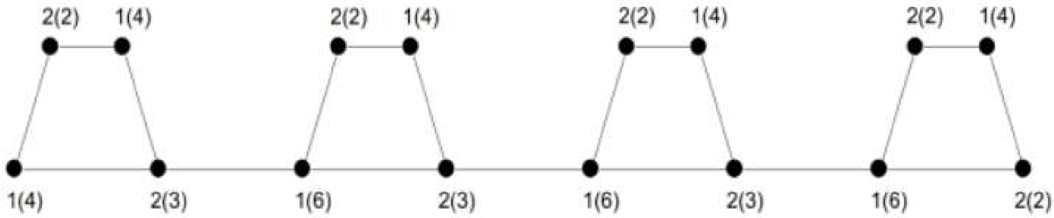

Figure 2.7: Lucky Alternate quadrilateral Snake Graph AQS_8

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

Sum of Labels of neighbour vertices:

$$s(u_{\mathrm{i}}) = \begin{cases} 2, & i = n \\ 3, & i < n, i \text{ even} \\ 4, & i = 1 \\ 6, & i > 1, i \text{ odd} \end{cases}$$

$$s(v_{\mathrm{i}}) = \begin{cases} 2, & i \text{ odd} \\ 4, & i \text{ even} \end{cases}$$

Therefore sum of adjacent vertices are not the same.

Therefore for $n > 1$ $\eta(AQS_{\mathrm{n}}) = 2$.

Therefore for AQS_{n} with $n > 1$ has $\eta(AQS_{\mathrm{n}}) = 2$ □

Theorem 2.0.0.9. *The double alternate quadrilateral snake graph $DAQS_n$ with $n > 1$ admits Lucky Labeling with lucky Number $\eta(DAQS_n)$=2*

Proof. Let $f : V(DAQS_{\mathrm{n}}) \to \{1,2\}$ defined by

Labeling Part:

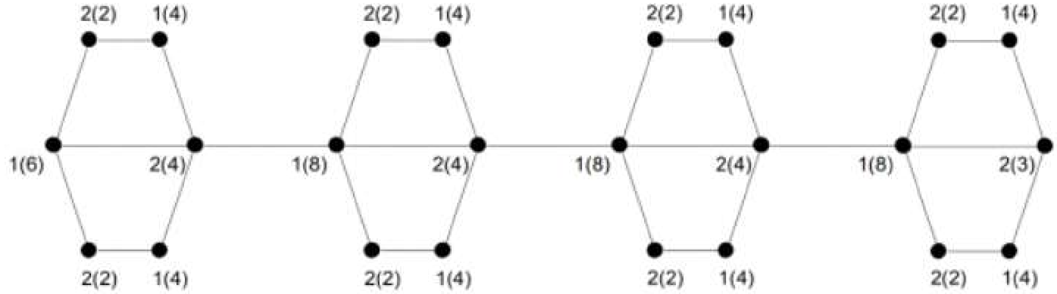

Figure 2.8: Lucky double alternate quadrilateral Snake Graph $DAQS_8$

$$f(u_i) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_i) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

Sum of Labels of neighbour vertices:

$$s(u_i) = \begin{cases} 3, & i = n \\ 5, & i < n, i \text{ even} \\ 6, i = 1 & \\ 8, & i > 1, i \text{ odd} \end{cases}$$

$$s(v_i) = \begin{cases} 2, & i \text{ odd} \\ 4, & i \text{ even} \end{cases}$$

$$s(w_i) = \begin{cases} 2, & i = n-1 \\ 3, & i \neq n-1, i \text{ odd} \\ 4, & i \text{ even} \end{cases}$$

Therefore sum of the adjacent vertices are not the same.

Therefore for $n > 1$ $\eta(DAQS_{\text{n}}) = 2$.

Therefore for $DAQS_{\text{n}}$ with $n > 1$ has $\eta(DAQS_{\text{n}}) = 2$ □

Theorem 2.0.0.10. *The mxn dimensional bloom graph $B_{m,n}$ admits lucky labeling and $\eta(B_{m,n}) = 2$, where $m > 5$ and $n > 3$.*

Proof. Since the bloom graph is 4-regular $\eta(B_{\text{m,n}}) \neq 1$

Case 1:When n is even.

Label the vertices in first row as 1 and 2 alternatively, without loss of generality $(0,0)$ is assigned $1, (0,1)$ is assigned 2 and so on. Similarly label the m^{th} row as 1 and 2 alternatively, without loss of generality $(m-1,0)$ is assigned 1, $(m-2,0)$ is assigned 2 and so on. Label the vertices in k^{th} row as 1 when k is odd and as 2 when k is even, $2 \leq k \leq m-1$.

Subcase(i):When m is even

The vertices in the first row will have the neighbourhood some as 8 and 6 alternatively i.e. $s(0,0) = 8$, $s(0,1) = 6$, and so on. All the vertices in the second row attains the neighbourhood sum as 5, i.e. $s(1,j) = 5, 0 \leq j \leq n-1$. All the vertices in the second row attains the neighbourhood sum as 8 when k is odd i.e. $s(2i,j) = 8$ when $1 \leq i \leq \frac{m-4}{2}$ and $0 \leq j \leq n-1$. All the vertices in the k^{th} row attain the neighbourhood sum as 4 when k is even i.e. $s(2i+1,j) = 4$ when $1 \leq i \leq \frac{m-4}{2}$ and $0 \leq j \leq n-1$. All the vertices in the $(m-1)^{\text{th}}$ row attains the neighbourhood sum as 7 i.e. $s(m-2,j) = 7, 0 \leq j \leq n-1$. All the vertices in the m^{th} row will have the neighbourhood sum as 6 and 4 alternatively i.e. $s(m-1,0) = 6$, $s(m-1,1) = 4$, and so on. It is evident that no two vertices have equal neighbourhood sum.

Subcase(ii):When m is odd

The vertices in the first row will have the neighbourhood sum as 8 and 6 alternatively i.e. $s(0,0) = 8$, $s(0,1) = 6$, and so on. All the vertices in the second row attains the

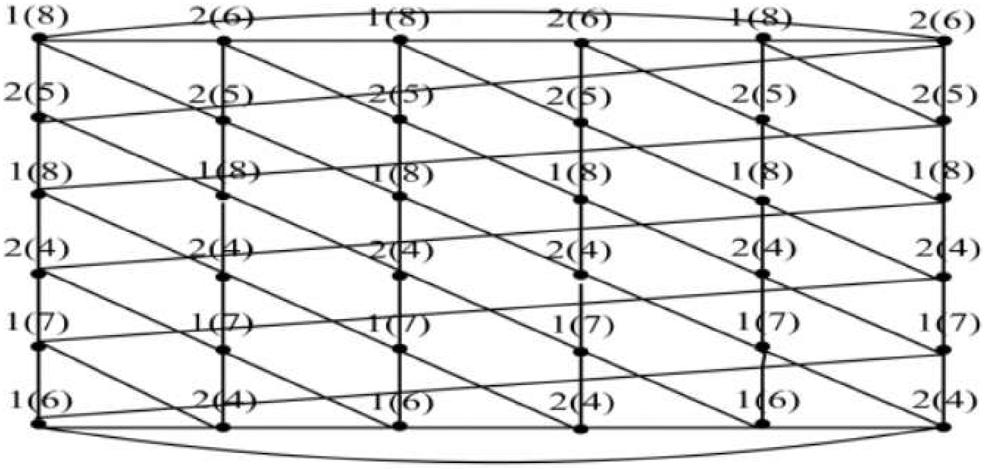

Figure 2.9: Lucky labeling of $B_{6,6}$

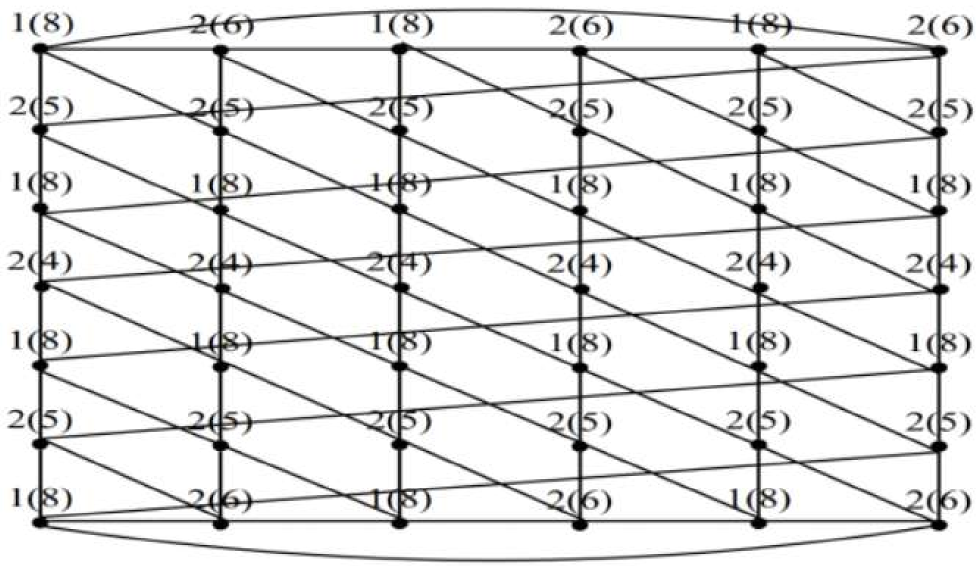

Figure 2.10: Lucky labeling of $B_{7,6}$

neighbourhood sum as 5 i.e. $s(1,j)=5, 0\leq j\leq n-1$. All the vertices in the k^{th} row attains the neighbourhood sum as 8 when k is odd i.e. $s(2i,j)=8$ when $1\leq i\leq \frac{m-4}{2}$ and $0\leq j\leq n-1$. All the vertices in the k^{th} row attains the neighbourhood sum as 4 when k is even i.e. $s(2i+1,j)=4$ when $1\leq i\leq \frac{m-4}{2}$ and $0\leq j\leq n-1$. All the vertices in the $(m-1)^{\text{th}}$ row attains the neighbourhood sum as 5 i.e. $s(m-2,j)=5, 0\leq j\leq n-1$. All the vertices in the m^{th} row will have neighbourhood sum as 8 and 6 alternatively i.e. $s(m-1,0)=8$, $s(m-1,1)=6$ and so on. It is evident that no two vertices have equal neighbourhood sum.

case (ii) When n is odd and m is odd.

Label the vertices in first row as 1 and 2 alternatively, i.e. $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on. In the second row, label the first three vertices as 2 and the rest as 1 i.e. $(1,0)\rightarrow 2$, $(1,1)\rightarrow 2$, $(1,2)\rightarrow 2$ and $(i,j)\rightarrow 1$ where $3\leq j\leq n-1$. In the $(m-1)^{\text{th}}$ row, label the last three vertices as 2 and the rest as 1 i.e. $(m-2,j)\rightarrow 1$ where $0\leq j\leq n-4$ and $(m-2,n-3)\rightarrow 2$, $(m-2,n-2)\rightarrow 2$, $(m-2,n-1) rightarrow 2$ i.e. $(m-2,j)\rightarrow 1$ where $0\leq j\leq n-4$ and $(m-2,n-3)\rightarrow 2$, $(m-2,n-2)\rightarrow 2$, $(m-2,n-1)\rightarrow 2$. Label the m^{th} row as 1 and 2 alternately, i.e. $(m-1,0)$ is assigned 1, $(m-1,1)$ is assigned 2 and so on. For the remaining vertices, label the vertices in k^{th} row as 1when k is odd and as 2 when k is even, $3\leq k\leq m-2$. It is evident that no two vertices have equal neighbourhood sum.

Hence $B_{m,n}$ admits lucky labeling with $\eta(B_{m,n})=2$

case (iii) When n is odd and m is even.

Label the vertices in first row as 1 and 2 alternatively, i.e. $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on. In the second row, label the first three vertices as 2 and the rest as 1 i.e. $(1,0)\rightarrow 2$, $(1,1)\rightarrow 2$, $(1,2)\rightarrow 2$ and $(i,j)\rightarrow 1$ where $3\leq j\leq n-1$. In the $(m-1)^{\text{th}}$ row, label the last three vertices as 1 and the rest as 2 i.e. $(m-2,j)\rightarrow 2$ where $0\leq j\leq n-4$ and $(m-2,n-3)\rightarrow 1$, $(m-2,n-2)\rightarrow 1$, $(m-2,n-1) rightarrow 1$. Label the m^{th} row as 2 and 1 alternately, i.e. $(m-1,0)$ is assigned 2, $(m-1,1)$ is

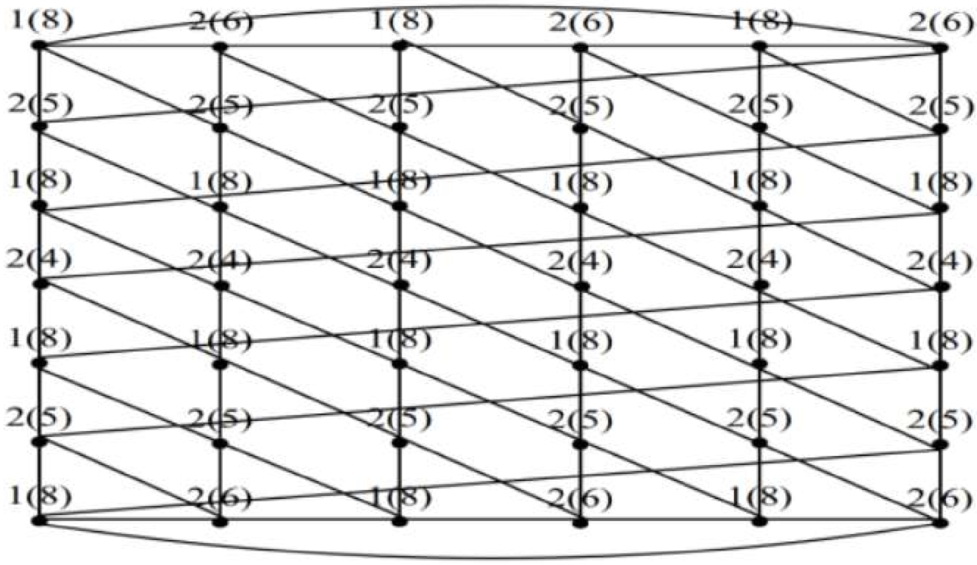

Figure 2.11: Lucky labeling of $B_{9,7}$

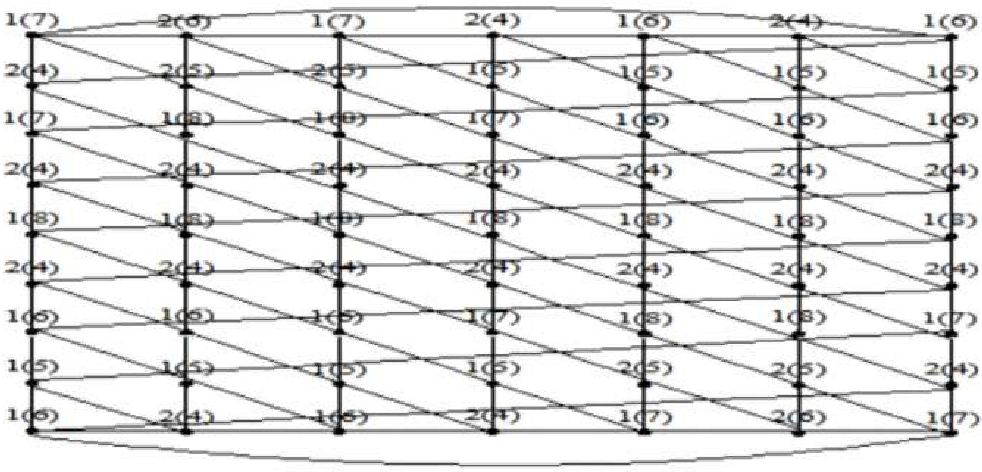

Figure 2.12: Lucky labeling of $B_{8,7}$

assigned 1 and so on. For the remaining vertices, label the vertices in k^{th} row as 1 when k is odd and as 2 when k is even, $3 \leq k \leq m-2$. It is evident that no two vertices have equal neighbourhood sum.
Hence $B_{m,n}$ admits lucky labeling with $\eta(B_{m,n}) = 2$ □

Theorem 2.0.0.11. $\eta(K_n) = n$

Proof. Let $V(K_n) = \{v_1, v_2, ..., v_n\}$. Label v_i with i for i varies from 1 to n.
$s(v_i) = \sum_{j=1 \neq i} j = \frac{n(n+1)}{2} - i$;
For $i \neq j$, $S(v_i) \neq S(v_j)$.
Hence it is Lucky labeling of K_n and lucky number of K_n is less than or equal to n. Suppose there exists a lucky labeling with maximum label strictly less than n. In that case at least one label must be repeated.
Let $f(v_i) = r = f(v_2)$. Then $f(v_1) = \sum_{i=3} f(v_i) + r = S(v_2)$, which is a contradiction.
Therefore $\eta(K_n) = n$ □

Chapter 3

PROPER LUCKY LABELING

Theorem 3.0.0.1. *The triangular snake TS_n with $n > 2$ is proper lucky graph with $\eta_p(TS_n) = \Delta(TS_n) - 1 = 3$.*

Proof. Let $f : V(TS_{\mathrm{n}}) \to \{1,2,3\}$ be defined by

Labeling Part:

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_{\mathrm{i}}) = 3$$

Sum of the Labels of the adjacent vertices:

$$s(u_{\mathrm{n}}) = \begin{cases} 4, & n \text{ even} \\ 5, & i = 1, n, \ n \text{ odd} \end{cases}$$

$$s(u_{\mathrm{i}}) = \begin{cases} 8, & i < n \ i \text{ even} \\ 10, & n > i > 1 \ i \text{ odd} \end{cases}$$

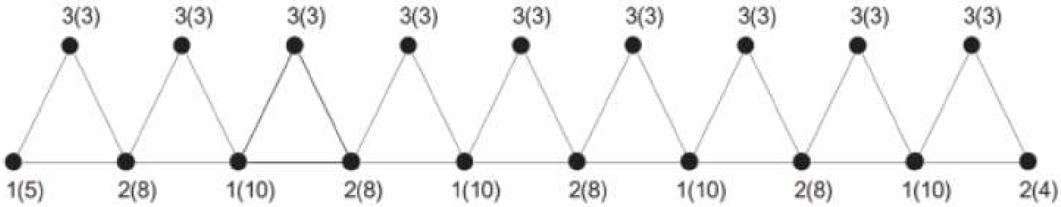

Figure 3.1: Proper Lucky triangular Snake Graph DTS_{10}

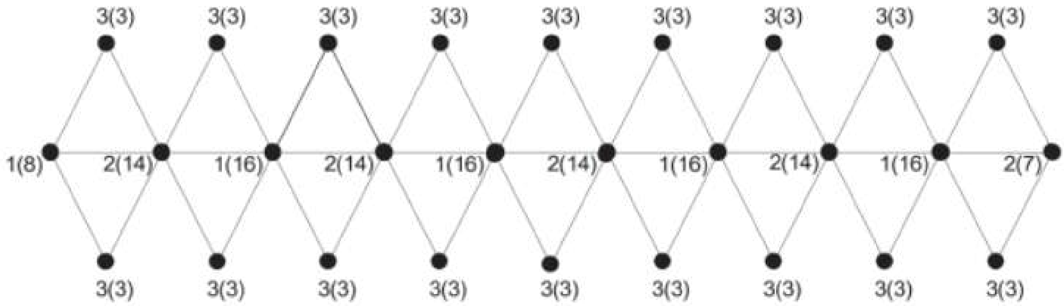

Figure 3.2: Proper Lucky double Triangular Snake Graph DTS_{10}

$$s(v_i) = 3$$

Therefore for TS_n with $n > 2$, proper lucky graph with $\eta(TS_n) = \Delta(TS_n) - 1$

Therefore the sum of the labels of the adjacent vertices are distinct □

Theorem 3.0.0.2. *The Double Triangular Snake graph DTS_n with $n > 2$ is proper lucky graph with $\eta_p(DTS_n) = \frac{\Delta(DTS_n)}{2} = 3$*

Proof. Let $f : V(ATS_n) \to \{1, 2, 3\}$ be defined by

Labeling Part:

$$f(u_i) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_i) = 3$$

$$f(w_i) = 3$$

Sum of the Labels of the adjacent vertices:

$$s(u_{\text{i}}) = \begin{cases} 7, & n \text{ even} \\ 8, & i = 1 \ n \text{ odd} \\ 14, & i < n, i \text{ even} \\ 16, & 1 < i < n, i \text{ odd} \end{cases} \qquad s(v_{\text{i}}) = 3$$

$$s(w_{\text{i}}) = 3$$

Therefore for DTS_{n} with $n > 2$, $\eta_p(DTS_{\text{n}}) = \frac{\Delta(DTS_{\text{n}})}{2} = 3$

Therefore the sum of the labels of the adjacent vertices are distinct

□

Theorem 3.0.0.3. *The alternate triangular snake graph ATS_n with $n > 4$ is properly lucky Graph with $\eta_p(ATS_n) = \Delta(ATS_n) + 1 = 4$*

Proof. Let $f : V(ATS_{\text{n}}) \to \{1, 2, 3, 4\}$ with $n > 4$ is proper lucky graph with $\eta_p(ATS_n) + 1 = 4$ be defined by

Labeling Part:

$$f(u_{\text{i}}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \neq n, i \text{ even} \\ 4, & i = n \end{cases}$$

$$f(v_{\text{i}}) = 3$$

Sum of the Labels of the adjacent vertices:

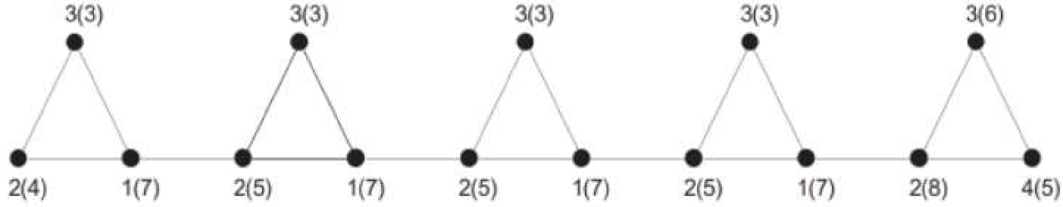

Figure 3.3: Proper Lucky Alternate triangular Snake Graph ATS_{10}

$$s(u_{\rm i}) = \begin{cases} 4, & i = 4 \\ 5, & i = n, 1 < i < n-1, i \text{ odd} \\ 7, & i < n, i \text{ even} \\ 8, & i = n-1 \end{cases}$$

$$s(v_{\rm i}) = \begin{cases} 3, & i \neq n \\ 6, & i = n \end{cases}$$

Therefore for $ATS_{\rm n}$ with $n > 4$, $\eta_p(ATS_{\rm n}) = \Delta(ATS_{\rm n}) + 1 = 4$

Therefore the sum of the labels of the adjacent vertices are distinct

□

Theorem 3.0.0.4. *The double alternate snake $DATS_n$ with $n > 2$ is proper lucky labeling with $\eta_p(DATS_n) = \Delta(ATS_n) - 1 = 3$*

Proof. Let $f : V(DATS_{\rm n}) \to \{1, 2, 3\}$ with $n > 2$ is proper lucky graph with $\eta_p(DATS_n) - 1 = 3$ be defined by

Labeling Part:

$$f(u_{\rm i}) = \begin{cases} 1, & \text{even } i < n \\ 2, & i \text{ odd} \\ 3, & i = n \end{cases}$$

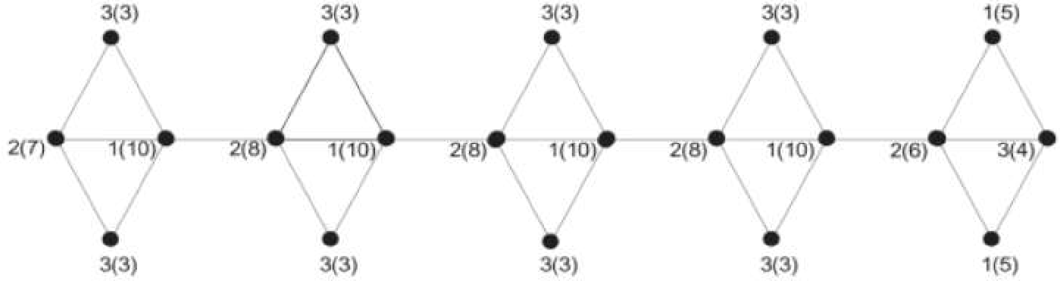

Figure 3.4: Proper Lucky Double Alternate Triangular Snake Graph $DATS_{10}$

$$f(v_i) = \begin{cases} 1, & i = \frac{n}{2} \\ 3, & i < \frac{n}{2} \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i = \frac{n}{2} \\ 3, & i < \frac{n}{2} \end{cases}$$

Sum of the Labels of the adjacent vertices:

$$s(u_i) = \begin{cases} 4, & i = n \\ 6, & i = n-1 \\ 7, & i = 1 \\ 8, & \text{odd } i > 1, i < n-1 \\ 10, & \text{even } i < n \end{cases}$$

$$s(v_i) = \begin{cases} 3, & i < \frac{n}{2} \\ 5, & i = \frac{n}{2} \end{cases}$$

$$s(w_i) = \begin{cases} 3, & i < \frac{n}{2} \\ 5, & i = \frac{n}{2} \end{cases}$$

Therefore the sum of the labels of the adjacent vertices are distinct

Therefore for $DATS_{\rm n}$ with $n > 2$, $\eta_p(DATS_{\rm n}) = \Delta(DATS_{\rm n}) - 1 = 3$

□

Theorem 3.0.0.5. *The Ladder Graph L_n for $n > 1$ is proper Lucky with $\eta_p(L_n) = 2$.*

Proof. Let $f : V(G) \to \{1,2\}$ for a ladder graph $L_{\rm n}$ for $n > 1$ be defined by,

$$f(u_{\rm i}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_{\rm j}) = \begin{cases} 1, & j \text{ even} \\ 2, & j \text{ odd} \end{cases}$$

$$s(a_{\rm n}) = \begin{cases} 2, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$$

$$s(b_{\rm n}) = \begin{cases} 2, & n \text{ odd} \\ 4, & n \text{ even} \end{cases}$$

$$s(u_{\rm i}) = \begin{cases} 3, & \text{even } i < n \\ 4, & i = 1 \\ 6, & 3 \leq i < n, \text{ odd } i \end{cases}$$

$$s(v_{\rm j}) = \begin{cases} 2, & i = 1 \\ 3, & 3 \leq j < n, \text{ odd } i \\ 6, & \text{even } j < n \end{cases}$$

The minimum value of $V(G)$ is 2. Therefore it is proper lucky $\eta_{\rm p}(L_{\rm n}) = 2$.

□

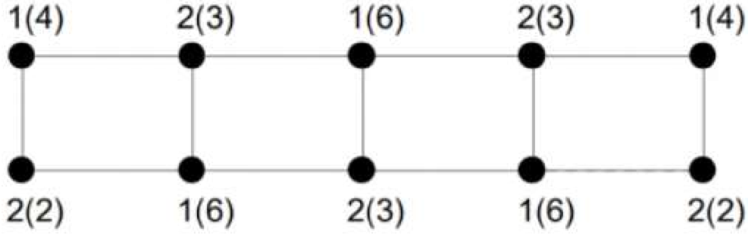

Figure 3.5: Proper lucky ladder graph L_5

Theorem 3.0.0.6. *The Open ladder graph OL_n for $n > 2$ is Proper Lucky with $\eta_p(OL_n) = 2$.*

Proof. Let $f : V(G) \to \{1,2\}$ for a ladder graph OL_n for $n > 2$ be defined by,

$$f(u_i) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j \text{ even} \\ 2, & j \text{ odd} \end{cases}$$

$$s(a_n) = \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

$$s(b_n) = \begin{cases} 1, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}$$

$$s(u_i) = \begin{cases} 2, & i = 1 \\ 3, & \text{even } i < n \\ 6, & 3 \le i < n, i \text{ odd} \end{cases}$$

$$s(v_j) = \begin{cases} 1, & i = 1 \\ 3, & 3 \le j < n, \text{odd } j \\ 6, & j < n, \text{even } j \end{cases}$$

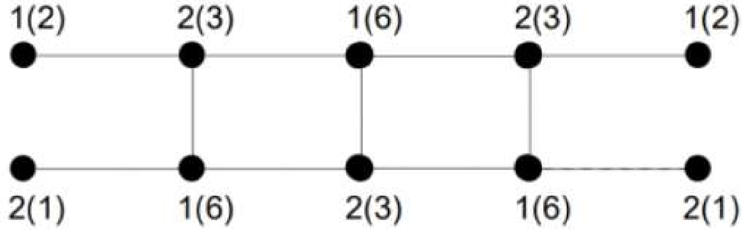

Figure 3.6: Proper lucky Open ladder graph OL_5

The minimum value of $V(G)$ is 2. Therefore it is proper lucky $\eta_{\mathrm{p}}(OL_{\mathrm{n}}) = 2$.

□

Theorem 3.0.0.7. *The Slanting Ladder Graph SL_n for $n > 1$ is proper Lucky with $\eta_p(SL_n) = 2$.*

Proof. Let $f : V(G) \rightarrow \{1,2\}$ for a ladder graph SL_{n} for $n > 1$ be defined by ,

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_{\mathrm{j}}) = \begin{cases} 1, & j \text{ odd} \\ 2, & j \text{ even} \end{cases}$$

$$s(a_{\mathrm{n}}) = \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

$$s(b_{\mathrm{n}}) = \begin{cases} 2, & n \text{ even} \\ 4, & n \text{ odd} \end{cases}$$

$$s(u_{\mathrm{i}}) = \begin{cases} 3, & i < n, i \text{ even} \\ 4, & i = 1 \\ 6, & 3 \leq i < n, i \text{ odd} \end{cases}$$

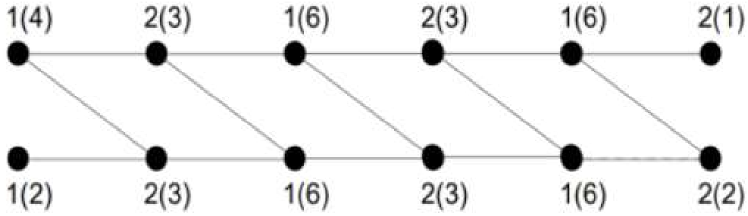

Figure 3.7: Proper lucky Slanting ladder graph SL_6

$$s(v_{\mathrm{j}}) = \begin{cases} 2, & i=1 \\ 3, & j<n,\ j \text{ even} \\ 6, & 3 \leq j < n,\ j \text{ odd} \end{cases}$$

The minimum value of $V(G)$ is 2. Therefore it is proper lucky $\eta_{\mathrm{p}}(SL_{\mathrm{n}}) = 2$.

□

Theorem 3.0.0.8. *The Triangular ladder graph TL_n for $n > 1$ is proper lucky with $\eta_p(TL_n) = 3$.*

Proof. Let $f : V(G) \to \{1,2,3\}$ for a ladder graph TL_{n} for $n > 1$ be defined by,

$$k \in \mathbf{N}$$

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i = 3k-1 \\ 2, & i = 3k-2 \\ 3, & i = 3k \end{cases}$$

$$f(v_{\mathrm{j}}) = \begin{cases} 1, & j = 3k \\ 2, & j = 3k-1 \\ 3, & j = 3k-2 \end{cases}$$

$$s(a_{\mathrm{n}}) = \begin{cases} 4, & n = 3k \\ 7, & n \neq 3k \end{cases}$$

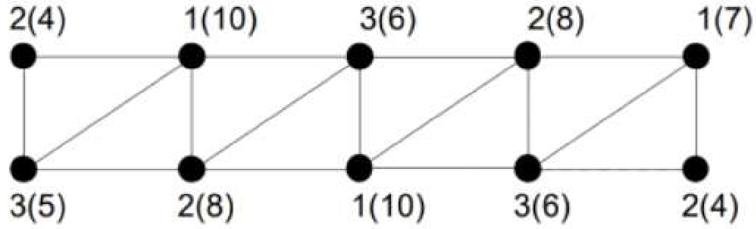

Figure 3.8: Proper lucky Triangular ladder graph TL_5

$$s(b_{\mathrm{n}}) = \begin{cases} 3, & n = 3k-2 \\ 4, & n = 3k-1 \\ 5, & n = 3k \end{cases}$$

$$s(u_{\mathrm{i}}) = \begin{cases} 4, & i = 1 \\ 6, & i = 3k, i < n \\ 8, & i = 3k-2, 4 \leq i < n \\ 10, & i = 3k-1, i < n \end{cases}$$

$$s(v_{\mathrm{j}}) = \begin{cases} 6, & j = 3k-2 \text{ and } 4 \leq j < n \\ 5, & i = 1 \\ 8, & j = 3k-1 \text{ and } j < n \\ 10, & j = 3k \text{ and } j < n \end{cases}$$

The minimum value of $V(G)$ is 3. Therefore it is proper lucky $\eta_{\mathrm{p}}(TL_{\mathrm{n}}) = 3$.

□

Theorem 3.0.0.9. *The Open Triangular Ladder graph TL_n for $n > 2$ is proper lucky with $\eta_p(OTL_n) = 3$*

Proof. Let $f : V(G) \to \{1, 2, 3\}$ for a ladder graph OTL_{n} for $n > 2$ be defined by,

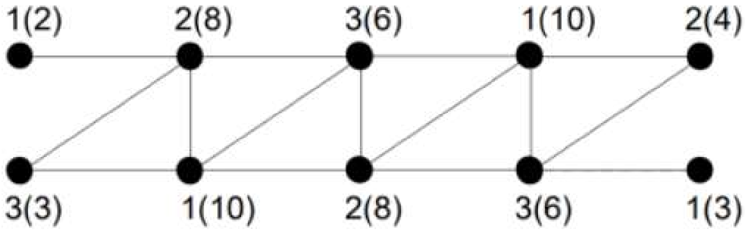

Figure 3.9: Proper lucky open ladder graph OTL_5

$$k \in \mathbf{N}\ f(u_i) = \begin{cases} 1, & i = 3k-2 \\ 2, & i = 3k-1 \\ 3, & i = 3k \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 3k-2 \\ 2, & j = 3k \\ 3, & j = 3k-2 \end{cases}$$

$$s(a_n) = \begin{cases} 3, & n = 3k \\ 4, & n = 3k+2 \\ 5, & n = 3k+1 \end{cases}$$

$$s(b_n) = \begin{cases} 1, & n = 3k \\ 2, & n = 3k+1 \\ 3, & n = 3k+2 \end{cases}$$

$$s(u_i) = \begin{cases} 2, & i = 1 \\ 6, & i = 3k, i < n \\ 8, & i = 3k-1, i < n \\ 10, & i = 3k-1, 4 \leq i < n \end{cases}$$

$$s(v_{\mathrm{j}}) = \begin{cases} 3, & i = 1 \\ 6, & j = 3k-2, 4 \leq j < n \\ 8, & j = 3k-1, j < n \\ 10, & j = 3k, j < n \end{cases}$$

The minimum value of $V(G)$ is 3. Therefore it is proper lucky $\eta_{\mathrm{p}}(OTL_{\mathrm{n}}) = 3$. □

Theorem 3.0.0.10. *The Diagonal Graph DL_n for $n > 1$ is proper lucky with $\eta_p(DL_n) = 4$*

Proof. Let $f : V(G) \to \{1,2,3,4\}$ for a ladder graph DL_{n} for $n > 1$ be defined by ,

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ odd} \\ 2, & i \text{ even} \end{cases}$$

$$f(v_{\mathrm{j}}) = \begin{cases} 3, & j \text{ odd} \\ 4, & j \text{ even} \end{cases}$$

$$s(a_{\mathrm{n}}) = \begin{cases} 8, & n \text{ even} \\ 9, & n \text{ odd} \end{cases}$$

$$s(b_{\mathrm{n}}) = \begin{cases} 6, & n \text{ even} \\ 7, & n \text{ odd} \end{cases}$$

$$s(u_{\mathrm{i}}) = \begin{cases} 9, & i = 1 \\ 12, & 2 \leq i < n, i \text{ even} \\ 15, & 3 \leq i < n, i \text{ odd} \end{cases}$$

$$s(v_{\mathrm{j}}) = \begin{cases} 7, & i = 1 \\ 10, & 2 \leq j < n, j \text{ even} \\ 13, & 3 \leq j < n, j \text{ odd} \end{cases}$$

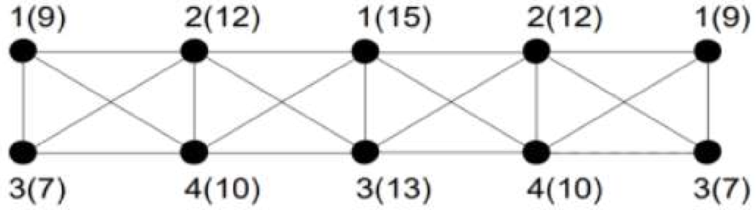

Figure 3.10: Proper Diagonal ladder graph DL_5

The minimum value of $V(G)$ is 4. Therefore it is proper lucky $\eta_{\mathrm{p}}(DL_{\mathrm{n}}) = 4$.

□

Theorem 3.0.0.11. *The Open diagonal ladder graph ODL_n for $n > 2$ is proper lucky with $\eta_p(ODL_n) = 4$.*

Proof. Let $f : V(G) \to \{1,2,3,4\}$ for a ladder graph ODL_{n} for $n > 2$ be defined by,

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$f(v_{\mathrm{j}}) = \begin{cases} 1, & j = 1 \\ 3, & j \text{ even} \\ 4, & j < 1, j \text{ odd} \end{cases}$$

$$s(u_{\mathrm{i}}) = \begin{cases} 5, & i = 1, n \\ 10, & i = 2 \\ 13, & 3 \leq i < n, i \text{ odd} \\ 14, & 4 \leq i < n, i \text{ even} \end{cases}$$

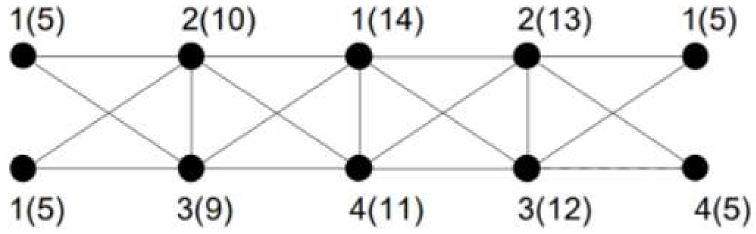

Figure 3.11: Proper Open Diagonal ladder graph ODL_5

$$s(v_j) = \begin{cases} 5, & j = 1, n \\ 9, & j = 2 \\ 11, & 3 \leq j < n, j \text{ odd} \\ 12, & 4 \leq j < n, j \text{ even} \end{cases}$$

The minimum value of $V(G)$ is 4. Therefore it is proper lucky $\eta_p(DL_n) = 4$. □

Theorem 3.0.0.12. *The Quadrilateral Snake QS_n with $n > 1$ admits proper lucky labeling with proper Lucky Number $\eta_p(QS_n)$=3.*

Proof. let $f : V(QS_n) \rightarrow \{1, 2, 3\}$ be defined by,

Labeling Part

$$f(u_i) = \begin{cases} 1, & i \text{ odd} \\ 3, & i \text{ even} \end{cases}$$

$$k \in \mathbf{N}$$

$$f(v_i) = \begin{cases} 1, & i = 4k-1, 4k-2 \\ 2, & i = 4k, 4k-2 \end{cases}$$

Sum of the labels of the adjacent vertices:

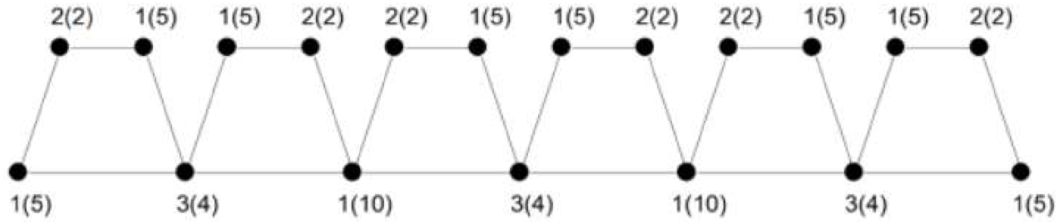

Figure 3.12: Lucky Double quadrilateral Snake Graph DQS_7

$$s(u_1) = 5$$

$$s(u_{\text{n}}) = \begin{cases} 2, & n \text{ even} \\ 5, & n \text{ odd} \end{cases}$$

$$s(u_{\text{i}}) = \begin{cases} 4, & \text{even } i < n, \\ 10, & \text{odd } i > 1, i < n \end{cases}$$

$$\text{for } k \in \mathbf{N}$$

$$s(v_{\text{i}}) = \begin{cases} 2, & i = 4k, 4k-3 \\ 5, & i = 4k-1, 4k-2 \end{cases}$$

Therefore the sum of the adjacent vertices and adjacent labeling are not the same. Hence QS_{n} with $n > 1$ for n odd has $\eta_p(QS_{\text{n}}) = 3$.

□

Theorem 3.0.0.13. *The Double quadrilateral Snake Graph $DAQS_n$ with $n > 1$ for odd n admits Lucky labeling with lucky labeling Number $\eta_p(DQS_n) = 2$*

Proof. Let $f : V(DQS_{\text{n}}) \rightarrow \{1, 2\}$ be defined by

Labeling Part

$$f(u_{\text{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$k \in \mathbf{N}$$

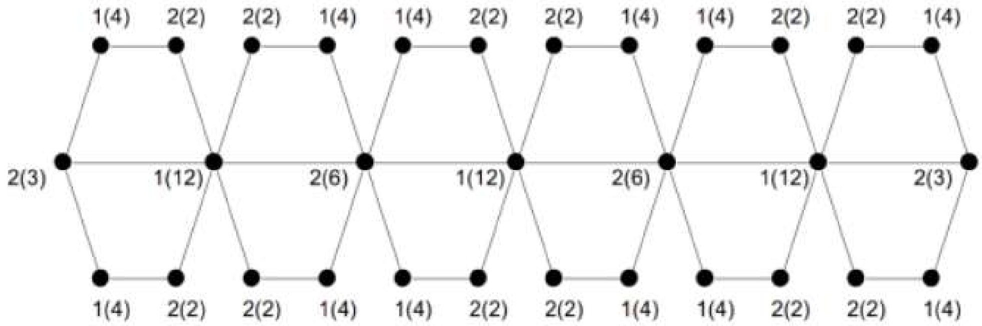

Figure 3.13: Lucky Double quadrilateral Snake Graph DQS_7

$$f(v_i) = \begin{cases} 1, & i = 4k,,4k-3 \\ 2, & i = 4k-1, 4k-2 \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i = 4k, 4k-3 \\ 2, & 4k-1, 4k-2 \end{cases}$$

Sum of the labels of the adjacent vertices:

$$\text{for } k \in \mathbf{N}$$

$$s(u_i) = \begin{cases} 3, & i = 1, n \\ 6, & 1 < i < n, i \text{ odd} \\ 12, & i \text{ even} \end{cases}$$

$$s(v_i) = \begin{cases} 2, & i = 4k, 4k-3 \\ 4, & i = 4k-1, 4k-2 \end{cases}$$

$$s(w_i) = \begin{cases} 2, & i = 4k, 4k-3 \\ 4, & i = 4k-1, 4k-2 \\ 4, & i = 1 \end{cases}$$

The DQS_{n} with $n > 1$ for n odd has $\eta_p(DQS_{\mathrm{n}}) = 2$.

Hence for the Double Quadrilateral Snake DQS_{n} with $n > 2$, $\eta(DQS_{\mathrm{n}}) = 2$ □

Theorem 3.0.0.14. *The Double quadrilateral Snake Graph $DAQS_n$ with $n > 1$ for n even admits proper Lucky labeling with lucky labeling Number $\eta_p(DQS_n) = 3$*

Proof. Let $f : V(DQS_{\mathrm{n}}) \to \{1, 2, 3\}$ be defined by

case(i) n even

Labeling Part

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$k \in \mathbf{N}$$

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i = 4k, 4k-3 i \neq 2n-2 \\ 2, & i = 4k-1, 4k-2 i \neq 2n-2 \\ 3, & 2n-2 \end{cases}$$

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i = 4k-1, 4k-2 i \neq 2n-3 \\ 2, & i = 4k-1, 4k-2 i \neq 2n-2 \\ 3, & 2n-2 \end{cases}$$

Sum of the labels of the adjacent vertices:

$$s(u_{\mathrm{i}}) = \begin{cases} 3, & i = 1 \\ 6, & \text{odd } i > 1 \\ 8, & i = n \\ 12, & \text{even } i < n \end{cases}$$

$$\text{for } k \in \mathbf{N}$$

$$s(v_{\mathrm{i}}) = \begin{cases} 2, & i = 4k-1, 4k-2, i \neq 2n-3 \\ 4, & i = 4k, 4k-3, i \neq 2n-3 \\ 5, & i = 2n-3 \end{cases}$$

$$s(w_{\mathrm{i}}) = \begin{cases} 2, & i = 4k-1, 4k-2, i \neq 2n-3 \\ 4, & i = 4k, 4k-3, i \neq 1 \\ 5, & i = 2n-3 \end{cases}$$

The DQS_{n} with $n > 1$ for n even has $\eta(DQS_{\mathrm{n}}) = 2$. case(ii)$n$ odd Let $f : V(DQS_{\mathrm{n}}) \to \{1,2\}$ be defined by

Labeling Part

$$f(u_{\mathrm{i}}) = \begin{cases} 1, & i \text{ even} \\ 2, & i \text{ odd} \end{cases}$$

$$f(v_{\mathrm{i}}) = \begin{cases} 1, & i \neq 2, n-1 \\ 2, & i = 2, n-1 \end{cases}$$

$$f(w_{\mathrm{i}}) = \begin{cases} 1, & i \neq 2, n-1 \\ 2, & i = 2, n-1 \end{cases}$$

The DQS_{n} with $n > 1$ for n even has $\eta(DQS_{\mathrm{n}}) = 2$.

Hence for the Double Quadrilateral Snake DQS_{n} with $n > 2$, $\eta_p(DQS_{\mathrm{n}}) = 3$ □

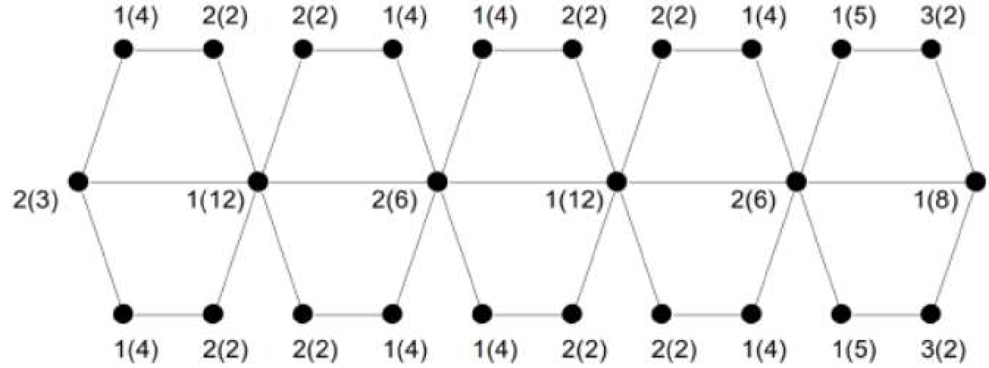

Figure 3.14: Lucky Double quadrilateral Snake Graph DQS_6

Theorem 3.0.0.15. *The mxn dimensional bloom graph $B_{m,n}$ admits proper lucky labeling and $\eta_p(B_{m,n}) = 3$, where $m, n \geq 3$.*

Proof. **Case(i)**When n and m are both even.

Label the vertices in first row as 1 and 2 alternately, without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on. Similarly label the m^{th} row as 2 and 3 alternately, without loss of generality $(m-1,0))$ is assigned $2, m-1, 1$ is assigned 3 and so on.Label the vertices in k^{th} row as 1 when k is odd and as 3 when k is even, $2 \leq k \leq m-1$. We notice that the vertices in the first row will have the neighbourhood sum as 10 and 8 alternately i.e, $(0,0) = 10$,$(0,1) = 8$, and so on.All the vertices in the second row attains the neighbourhood sum as 5 i.e, $(1, j) = 5$, $0 \leq j \leq n-1$. All the vertices in the k^{th} row attains the neighbourhood sum as 12 when k is odd i.e, $(i, j) = 12$ when $1 \leq \text{i} \leq \frac{m-4}{2}$ and $0 \leq j \leq n-1$. All the vertices in the k^{th} row attains the neighbourhood sum as 4 when k is even i.e, $(2i+1, j) = 4$ when $1 \leq \text{i} \leq \frac{m-4}{2}$ and $0 \leq j \leq n-1$. All the vertices in the $(m-1)^{\text{th}}$ row attains the neighbourhood sum as 11 i.e $(m-2j) = 11$,$0 \leq j \leq n-1$. All the vertices in the m^{th} row will have the neighbourhood sum as 8 and 6 alternately i.e, $(m-1,0) = 8$,$(n-1,1) = 6$, and so on. It is evident that no two adjacent vertices have equal neighbourhood sum. **case(ii)**When n is even and m is odd

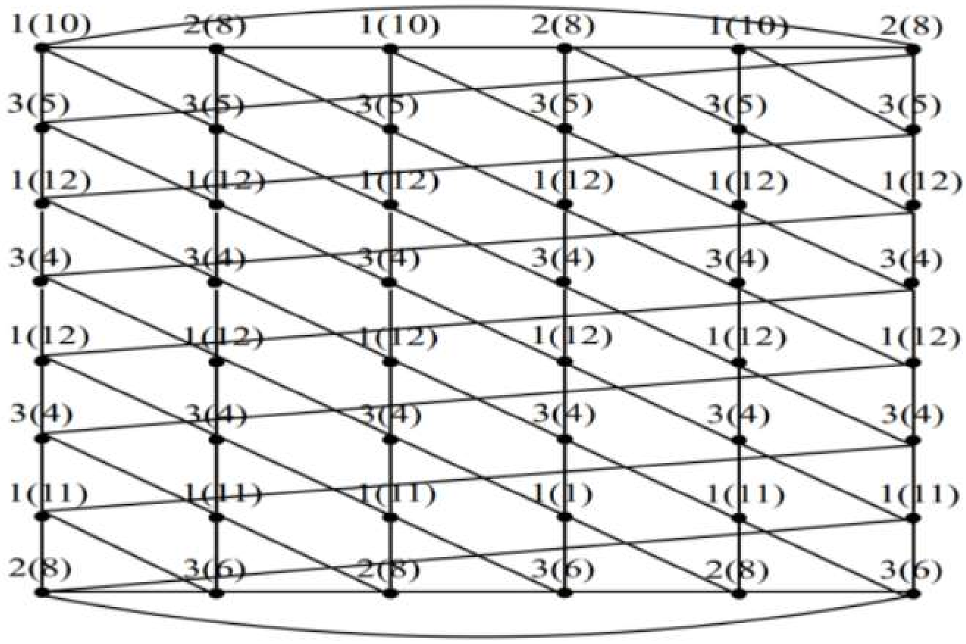

Figure 3.15: Proper Lucky labeling of $B_{8,6}$

Label the vertices in first row as 1 and 2 alternately ,without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on.Similarly label the m^{th} row as 1 and 2 alternately, without loss of generality $(m-1,0)$ is assigned 1,m-1,1 is assigned 2 and so on. Label the vertices in k^{th} as 1 when k is odd and as 3 when k is even, $2 \leq k \leq m-1$. It is evident that no two adjacent vertices have equal neighbourhood sum. **case(iii)**When n is odd and m is odd.

subcase(i) When $m \equiv 0 \pmod 3$ 0

Label the vertices in first row 1 and 2 alternately and the n^{th} column as 3 without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on and $(0,m)$ is assigned 3.Similarly label the m^{th} row the first column as 3 and remaining column as 1 and 2 alternately. without loss of generality $(m-1,0)$ is assigned 3,$(m-1,1)$ is assigned 1 and $(m-1,2)$ is assigned 2 and so on. Label the vertices in $(3k-1)^{\text{th}}$ row as follows: The First column is assigned as 2 and the last column is assigned as 1 and the rest of the column as 3, $1 \leq k \leq m-2$.Label the vertices in $(3k)^{\text{th}}$ row as follows:The first

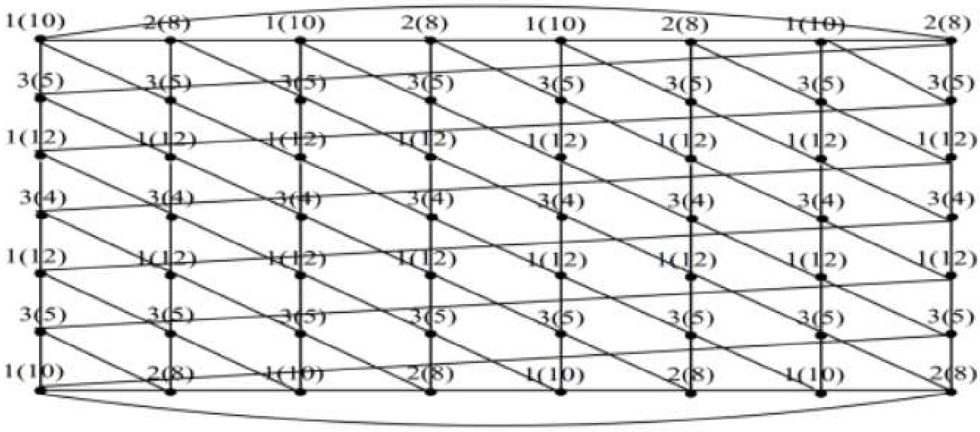

Figure 3.16: Proper Lucky labeling of $B_{7,8}$

column is assigned as 3 and the last column is assigned as 2 and the rest of columns as $1, 1 \leq k \leq m-2$.Label the vertices in $(3k+1)^{th}$ row as follows:The first column is assigned as 1 and the last column is assigned as 3and the rest of the column as $2, 1 \leq k \leq m-2$.It is evident that no two adjacent vertices have equal neighbourhood sum.

subcase(ii)when $m \equiv 0 \pmod 3$ 1

Label the vertices in first row as 1 and 2 alternately and m^{th} column as 3 without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on and $(0,n)$ is assigned 3.Similarly label the m^{th} row the first column as 1 and remaining column as 2 and 3 alternately. without loss of generality $(m-1,0)$ is assigned 1, $(m-1,1)$ is assigned 2 and $(m-1,2)$ is assigned 3 and so on. Label the vertices in $(3k-1)^{th}$ rows as follows:The first column is assigned as 2 and last column is assigned as 1 and the rest of the columns as 3, $1 \leq k \leq m-2$.Label the vertices in $(3k)^{th}$ rows as follows:The first column is assigned as 3 and the last column is assigned as 2 and the rest of the columns as 1, $1 \leq k \leq m-2$.Label the vertices in $(3k+1)^{th}$ row as follows:The first column is assigned as 1 and the last column as 3 and the rest of the columns as $2, 1 \leq k \leq m-2$. It is evident that no two adjacent vertices have equal neighbourhood sum.

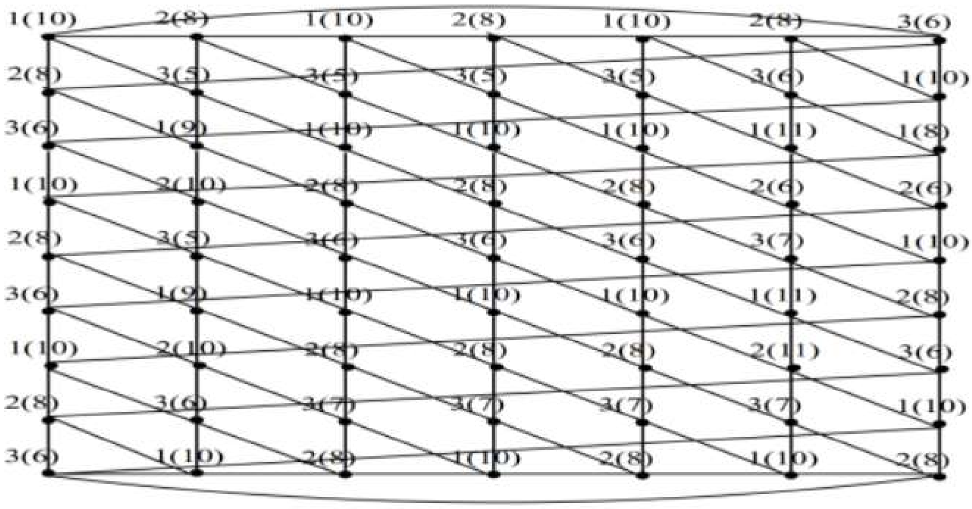

Figure 3.17: Proper Lucky labeling of $B_{9,7}$

subcase(iii) When $m \equiv 0 \pmod 3$ 1

Label the vertices in first row as 1 and 2 alternately and the n$^{\text{th}}$ column as 3 without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on and $(0,n)$ is assigned 3.Similarly label the $(m-1)^{\text{th}}$ row the first column as 2 second column as 3 and remaining column as 1 and 2 alternately.without loss of generality $(m-1,0)$ is assigned 2, $(m-1,1)$ is assigned 3 and rest of the column is assigned as 1 and 2 alternately.$(m-)^{\text{th}}$ rows is assigned as follows the first column as 1 and second column as 2 and the rest of the columns as 3. $(m-)^{\text{th}}$ row is assigned as follows: the first column as 1 second column as 2 and the rest of the column as 3.Label the vertices in $(3k-1)^{\text{th}}$ row as follows:The first column is assigned as 2 and the last column is assigned as 1 and the rest of the columns as 3, $1 \leq k \leq m-3$.Label the vertices in $(3k)^{\text{th}}$ row as follows:The first column is assigned as 3 and last column is assigned 2 and rest columns as 1,$1 \leq k \leq m-3$. Label the vertices in $(3k+1)^{\text{th}}$ row as follows: The first column is assigned as 1 and the last column is assigned 3 and the rest of the column as 2,$1 \leq k \leq m-3$.It is evident that no two adjacent vertices have equal neighbourhood sum. **case(iv)**When m is odd and n is even.

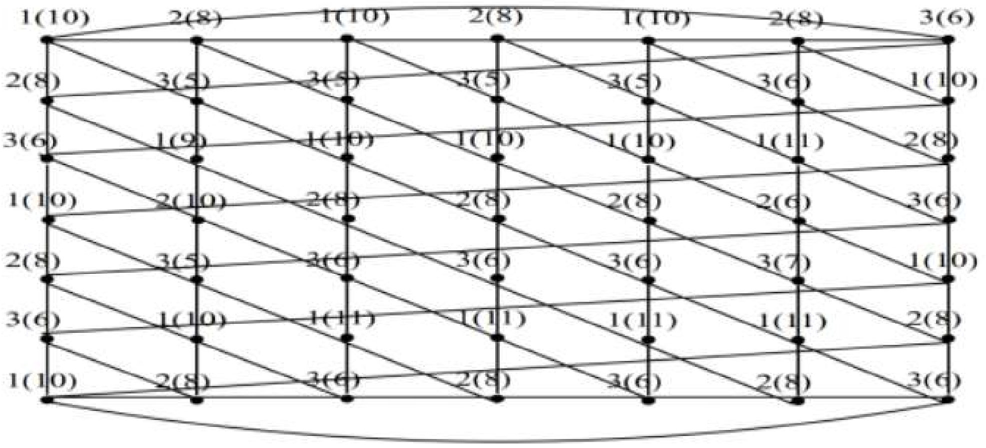

Figure 3.18: Proper Lucky labeling of $B_{7,7}$

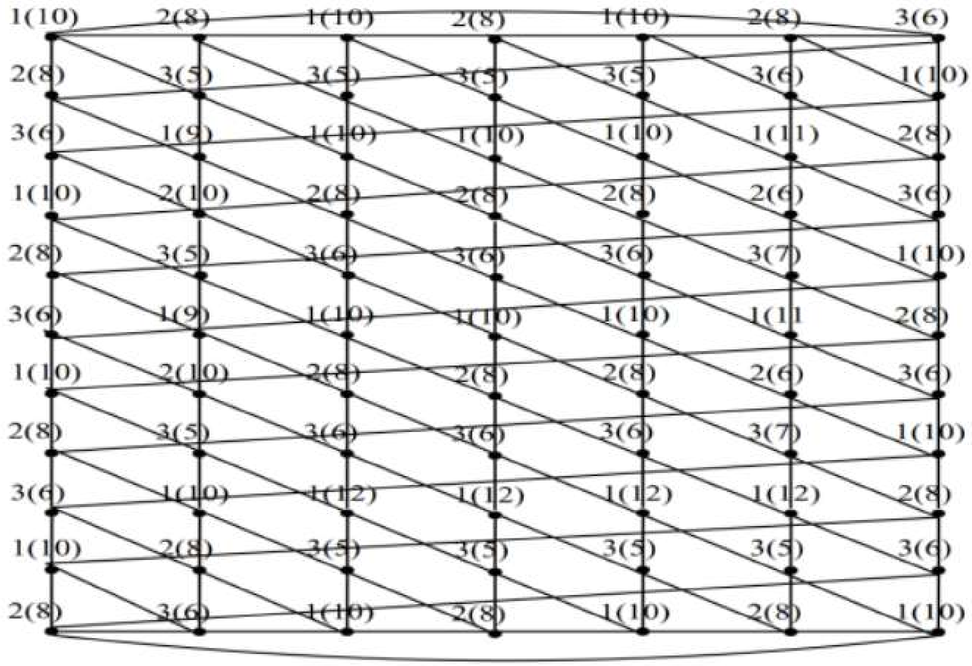

Figure 3.19: Proper Lucky labeling of $B_{11,7}$

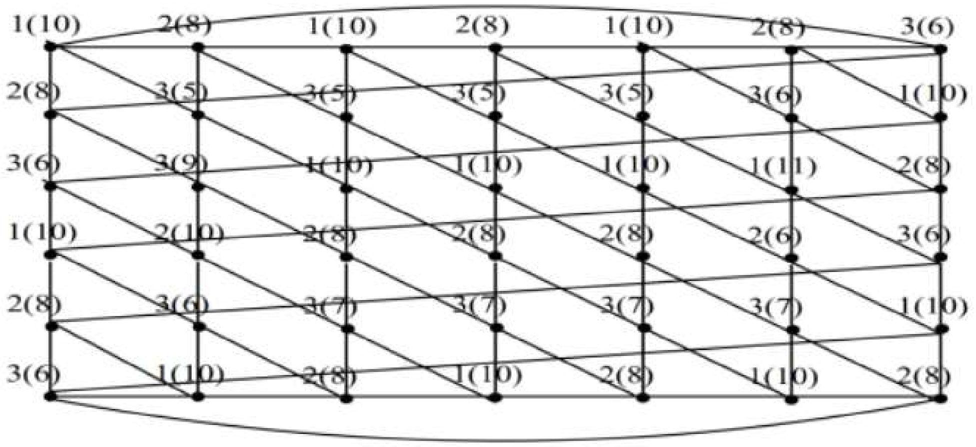

Figure 3.20: Proper Lucky labeling of $B_{6,7}$

subcase(i) When $m \equiv 0 \pmod 3$ 0.

Label the vertices in first row as 1 and 2 alternately and the n[th] column as 3 without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on and $(0,n)$ is assigned 6. Similarly label the m^{th} row as the first column as 3 and remaining columns as 1 and 2 alternately without loss of generality $(m-1,0)$ is assigned 3 ,$(m-1,1)$ is assigned 1 and $(m-1,2)$ is assigned 2 and so on. Label the vertices in $(3k-1)^{\text{th}}$ as follows: the first column is assigned as 2 and the last column is assigned as 1 the rest of the columns as 3, 1≤l≤m-2. Label the vertices in $(3k)^{\text{th}}$ row as follows: The first column is asssigned as 3 and the last column is assigned as 2 and the rest of the columns as 1 1≤k≤2 . Label the vertices in $(3k-1)^{\text{th}}$ row as follows:The first column is assigned as 1 and the last column is assigned as 3 and the rest of the columns as 2, $1 \leq k \leq m-1$. It is evident that no two adjacent vertices have equal neighbourhood sum. **subcase(ii)**When $m \equiv 0 \pmod 3$ 1

Label the vertices in first row as 1 and 2 alternately and the n[th] column as 3 without loss of generality $(0,0)$ is assigned 1, $(0,1)$ is assigned 2 and so on and $(0,n)$ is assigned 3. Similarly label the m^{th} row the first column as 1 and remaining columns as 2 and 3 alternately. without loss of generality $(m-1,0)$ is assigned 1 , $(m-1,1)$ is assigned 2

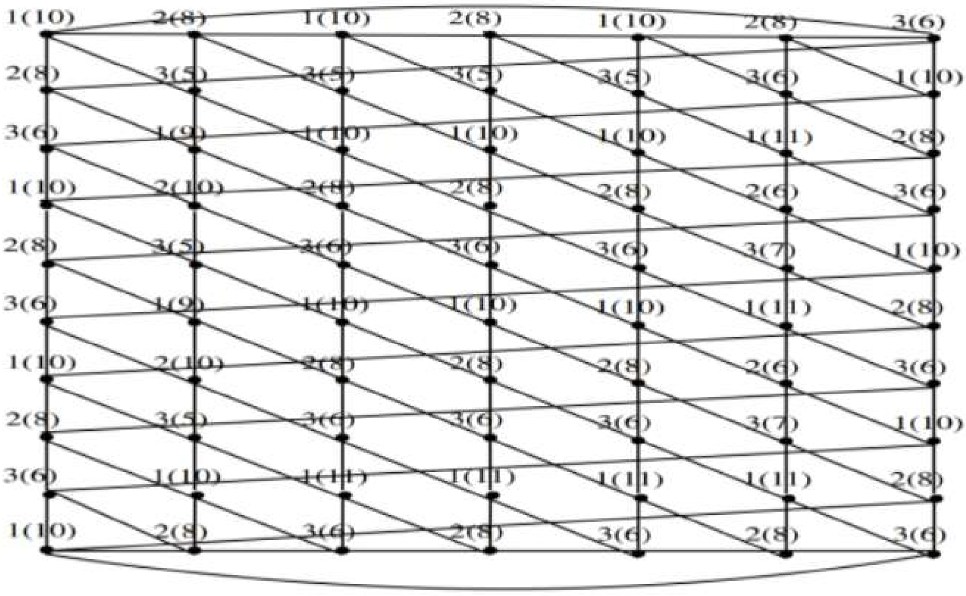

Figure 3.21: Proper Lucky labeling of $B_{10,7}$

and $(m-1,2)$ is assigned 3 and so on.Label the vertices in $(3k-1)^{\text{th}}$ row as follows: The first column is assigned as 2 the last column is assigned as 1 and the rest of the column as 3, $1 \leq k \leq m-2$. label the vertices in $(3k)^{\text{th}}$ row as follows:The first column is assigned as 3 and the last is assigned as 2 and the rest of the columns as 1, $1 \leq k \leq m-2$. Label the vertices in $(3k+1)^{\text{th}}$ rows as follows:The first column is assigned as 1 and the last column is assigned as 3 and the rest of the column as 2, $1 \leq k \leq m-2$.It is evident that no two adjacent vertices have equal neighbourhood sum. **subcase(iii)**When $m \equiv 0$ (mod 3) 2

Label the vertices in first row as 1 and 2 alternately and the n$^{\text{th}}$ column as 3 without loss of generality $(0,0)$ is assigned 1 , $(0,1)$ is assigned 2 and so on and $(0,n)$ is assigned 3.Similarly label the $(m-1)^{\text{th}}$ the first column as 2 second column as 3 and remaining column as 1 and 2 alternately. without loss of generality $(m-1,0)$ is assigned 2, $(m-1,1)$ is assigned 3 and rest of the column is assigned as 1 and 2 alternately. $(m-)^{\text{th}}$ row is assigned as follows: The first column as 1 second column as 2 and the rest of the column as 3.Label the vertices in $(3k-1)^{\text{th}}$ row as follows: The first column

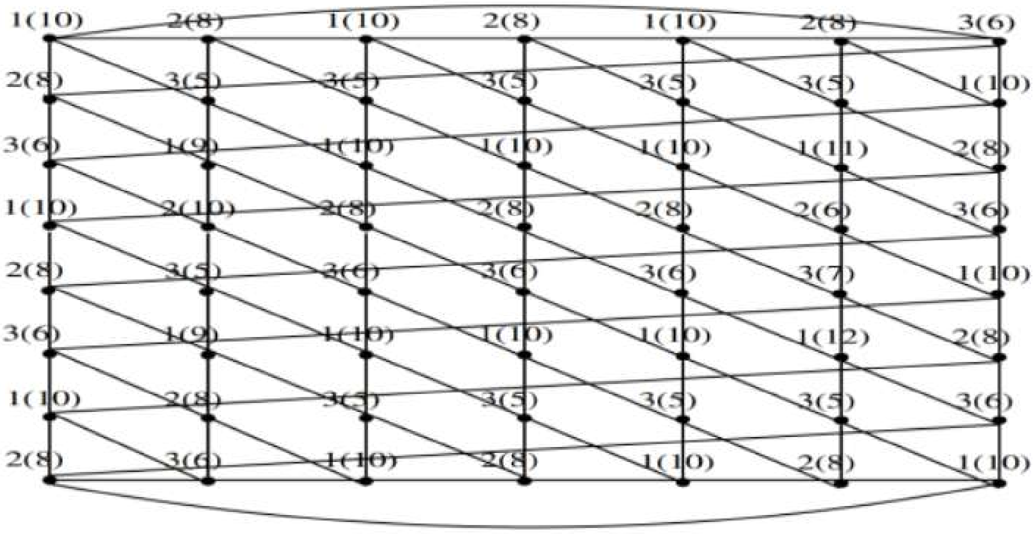

Figure 3.22: Proper Lucky labeling of $B_{8,7}$

is assigned as 2 and the last column is assigned as 1 and the rest of the column as 3 ,$1 \leq k \leq m-3$. Label the vertices in $(3k)^{\text{th}}$ as follows: The first column is assigned as 3 , the last column is assigned as 2 and the rest of the column as 1 ,$1 \leq k \leq m-3$.Label the vertices in $(3k+1)^{\text{th}}$ row as follows:The first column assigned as 1 , the last column assigned as 3 and the rest of the columns as 2, $1 \leq k \leq m-3$.It is evident that no two adjacent vertices have equal neighbourhood sum. □

Theorem 3.0.0.16. *Let G be a mesh M_{nxn}.Then the Proper Lucky number of G is* $\eta_p(G) = 2$

Proof. Define mapping $f : V(G) \in \{1,2\}$ as follows, Let $v_{\text{ij}} \rightarrow V$ then the vertex v_{ij} does not receive the value i.e, if vertex v_{ij} is mapped to 1, its adjacent vertices is mapped to 2 and vise versa.Clearly, $f(u) \neq f(v)$, for all $(u,v) \in E(G)$. Hence the given labeling is a proper Labeling. Next we claim that the given mapping is a lucky labeling. That is to prove $s(u) \neq s(v)$ for all $(u,v) \in E(G)$.

We obtain $s(v_{\text{ij}})$, the sum of labels over all neighbours of vertex v_{ij} as follows

$s(v_{\text{ij}}) = f(v_{\text{ij+1}}) + f(v_{\text{i+1j}}); i = 1, j = 1$

$s(v_{ij}) = f(v_{ij+1}) + f(v_{i-1j}; i = m, j = 1$

$s(v_{ij}) = f(v_{ij-1}) + f(v_{i+1j}); i = 1, j = n$

$s(v_{ij}) = f(v_{ij-1}) + f(v_{i-1j}); i = m, j = n$

$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1j}) + f(v_{ij+1}); i = 2,3,...,m-1, j = 1$

$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1j}) + f(v_{ij-1}); i = 2,3,...,m-1 and j = n$

$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1j}) + f(v_{ij+1} + f(v_{ij-1}); i = 2,3,...,m-1 and j = 2,3,...,n-1$

$s(v_{ij}) = f(v_{ij-1}) + f(v_{ij+1}) + f(v_{i+1j}); i = 1, j = 2,3,...,n-1$

$s(v_{ij}) = f(v_{ij-1}) + f(v_{ij+1}) + f(v_{i-1j}); i = n, j = 2,3,...,n-1$

case(i) Inner part of the Mesh

$s(v_{2i,2j})$=8; $i = 1,2...\lceil \frac{m}{2} \rceil$

$s(v_{2i+1,2j+1})$=8; $i = 1,2,...\lceil \frac{m}{2} \rceil - 1,\ j = 1,2,...\lceil \frac{n}{2} \rceil - 1$

$s(v_{2i,2j+1})$=4; $i = 1,2,...\lceil \frac{m}{2} \rceil,\ j = 1,2,...\lceil \frac{n}{2} \rceil - 1$

$s(v_{2i+1,2j})$=4; $i = 1,2,...\lceil \frac{m}{2} \rceil - 1,\ j = 1,2,...\lceil \frac{n}{2} \rceil$

case(ii) Upper part of the Mesh

$s(v_{i,2j})$=3; $i = 1\ j = 1,2,...\lceil \frac{n}{2} \rceil$

$s(v_{i,2j+1})$=6; $i = 1\ j = 1,2,...\lceil \frac{n}{2} \rceil - 1$

upper part and Lower part of the Mesh have same labeling when $i = m$.

case(iii) Left side of the grid

$s(v_{2j,i})$=3; $i = 1,2,...\lceil \frac{m}{2} \rceil j = 1$

$s(v_{i,2j+1})$=6; $i = 1,2,...\lceil \frac{m}{2} \rceil - 1 j = 1,$

upper part and Lower part of the Mesh have same labeling when $i = m$.

Left side and right side of the mesh have the same labeling (When j=n). All the corner vertices receives the same labeling. From the above cases we see that $s(u) \neq s(v)$ for all $(u,v) \in E(G)$. Therefore $\eta_p \leq 2$

Hence the clique number ω of M_{mxn} is 2 , from theorem 1.3.0.4 we have $\eta_p \geq \omega$.

Hence $\eta_p(G) \geq 2$

Therefore by $\eta_p \leq 2$ and $\eta_p \geq \omega$ we have $\eta_p(G) = 2$

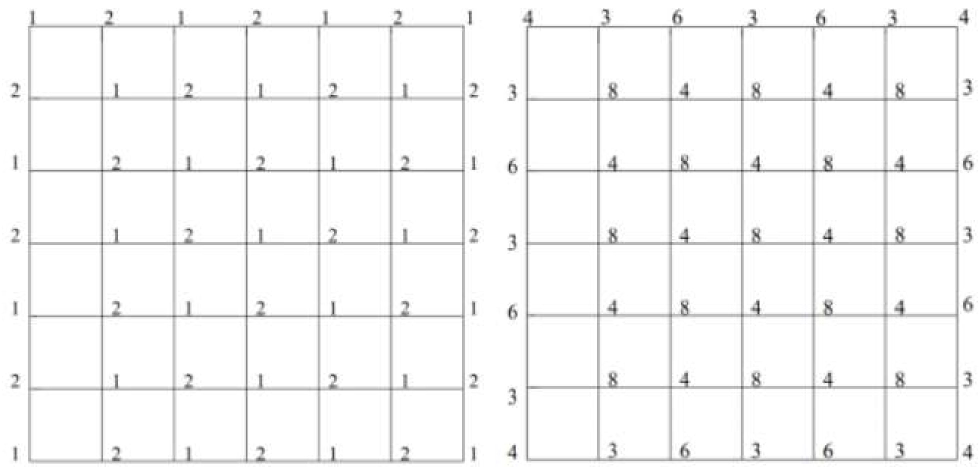

Figure 3.23: Proper Lucky labeling of Mesh M_{7x7} and its sum of neighbourhood

□

Theorem 3.0.0.17. *Let G be an Extended Mesh EX_{mxn}. Then the proper lucky number of G is $\eta_p(G) = 4$*

Proof. We Partition the vertex set $V(G_{\mathrm{nxn}})$ into 2 disjoint sets V_1 and V_2.

Let $V_1 = v_{\mathrm{2m(j\text{-}1)+i}}\ i = 1,2,...m, j = 1,2,...n$ and $V_1 = v_{\mathrm{2m(j\text{-}1)+i}}\ i = 1,2,...m, j = 1,2,...n$.

Define a mapping $f : V(G) \in N$ as follows

$$f(v_{\mathrm{2m(j\text{-}1)+i}}) = \begin{cases} 1 & \text{if } i \text{ is odd where } i = 1,2,...m\ \ , j = 1,2,..\lceil \frac{n}{2} \rceil \\ 2 & \text{if } i \text{ is even where } i = 1,2,...m\ \ , j = 1,2,..\lceil \frac{n}{2} \rceil \end{cases}$$

$$f(v_{\mathrm{2m(j\text{-}1)+i}}) = \begin{cases} 3 & \text{if } i \text{ is odd where } i = 1,2,...m\ \ , j = 1,2,..\lfloor \frac{n}{2} \rfloor \\ 4 & \text{if } i \text{ is even where } i = 1,2,...m\ \ , j = 1,2,..\lfloor \frac{n}{2} \rfloor \end{cases}$$

Claim: f is a proper labeling. We need to verify that $f(u) \neq f(v)$, for all $(u,v) \in E(G)$.Let $e = uw$ be an edge in G.

case(i) Suppose $u,w \in V_1$.Then $u = v_{\mathrm{2m(l\text{-}1)+s}}$ and $w = v_{\mathrm{2m(l\text{-}1)+t}}$, $1 \leq l \leq \lceil \frac{n}{2} \rceil$, $1 \leq s,t \leq m$.Since $e = uw$ is an edge , we have t=s+1. Therefore $f(u) = 1, f(w) = 2 or f(u) = 2, f(w) = 1$.Hence $f(u) \neq f(v)$. **case(ii)**Suppose $u,w \in V_2$.Then $u = v_{\mathrm{m(l\text{-}1)+s}}$ and $w = v_{\mathrm{m(l\text{-}1)+t}}$, $1 \leq l \leq \lceil \frac{n}{2} \rceil$, $1 \leq s,t \leq m$. Since $e = uw$ is an edge, we have $t = s+1$.

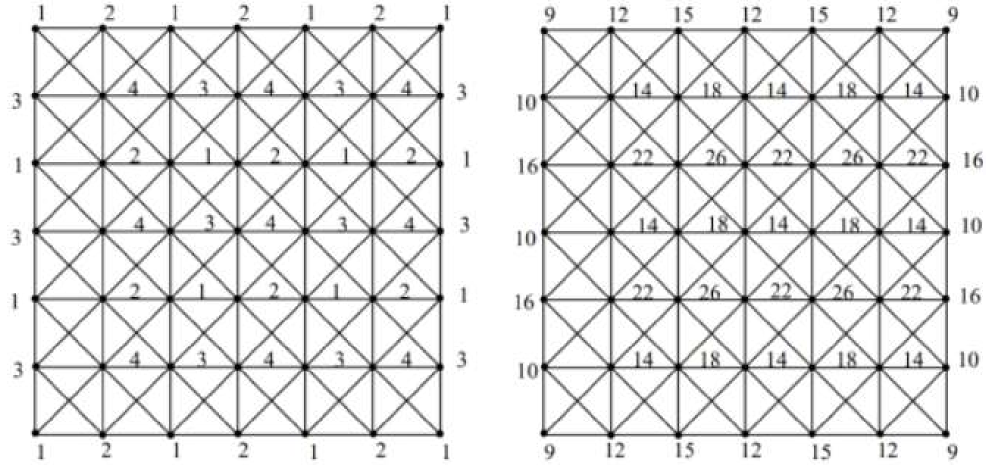

Figure 3.24: Proper Lucky Labeling of an extended mesh EX_{7x7} and its sum of neighbourhood

Therefore $f(u) = 3, f(w) = 4$ or $f(u) = 4, f(w) = 3$. Hence $f(u) \neq f(v)$.

case(ii) Suppose $u \in V_1, w \in V_2$.Then $u = v_{m(2i-1)+s}$ and $w = v_{m(2l-1)+t}$. Since the vertex V_1 is labeled 1,2 and V_2 Hence $f(u) \neq f(v)$ for all $(u,v) \in E(G)$.Hence the labeling is proper labeling.

Next we claim that the given mapping is a proper lucky labeling.That is, to prove $s(u) \neq s(v)$ for all $(u,v) \in E(G)$.

The sum of open neighbourhood of v_{ij} are defined below as follows, where $i = 1, m$ and $j = 1, n$

$s(v_{ij}) = f(v_{ij+1}) + f(v_{i+1j+1}).i = 1,2,...m-1, j = 1,n$

$s(v_{ij}) = f(v_{i-1j1}) + f(v_{i+1j}) + f(v_{i-1j+1}) + f(v_{i+1j+1}).i = 2,3,...mj = 2,3,...n-1$

$s(v_{ij}) = f(v_{i-1j}) + f(v_{i+1j}) + f(v_{i+1j+1}) + f(v_{i-1j+1}) + f(v_{ij+1}) + f(v_{i+1j-1}) + f(v_{ij-1}) + f(v_{i-1j-1}).i = 1, mj = 2,3,...n-1.$

$s(v_{ij}) = f(v_{ij-1}) + f(v_{ij+j}) + f(v_{i+1i-1}) + f(v_{i+1j}) + f(v_{i+1j+1})$

from the above mapping we obtained values for each neighbourhood of v_{ij}

case(iii) Inner Part of the Mesh

$s(v_{2i,2j})$=14; $i = 1,2...\lceil \frac{m}{2} \rceil j = 1,2...\lceil \frac{n}{2} \rceil$

$s(v_{2i+1,2j+1})$=18; $i = 1,2,...\lceil\frac{m}{2}\rceil - 1$, $j = 1,2,...\lceil\frac{n}{2}\rceil - 1$

$s(v_{2i,2j+1})$=22; $i = 1,2,...\lceil\frac{m}{2}\rceil - 1$, $j = 1,2,...\lceil\frac{n}{2}\rceil$

$s(v_{2i+1,2j})$=26; $i = 1,2,...\lceil\frac{m}{2}\rceil - 1$, $j = 1,2,...\lceil\frac{n}{2}\rceil - 1$

case(iv) Upper part of the Mesh

$s(v_{i,2j})$=12; $i = 1$, $j = 1,2,...\lceil\frac{n}{2}\rceil$

$s(v_{i,2j+1})$=15; $i = 1$, $j = 1,2,...\lceil\frac{n}{2}\rceil - 1$

upper part and Lower part of the Mesh have same labeling when $i = m$.

case(v) Left side of the grid

$s(v_{2j,i})$=10; $i = 1,2,...\lceil\frac{m}{2}\rceil$, $j = 1$

$s(v_{i,2j+1})$=16; $i = 1,2,...\lceil\frac{m}{2}\rceil - 1$, $j = 1$

Left side and right side of the mesh have the same labeling (When $j = n$). All the corner vertices receives the same labeling. From the above cases we see that $s(u) \neq s(v)$ for all $(u,v) \in E(G)$. Therefore $\eta_p \leq 4$

We note that the maximal complete subgraph of EX_{mxn} is K_4. Hence the clique number ω of EX_{mxn} is 4, By theorem 1.3.0.4 □

Chapter 4

D-LUCKY LABELING OF GRAPHS

Theorem 4.0.0.1. *The n-dimensional Butterfly Network $BF(n)$ admits d-lucky labeling and $\eta_d(BF(n)) = 2$.*

Proof. Label the vertices in consecutive levels of $BF(n)$ as 1 and 2 alternately, beginning from level 0. We note that every edge $e = (u, v) \in BF(n)$ has one end at level i and the other end at level $i+1$ or level $i-1$ (if it exists), $0 \leq i \leq n$.

Case (i): Suppose u is in level 0, then u is incident on one cross edge and one straight edge with the other ends at level 1. Since $f(u) = 1$ and each member of $N(u)$ is labeled 2, we have $c(u) = \sum_{v \in N(u)} f(v) + d(u) = 6$, where $d(u)$ is the degree of u. Since $f(v) = 2$ and each member of $N(v)$ is labeled 1, we have $c(v) = \sum_{u \in N(v)} f(u) + d(v) = 8$. Thus, $c(u) \neq c(v)$. The same argument holds when u is in level n.

Case (ii): Suppose u is in level i, where i is even, $0 \leq i \leq n$, and v is in the level $i+1$. Then u is incident on one cross edge and one straight edge with the other ends at level $i+1$, and also incident on one cross edge and straight edge with the other ends at level $i-1$. Since $f(u) = 2$ and each member of $N(u)$ is labeled 1, we have $c(u) = \sum_{v \in N(u)} f(v) + d(u) = 8$. Further, $f(v) = 1$ and each member of $N(v)$ is labeled

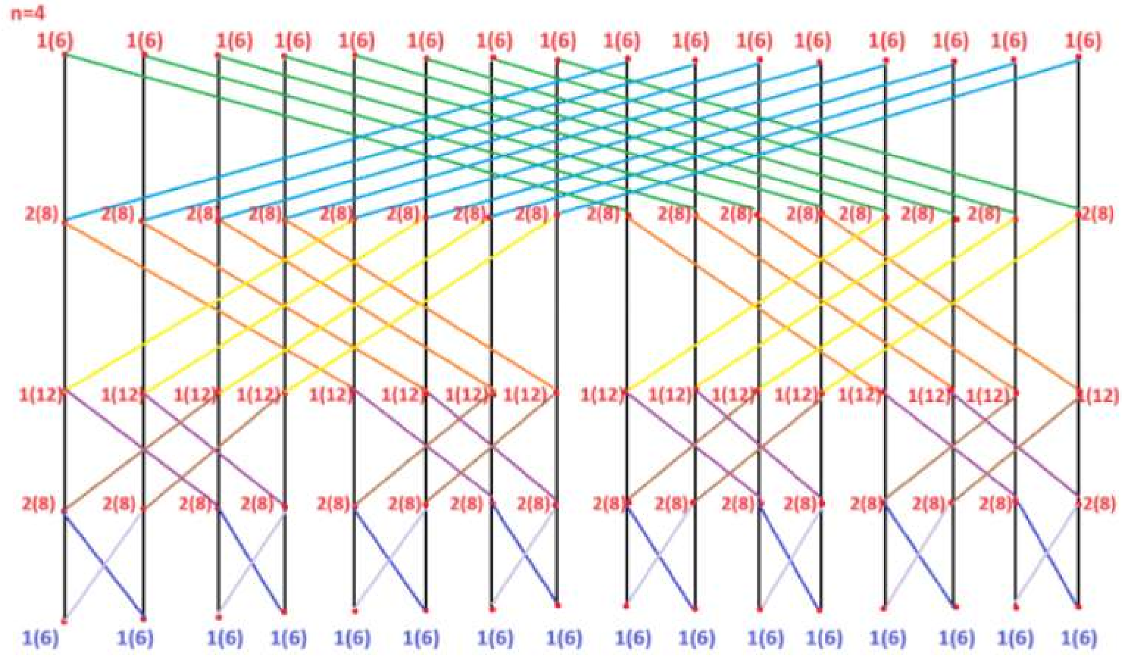

Figure 4.1: d-Lucky labeling of $BF(3)$

2. Therefore, we have $c(v) = \sum_{u \in N(v)} f(u) + d(v) = 12$. Thus, $c(u) \neq c(v)$. A similar argument shows that $c(u) \neq c(v)$ if v is in level $i-1$. The case when i is odd is also similar. Hence, the n-dimensional butterfly network admits d-lucky labeling. □

Theorem 4.0.0.2. *The Mesh Network denoted by M_{mxn} admits d lucky labeling and $\eta_{dl}(M_{mxn}) = 2$.*

Proof. Let G be a mesh M_{mn} where $m,n \leq 2$.Then G admits d-lucky labeling and $\eta_{\mathrm{dl}}(G) = 2$. Label the vertices in row i even, as 1 and 2 alternately, beginning with label 1 from left to right. Label all the vertices in row i, i odd, as 2. Edges with both ends in the same row are called horizontal edges. Edges with one end in row i and the other end in row $(i+1)$ or row $(i-1)$ are called vertical edges.

case(i) Suppose u and v are in row 1, where $d(u) = 2$, then u has one horizontal edge and one vertical edge incident at it. If $f(u) = 2$, by labeling of G the adjacent vertices on

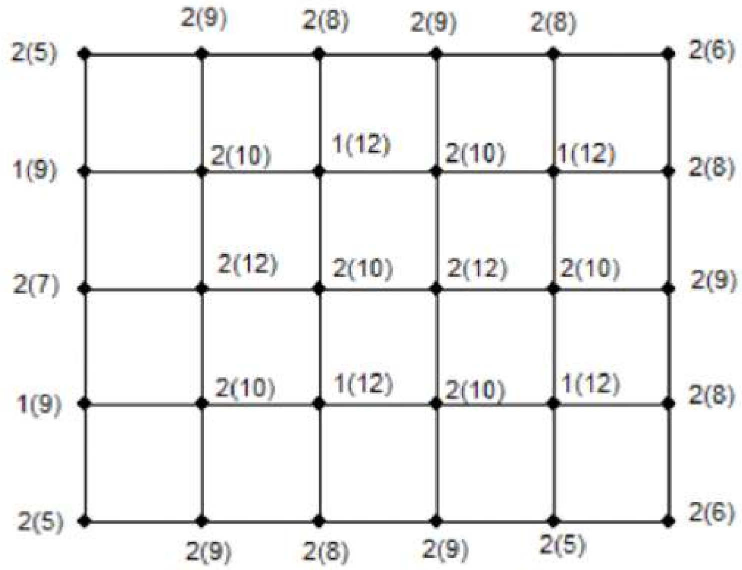

Figure 4.2: d-Lucky labeling of M_{5x6} Mesh Network

the horizontal and vertices edges incident with u are labeled as 2 and 1 respectively.We have $c(u) = \sum_{v \in N(u)} f(v) + d(u) = 5$. On the other hand, if v and each member of $N(v)$ is labeled 2, we have $c(v) = \sum_{u \in N(v)} f(u) + d(v) = 9$. Thus $c(u) \neq c(v)$. A similar argument shows that $c(u) \neq c(v)$ when $f(u) = 1$ or u is in row n.

case(ii) Suppose u and v are in row i, i even, where $d(u) = 3$ and $d(v) = 4$, u has vertical edges in row $i+1, i-1$ and one horizontal edge with the other end in row i incident with it. Since $f(u) = 1$, each member of $N(u)$ is labeled 2.Therefore we have $c(u) = \sum_{v \in N(u)} f(v) + d(u) = 9$. On the other hand suppose v is in row i, then v has 2 edges in vertical edges in row $i-1$ and $i+1$ and two horizontal edges with the other end in i incident with it. Since $f(v) = 2$, each member of $N(v)$ in the horizontal row is labeled 1 and $N(v)$ in the vertical column is labeled 2, Therefore we have $c(v) = \sum_{u \in N(v)} f(u) + d(v) = 10$. The vertex sums are distinct. A similar argument holds when v is in row $n-1$.

case(iii) Suppose u and v are in row i, i odd, where $d(u) = 3$ and $d(v) = 4$, u has vertical edges in row $i+1$, $i-1$ and one horizontal edge with the other end in row$i$incident with it. Since $f(u) = 2$, by labeling of G the adjacent vertices on the horizontal and vertical edges incident with u are labeled 2 and 1 respectively. Therefore we have $c(u) = \sum_{v\in N(u)} f(v) + d(u) = 7$. On the other hand suppose v is in row i, then v has 2 edges in vertical edges in row $i-1$ and $i+1$ and two horizontal edges with the other end in i incident with it.Since $f(v) = 2$, each member of $N(v)$ is labeled 2, Therefore we have $c(v) = \sum_{u\in N(v)} f(u) + d(v) = 12$.The vertex sums are distinct.A similar argument holds when v is in row $n-1$.

□

Theorem 4.0.0.3. *The n-dimensional benes network BB(n) admits d-lucky labeling and* $\eta_{dl}(BB(n)) = 2$.

Proof. Label the vertices in consecutive levels of $BF(n)$ as 1 and 2 alternately, beginning from level 0. We note that every edge $e = (u, v)$ in $BF(n)$ has end at level i and the other end at level $i+1$ or level $i-1$ (If it exists), $0 \leq i \leq n$.

Case(i):Suppose u is in level 0, then u is incident on one cross edge and one straight edge with the other ends at level 1. Since $f(u) = 2$ and each member of $N(u)$ is labeled 1 we have $c(u) = \sum_{v\in N(u)} f(v) + d(u) = 4$, where $d(u)$ is the degree of u. Since $f(v) = 1$ and each member of $N(v)$ is labeled 2, We have $c(v) = \sum_{u\in N(v)} f(u) + d(v) = 12$. Thus $c(u) \neq c(v)$.The same argument holds good when u is in level n.

case(iii) Suppose u is in level i, i is even, $0 < i < 1$ and v is in level $i+1$. Then u is incident on one cross edge and one horizontal edge with the other ends at level $i-1$. Since $f(u) = 2$, each member of $N(u)$ is labeled 1, Therefore we have $c(u) = \sum_{v\in N(u)} f(v) + d(u) = 8$. Further $f(v) = 1$, each member of $N(v)$ is labeled 2, Therefore we have $c(v) = \sum_{u\in N(v)} f(u) + d(v) = 12$. The vertex sums are distinct. A similar argument shows that $c(u) \neq c(v)$ if v is in level $i-1$.The case when i is odd is also Similar.

□

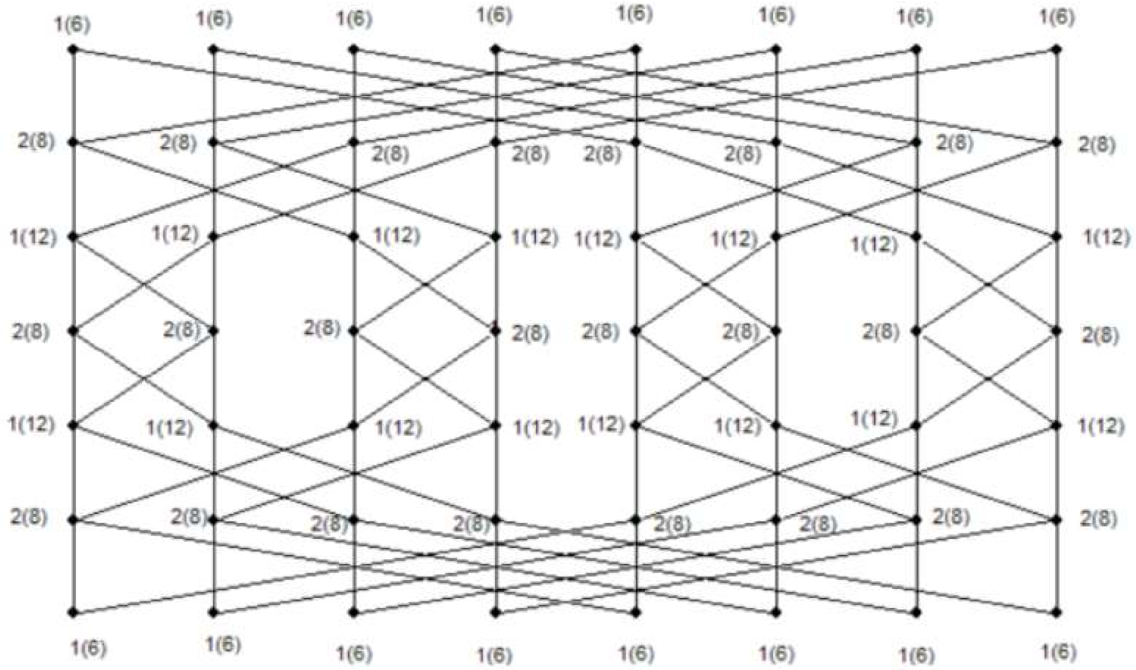

Figure 4.3: d-Lucky labeling of BB(3)

Theorem 4.0.0.4. *The r-level hypertree HT_r admits d-lucky labeling and* $\eta_{dl}(HT(R)) = 2$.

Illustration

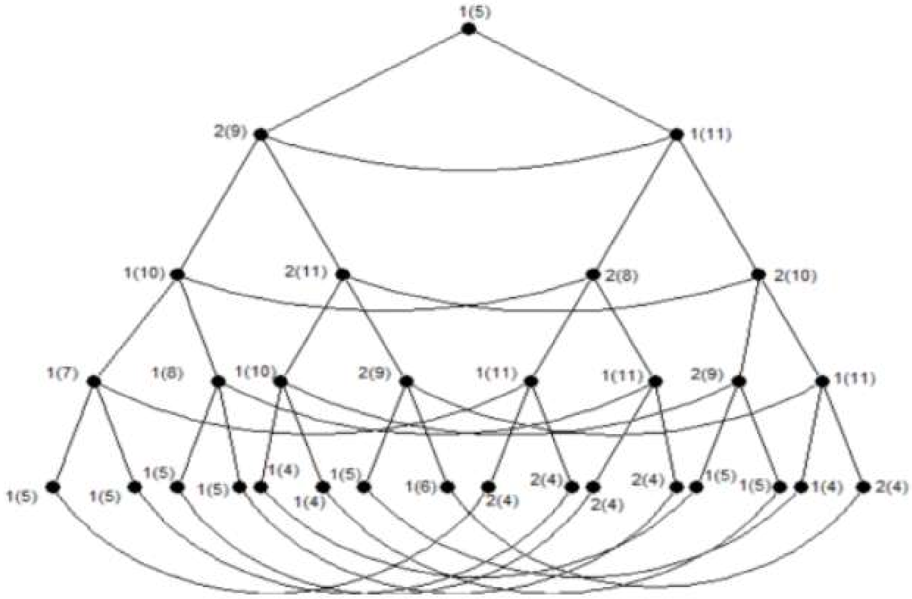

Figure 4.4: d-Lucky labeling of hypertree $HT(4)$

Theorem 4.0.0.5. *The X-Tree XT_r admits d-lucky labeling and $\eta_{dl}(XT_r) = 2$.*

Illustration

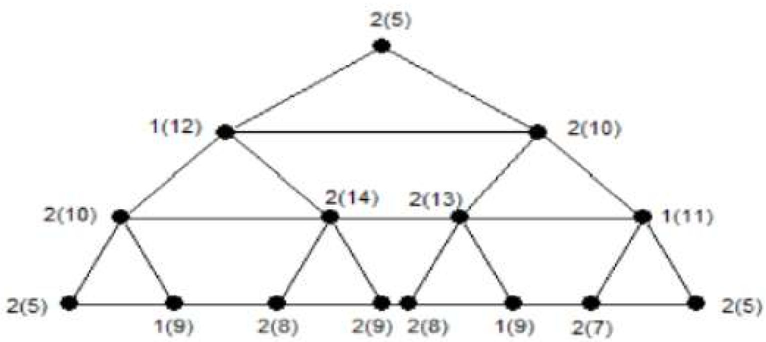

Figure 4.5: d-Lucky labeling X-Tree , $XT(3)$

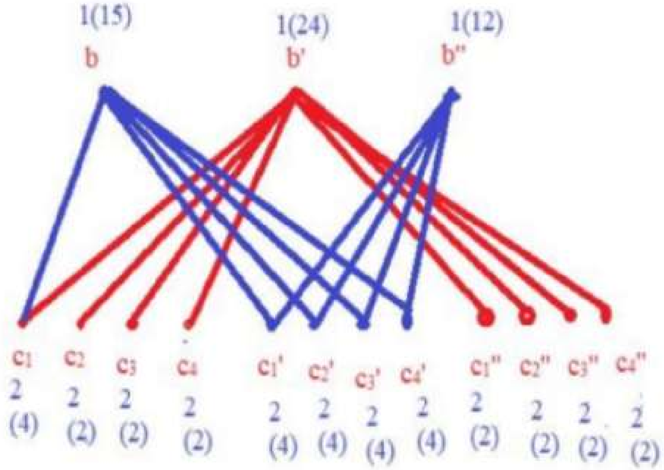

Figure 4.6: d-lucky labeling of Extended Triplicate of Star graph $ETG(K_{1,4})$

Theorem 4.0.0.6. *The extended Triplicate Star of Graph admits d-lucky labeling with lucky Number* $\eta_{dl}(ETG(K_{1,n})) = 3$

Proof. To prove that $ETG(k_{1,n})$ admits d-lucky labeling.
Define a function $S : \zeta(G) \to 1,2$ to label the vertices, such that $S(b) = S(b') = S(b'') = 1$ and $S(c_i) = S(c'_i) = S(c''_i) = 2$; $1 \le i \le n$, The degree of the vertices are

$d(b) = n+1$,

$d(b') = 2n$

$d(b'') = n$

for $1 \le i \le n$,

$d(c'_i) = 2$ and $d(c''_i) = 1$

Then by $\mu(b) = d(b) + \sum_{\delta \in N(b)} l(c)$ we obtain a labeling

$\mu(b) = 3(n+1), \mu(b') = 6n, \mu(b'') = 3n, \mu(c_1) = 4, \mu(c_i) = 2; 2 \le i \le n$

For $1 \le i \le n$, $\mu(c'_i) = 4$ and $\mu(c''_i) = 2$. For every pair of adjacent vertices of b and c in $ETG(K_{1,n}$ are distinct. Therefore, $\mu(b) \ne \mu(c)$

Hence, the extended triplicate graph of star admits d-lucky labeling with lucky number $\eta_{dl}(ETG(K_{i,n})) = 2$ □

Chapter 5

PROPOSED THEOREM

Theorem 5.0.0.1. *The n-dimensional butterfly network $BF(n)$ admits lucky labeling and $\eta(BF(n)) = 2$ for $n \neq 2$*

Proof. For $n > 2$ Label the vertices in consecutive levels of $BF(n)$ as

$$f(u_{i;j}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.1}$$

for $j = 0, 2, ..., n-1$ when n is odd, $j = 0, 2, ..., n-2$ when n is even and

$$f(u_{i;j}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.2}$$

for $j = 1, 3, ..., n-21$ when n is even, $j = 0, 2, ..., n-2$ when n is odd.

Note that $j = n$ will receive the same label as that of $j = n-1$.

We note that every edge $e = (u, v)$ in $BF(n)$ has one end at level j and other end at level

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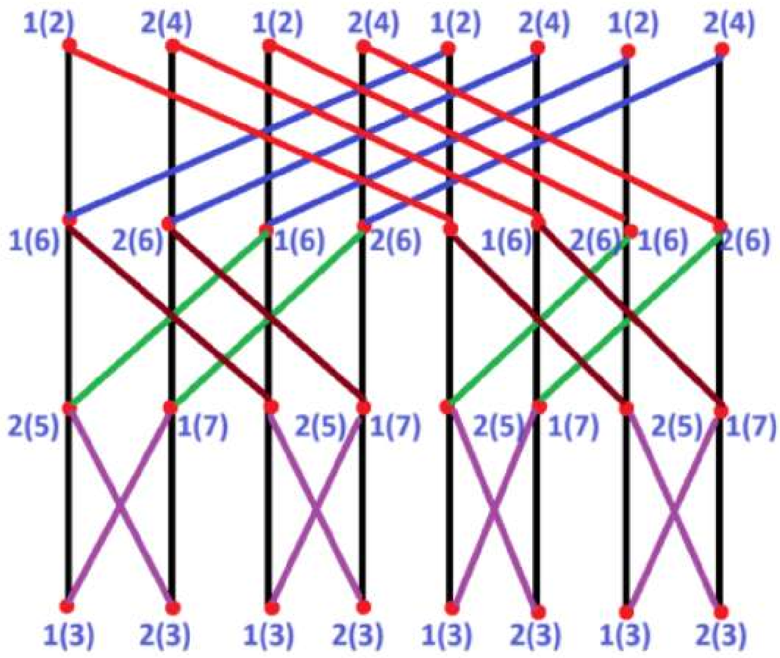

Figure 5.1: Lucky labeling of BF(3)

$j+1$ or $j-1$ (if exists), $0 \leq j \leq n$.

case(i) When u is in level 0

$$f(u_{\mathrm{i};0}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.3}$$

and each member of $N(u_{\mathrm{i};0})$ is labeled

$$f(v_{\mathrm{i};1}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.4}$$

We've $s(u) = \sum_{u \in N(u_{\mathrm{i};0})} f(v_{\mathrm{i};1}) = 3$.

Since $f(v_{\mathrm{i};1}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases}$

and each member of $N(v_{i,1})$ is labeled $f(u_{i;0}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases}$

We've

$$s(v) = \sum_{v \in N(v_{i;1})} f(u_{i;0}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.5}$$

Thus when u is in level 0, $s(u) \neq s(v)$ $\forall$ $n > 3$ and $n \in \mathbb{N}$

case(ii) When u is in level $j = n$

subcase(i)

When u is in level $j = n$ and n is odd, then u is incident on one cross edge and one straight edge with ends at level $j = n - 1$. Since if n odd,

$$f(u_{i;n}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.6}$$

and each member of $N(u_{i;n})$ is labeled as

$$f(v_{i;n\text{-}1}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.7}$$

$$s(u) = \sum_{u \in N(u_{i;n})} f(v_{i;n\text{-}1}) = \begin{cases} 2 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.8}$$

Then we've $s(v) = \sum_{u \in N(u_{i;n})} f(v_{i;n\text{-}1}) = 6$

Therefore subcase(i) along with case(i) Proves that $s(u) \neq s(v)$ for all $(u, v) \in BF(3)$

subcase(ii)

When u is in level $j = n$ and n is even, then u is incident on one cross edge and one straight edge with ends at level $j = n-1$.

$$f(u_{\text{i;n}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.9}$$

and each member of $N(u_{\text{i;n}})$ is labeled as

$$f(v_{\text{i;n-1}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.10}$$

$$s(u) = \sum_{u \in N(u_{\text{i;n}})} f(v_{\text{i;n-1}}) = \begin{cases} 4 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.11}$$

for $n > 2$,

$s(v) = \sum_{u \in N(u_{\text{i;n}})} f(v_{\text{i;n-1}}) = 6$

$s(u) \neq s(v)$

case(iii) When u is in level $j = 2$ and for $n > 3$ Then u is incident on one cross edge and one straight edge with the other ends at level $j+1$ and also one straight edge and cross edge with the other ends at level $j-1$.

subcase(i) When n is odd

Since

$$f(u_{\text{i;2}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.12}$$

and each member of $N(u_{\text{i;2}})$ is labeled

$$f(v_{\text{i;1,3}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.13}$$

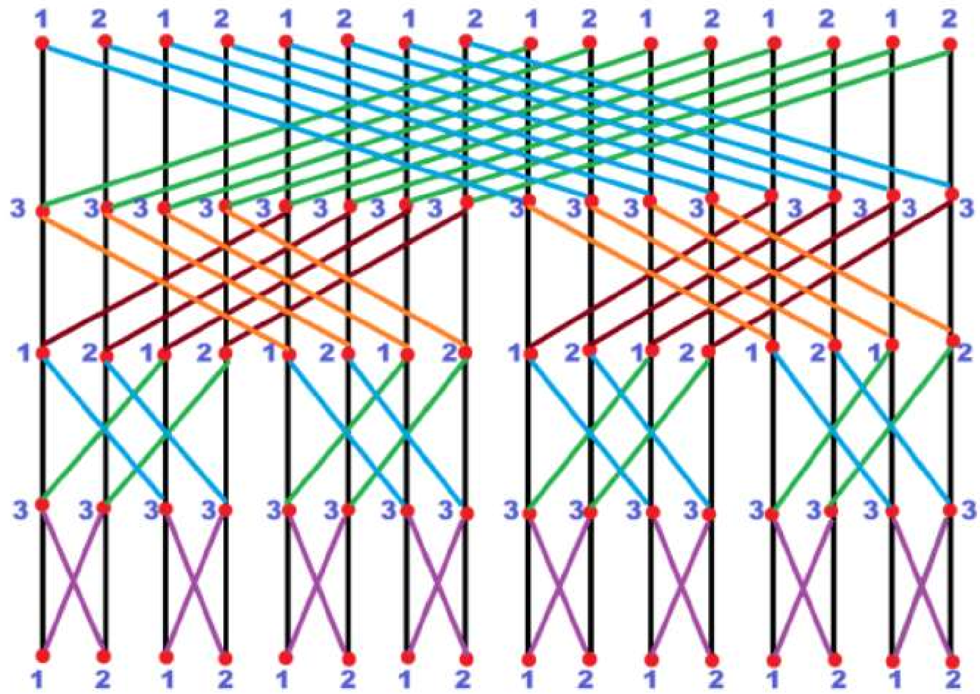

Figure 5.2: Lucky labeling of $BF(4)$

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Therefore $s(u) = \sum_{u \in N(u_{i;2})} f(v_{i;1,3}) = \begin{cases} 8 & i \text{ odd} \\ 4 & i \text{ even} \end{cases}$

subsubcase(i):when v is in level 3,

$$s(v) = \sum_{v \in N(v_{i;3})} f(u_{i;2}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.14}$$

subsubcase(ii):When v is in level 1,

$$s(v) = \sum_{v \in N(v_{i})} f(u_{i;2}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.15}$$

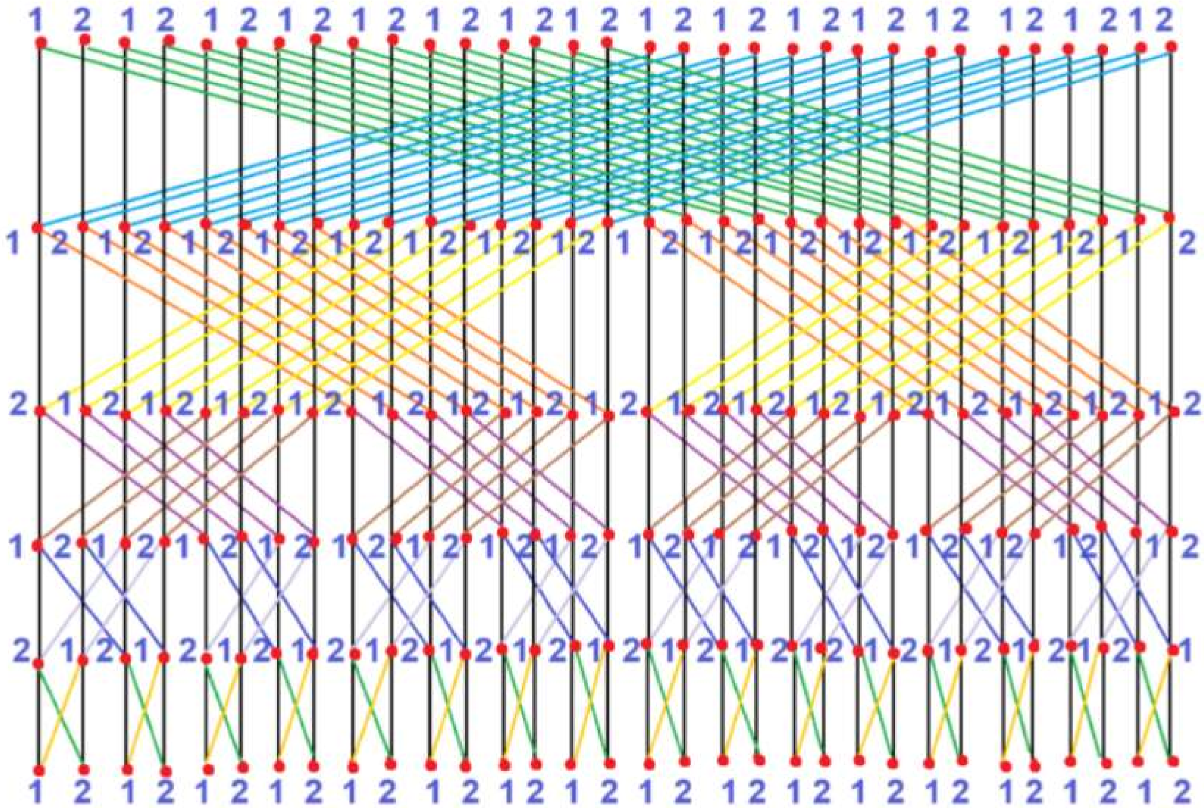

Figure 5.3: Lucky labeling of $BF(5)$

combining above 2 subsubcases, subcase(i) of case(ii) and case(i) we have $s(u) \neq s(v)$ for all $(u,v) \in E(BF(5))$

subcase(ii) When n is even.

Since

$$f(u_{\text{i;n}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.16}$$

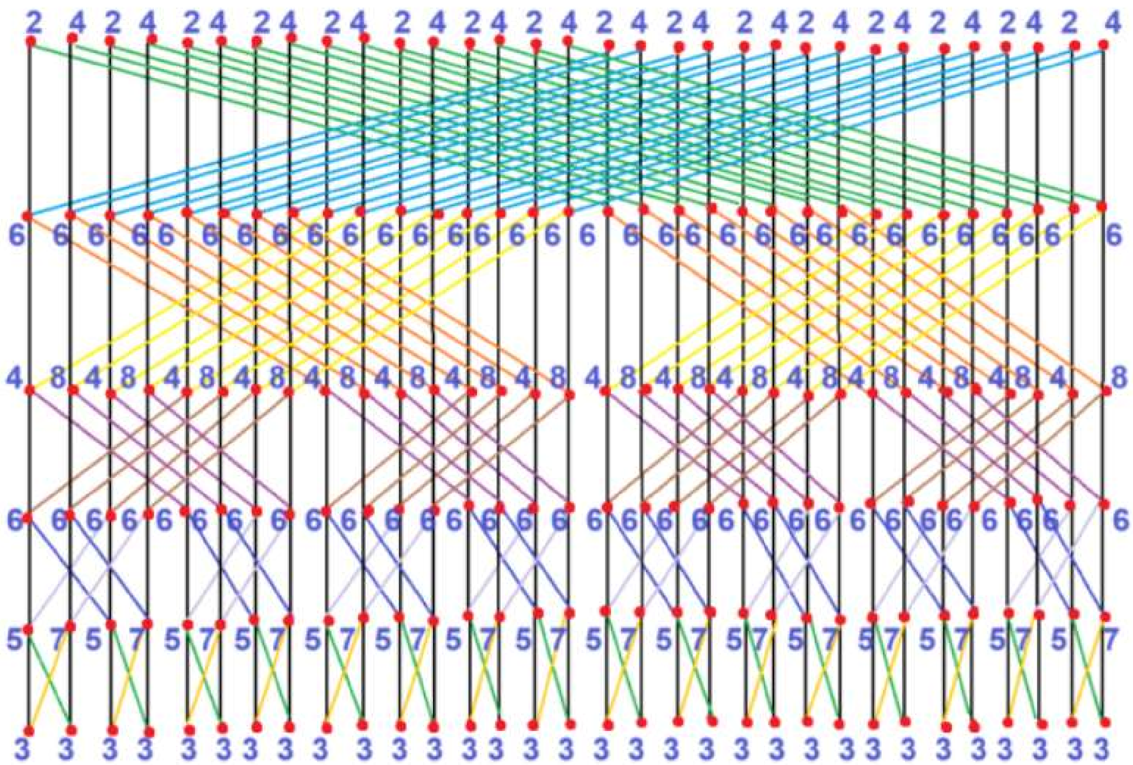

Figure 5.4: sum of neighbourhood of $BF(5)$

and each member of $N(u_{\mathrm{i};2}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases}$.

Therefore $s(u) = \sum_{u \in N(u_{\mathrm{i};2})} f(v_{\mathrm{i};1,3}) = \begin{cases} 8 & i \text{ odd} \\ 4 & i \text{ even} \end{cases}$

subsubcase(i):When v is in level 1,

$$s(v) = \sum_{v \in N(v_{\mathrm{i};1})} f(u_{\mathrm{i};2}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.17}$$

subsubcase(ii) When v is in level 3,

for $n > 4$

$$s(v) = \sum_{v \in N(v_{\mathrm{i};1,3})} f(u_{\mathrm{i};2}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.18}$$

and for $n = 4$

$s(v) = \sum_{v \in N(v_{\mathrm{i};1,3})} f(u_{\mathrm{i};2}) = 6$

Combining above subsubcase with case(i) and subcase(ii) of case(ii) we have $s(u) \neq s(v)$ for all $(u, v) \in E(BF(4))$

case(iv) When $n > 5$ and u is in level j , $3 < j < n-1$

Then u is incident on one cross edge and one straight edge with the other ends at level $j+1$ and also one straight edge and cross edge with the other ends at level $j-1$. Since

subcase(i) j is even and $3 < j < n-1$.

Since

$$f(u_{\mathrm{i;j}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.19}$$

and each members of $N(u_{\text{i;j}})$ are labeled as

$$f(v_{\text{i;j-1,j+1}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.20}$$

Therefore

$$s(u) = \sum_{u \in N(u_{\text{i}};j)} f(v_{\text{i;j+1,j-1}}) = \begin{cases} 8 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.21}$$

Combining above subcase with case(i) and subcase(ii) of case(ii) and subcase(i) of case(iii) we have $s(u) \neq s(v)$ for all $(u,v) \in E(BF(6))$ **subcase(i)** j is odd and $3 < j < n-1$.

Since

$$f(u_{\text{i;j}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.22}$$

and each members of $N(u_{\text{i;j}})$ are labeled as

$$f(v_{\text{i;j-1,j+1}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.23}$$

Therefore

$$s(u) = \sum_{u \in N(u_{\text{i}};j)} f(v_{\text{i;j+1,j-1}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.24}$$

subsubcase(i)for $n > 6$

$$s(v) = \sum_{v \in N(v_{\text{i;j-+1,j-1}})} f(u_{\text{i;j}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases}$$

subcase(ii) j is even $4 < j < n-1$. Since

$$f(u_{\mathrm{i;j}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.25}$$

$$and\, f(v_{\mathrm{i;j+1,j-1}}) = \begin{cases} 2 & i \text{ odd} \\ 1 & i \text{ even} \end{cases} \tag{5.26}$$

Therefore

$$s(u) = \sum_{u \in N(u_{\mathrm{i}};j)} f(v_{\mathrm{i;j+1,j-1}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.27}$$

$$s(v) = \sum_{v \in N(v_{\mathrm{i}};i-1,j+1)} f(u_{\mathrm{i;j}}) = \begin{cases} 8 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.28}$$

Therefore in all the cases $s(u) \neq s(u)$.
$\eta(BF(n)) = 2$ for $n \neq 2$ □

Theorem 5.0.0.2. *The n-dimensional Butterfly Network $BF(n)$ admits Proper Lucky labeling and $\eta_p(BF(n)) = 3$ for $n > 1$*

Proof. Define a Mapping $f : V(G) \to \{1,2,3\}$ then the vertex u_{i} is mapped to 1 or 2 under f such that for $j = 0,1,2,...\lfloor \frac{n}{2} \rfloor$.

$$f(u_{\mathrm{i;2j}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.29}$$

for even levels i.e, and v_{i} is mapped to 3 on odd levels i.e, for $j = 0,1,2,...\lfloor \frac{n}{2} \rfloor$. Clearly $f(u) \neq f(v)$ for all $(u,v) \in E(BF(n))$.
Now we need to prove that $s(u) \neq s(v)$ Label the vertices in consecutive levels of $BF(n)$

as for $j = 0, 1, 2, ... \lfloor \frac{n}{2} \rfloor$.

$$f(u_{\text{i;2j}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.30}$$

and; $f(u_{\text{i;2j+1}}) = 3$. We note that every edge $e = (u, v)$ in $BF(n)$ has one end at level $j$and other end at level $j+1$ or $j-1$(if exists), $0 \leq j \leq n$.

case(i) Suppose u is in level 0 , then $u$is incident on one cross edge and one straight edge with the other ends at $j = 1$. Since

$$f(u_{\text{i;0}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.31}$$

and each member of $N(u_{\text{i;0}})$ is labeled 3, we have $s(u) = \sum_{v \in N(u_{\text{i,0}})} f(v_{\text{i;1}}) = 6$. Since $f(v_{\text{i;1}}) = 3$ and each member of $N(v_{\text{i;1}})$ is labeled

$$f(u_{\text{i;0}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.32}$$

we have We've

$$s(v) = \sum_{v \in N(v_{\text{i;1}})} f(u_{\text{i;0}}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.33}$$

Thus $s(u) \neq s(v)$.

case(ii) When u is in level n, then u is incident on one cross edge and one straight edge with ends at $j = n - 1$.

subcase(i): If n is odd

$f(u_{\text{i;n}}) = 3$ and each member of $N(u_{\text{i;n}})$ is labeled

$$f(v_{\text{i;n-1}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.34}$$

and

$$s(u) = \sum_{v \in N(v_{\text{i;n-1}})} f(u_{\text{i;n}}) = \begin{cases} 2 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.35}$$

and $s(v) = \sum_{u \in N(u_{\text{i;n}})} f(v_{\text{i;n-1}}) = 12$

Therefore $s(u) \neq s(v)$.

subcase(ii): If n is even ,

$$f(u_{\text{i;n}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.36}$$

$N(u_{\text{i;n}})$ is labeled as $f(v_{\text{i,n-1}}) = 3$ Therefore $s(u) = \sum_{v \in N(u_{\text{i;n}})} f(v_{\text{i;n-1}}) = 6$

subsubcase(i):$n = 2$

$$s(v) = \sum_{v \in N(v_{\text{i;1}})} f(u_{\text{i;2}}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.37}$$

Then by case(i) for $n = 2$ $s(u) \neq s(v)$ for all $(u, v) \in E(BF(2))$

subsubcase(ii):$n > 2$

$$s(v) = \sum_{v \in N(v_{\text{i;n-1}})} f(u_{\text{i;n}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.38}$$

Therefore $s(u) \neq s(v)$

case(iii) When u is in level $j = 2$ and for $n > 2$ Then u is incident on one cross edge and

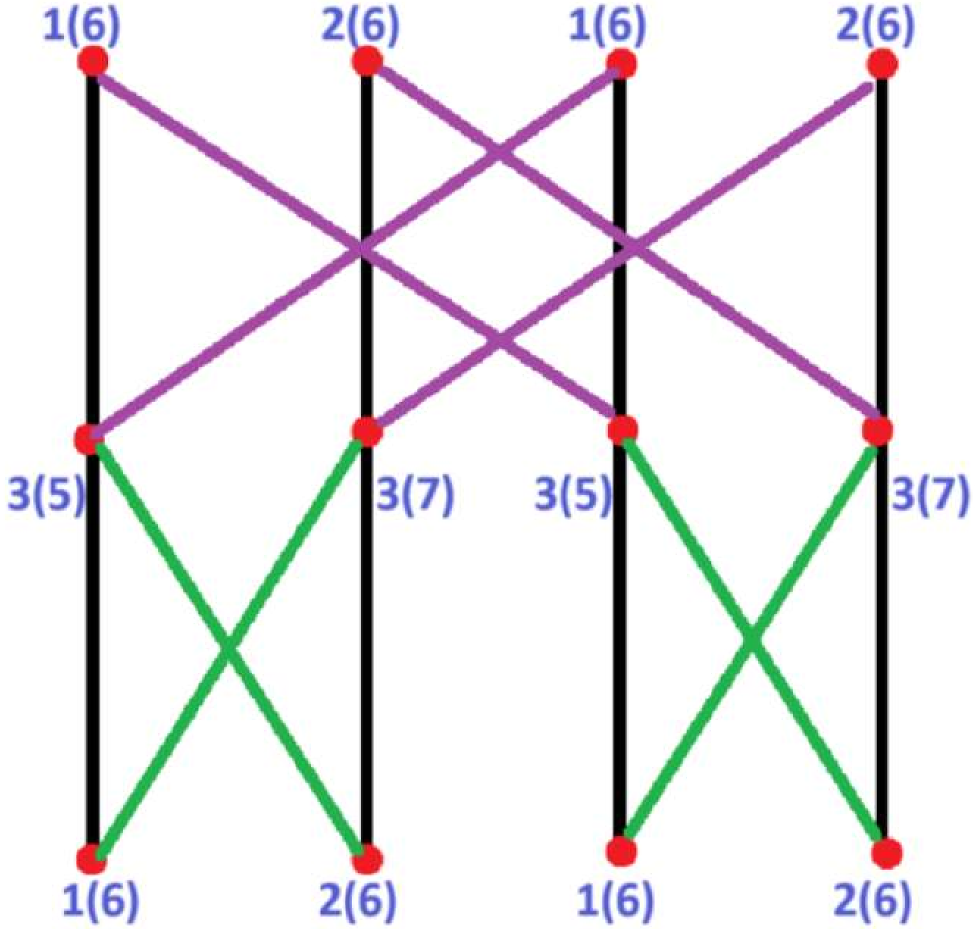

Figure 5.5: Proper Lucky labeling of $BF(2)$

one straight edge with the other ends at level $j+1$ and also one straight edge and cross edge with the other ends at level $j-1$.

subcase(i) When n is odd

Since

$$f(u_{\text{i};2}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.39}$$

and each member of $N(u_{\text{i};1,3})$ is labeled f($v_{i;2,3}$)=3.

Therefore $s(u) = \sum_{u \in N(u_{\text{i};2})} f(v_{\text{i};1,3}) = 12$

subsubcase(i):when v is in level 3 ,

$$s(v) = \sum_{v \in N(v_{\text{i};3})} f(u_{\text{i};2}) = \begin{cases} 2 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.40}$$

subsubcase(ii):When $v$is in level 1,

$$s(v) = \sum_{v \in N(v_{\text{i}})} f(u_{\text{i};2}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.41}$$

combining above 2 subsubcases and case(i) we have $s(u) \neq s(v)$ for all $(u,v) \in E(BF(3))$

This also covers the case when $n = 3$

subcase(ii) When n is even.

Since

$$f(u_{\text{i}};n) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.42}$$

and each member of $N(u_{\text{i}};1,3) = 3$.

Therefore $s(u) = \sum_{u \in N(u_{\text{i};2})} f(v_{\text{i};1,3}) = 12$

subsubcase(i)then v is in level 3,

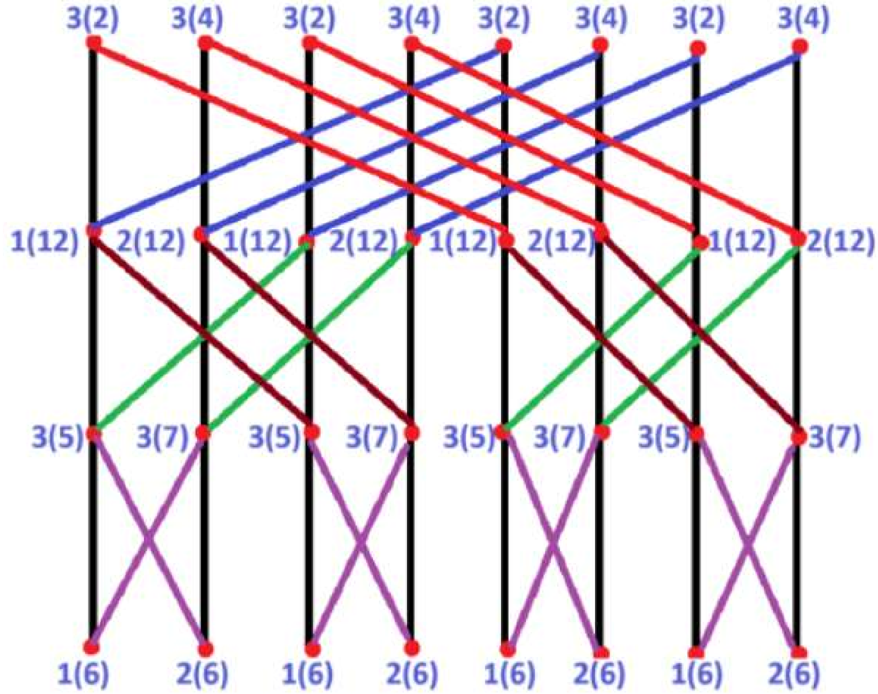

Figure 5.6: Proper Lucky labeling of $BF(3)$

$$s(v) = \sum_{v \in N(v_{\text{i;3}})} f(u_{\text{i;2}}) = \begin{cases} 4 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.43}$$

subsubcase(ii):When v is in level 1,

$$s(v) = \sum_{v \in N(v_{\text{i;1}})} f(u_{\text{i;2}}) = \begin{cases} 5 & i \text{ odd} \\ 7 & i \text{ even} \end{cases} \tag{5.44}$$

Combining subcase(ii) of case(ii) we have $s(u) \neq s(v)$ for all $(u,v) \in E(BF(4))$

case(iii)when $u$is in level $n-1$ and for $n > 4$

subcase(i):when n is odd

$$f(u_{\text{i;n-1}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.45}$$

and each member of $N(u_{\text{i;n-1}})$ is labeled $f(v_{\text{i;n,n+1}}) = 3$ .Therefore $s(v) = \sum_{v \in N(v_{\text{i}})} f(u_{\text{i}})$=12 and since

$$f(v_{\text{i;n,n+1}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.46}$$

.Therefore

$$s(v) = \sum_{v \in N(v_{\text{i;n,n-1}})} f(u_{\text{i;n-1}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.47}$$

. combining above 2 subcases and case(i) we have $s(u) \neq s(v)$ for all $(u,v) \in E(BF(5))$, Therefore $s(u) \neq s(v)$

case(iv) When $n > 5$ and u is in level j, $2 < j < n-1$. Then u is incident on one cross edge and one straight edge with the other ends at level $j+1$ and also one straight edge and cross edge with the other ends at level $j-1$.

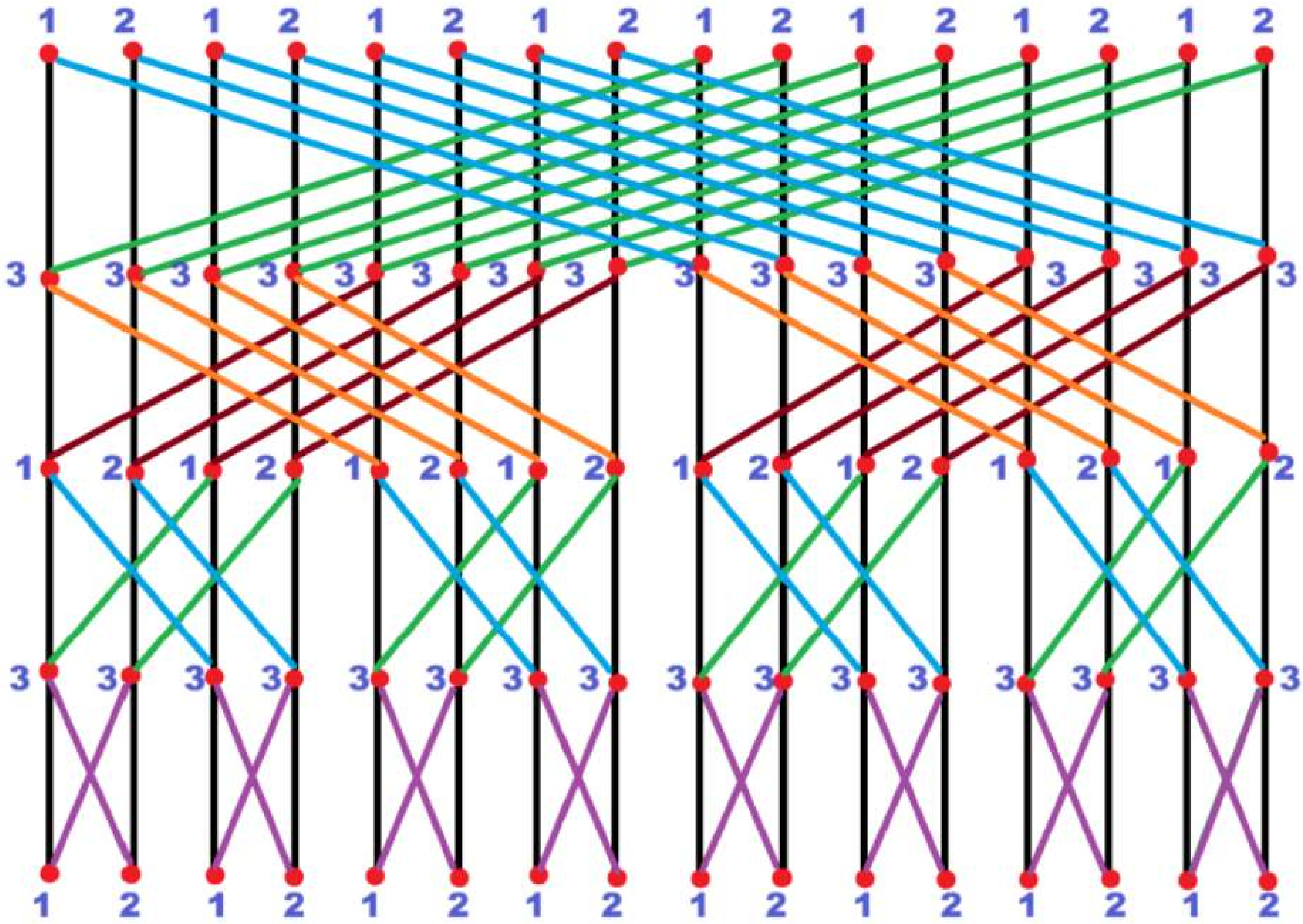

Figure 5.7: Proper Lucky labeling of $BF(4)$

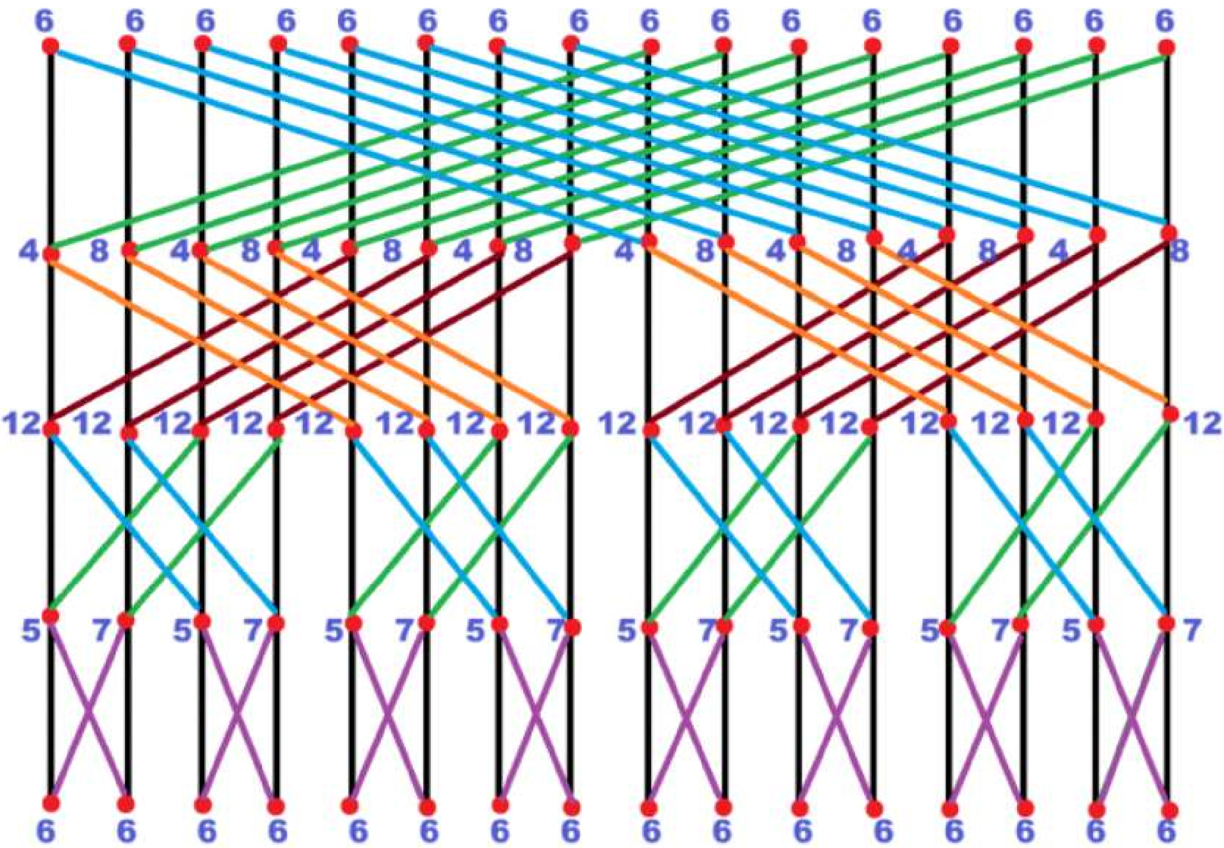

Figure 5.8: Sum of neighbourhood of proper $BF(4)$

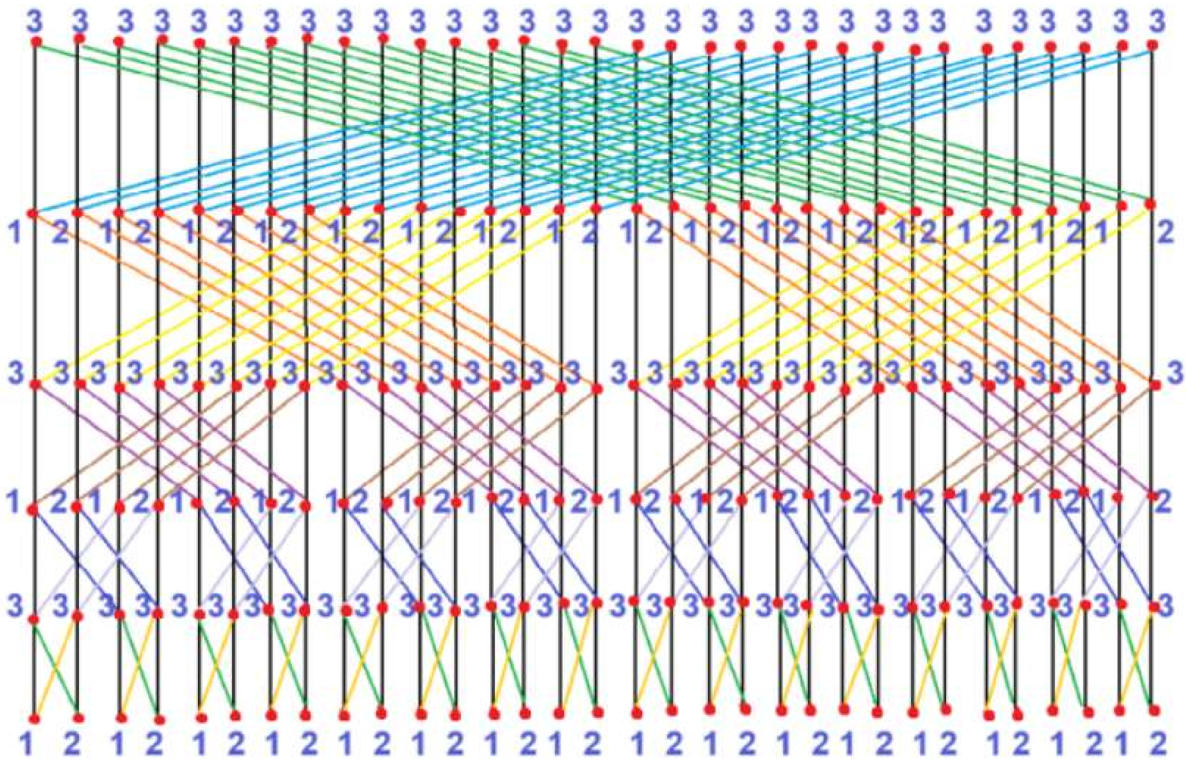

Figure 5.9: Proper lucky labeling of $BF(5)$

subcase(i) j=odd $2 < j < n-1$.

Since $f(u_{\text{i}}; j) = 3$ and each members of $N(u_{\text{i}})$ are labeled as

$$f(v_{\text{i;j-1,j+1}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.48}$$

Therefore

$$s(u) = \sum_{u \in N(u_{\text{i}})} f(v_{\text{i;j+1,j-1}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.49}$$

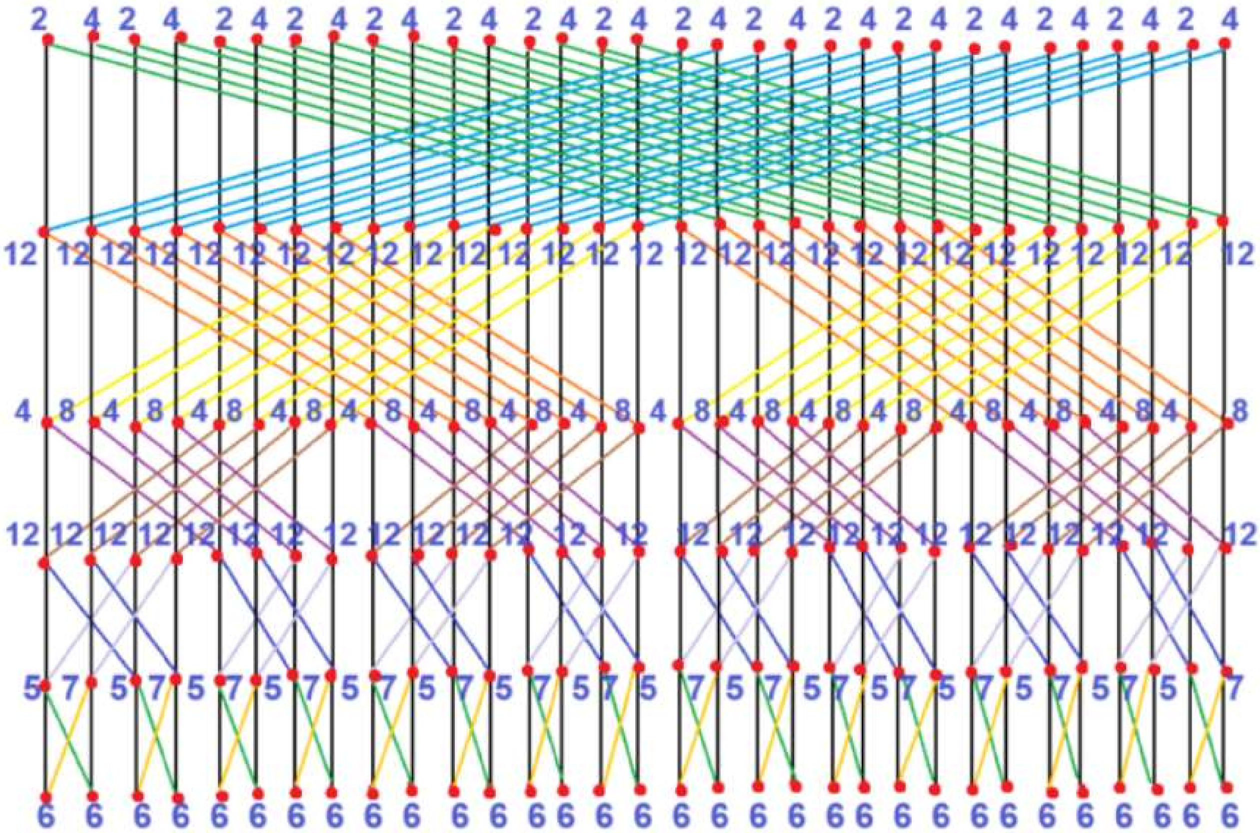

Figure 5.10: Sum of neighbourhood of proper $BF(5)$

$s(v) = \sum_{v \in N(v_{\mathrm{i}})} f(u_{\mathrm{i;j+1,j-1}})$=12 if n=6, this case along with case(i) and subsubcase(ii) of subcase(ii) proves that $s(u) \neq s(v)$ for all $(u,v) \in E(BF(6))$ and for each j=odd $2 < j < n-1$

subcase(ii) j is even $2 < j < n-1$. Since

$$f(u_{\mathrm{i;j}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.50}$$

Therefore

$$s(u) = \sum_{u \in N(u_{\mathrm{i}};j+1,j-1)} f(v_{\mathrm{i;j+1,j-1}}) = 12 \tag{5.51}$$

$$s(v) = \sum_{v \in N(v_{\mathrm{i}};i-1,j+1)} f(u_{\mathrm{i;j+1,j-1}}) = \begin{cases} 4 & i \text{ odd} \\ 8 & i \text{ even} \end{cases} \tag{5.52}$$

Therefore in all the cases $s(u) \neq s(u)$.

$\eta_{\mathrm{p}}(BF(n)) = 3$ for $n > 1$ □

Theorem 5.0.0.3. *The Wheel Graph W_n with $n > 3$ admits lucky labeling with lucky labeling number*

$$\eta(W_n) = \begin{cases} 2, & \text{if nodd} \\ 3, & n \text{ even} \\ 4, & n{=}4 \end{cases} \tag{5.53}$$

Proof. Let the $V(W_{\mathrm{n}})$ be denoted by $\{v_0, v_1, ..., v_{\mathrm{n}}\}$ where the rim vertices are $v_1, v_2, ..., v_{\mathrm{n}}$ and the vertex placed at center is v_0 . For $n = 4$, Since $K_4 = W_4$, Therefore $\eta(W_4) = 4$ let $V(W_{\mathrm{n}}) \to \{1,2,3\}$ be defined by

case(i) n is odd

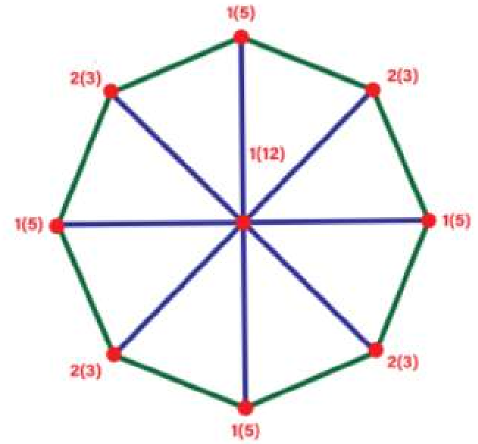

Figure 5.11: Lucky labeling of W_{10}

$$f(v_{\mathrm{i}}) = \begin{cases} 1 & \text{if } i = 0, odd \\ 2 & \text{if i} even \end{cases} \tag{5.54}$$

$$s(v_{\mathrm{i}}) = \begin{cases} 3 & i \text{ even} \\ 5 & i \text{ odd} \\ 3\lceil \frac{n}{2} \rceil & \text{if i=0} \end{cases} \tag{5.55}$$

case(ii) When n is even

$$f(v_{\mathrm{i}}) = \begin{cases} 1 & \text{if } i = 0, 1 \\ 2 & \text{if } i = i = 4k, 4k-1, 4k-2 \\ 3 & \text{if } i = 4k+1, k \in N \end{cases} \tag{5.56}$$

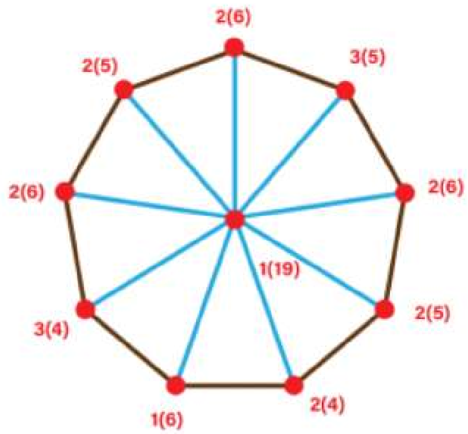

Figure 5.12: Lucky labeling of W_{10}

subcase(i)when $n = 2m, m = 3, 5, ...$

$$s(v_i) = \begin{cases} 4 & \text{if } i = n, 2 \\ 5 & \text{if } i = 3, \lceil \frac{2k+4}{2} \rceil, k = 3, 5, .. \\ 6 & \text{if } i = 1, 4k, 4k+2k = 1, 2, 3, ... \\ 9\lceil \frac{k}{2} \rceil + 1 & \text{if } i = 0 \end{cases} \quad (5.57)$$

subcase(ii) When $n = 2k, k = 4, 6, ...$

$$s(v_i) = \begin{cases} 4 & \text{if } i = n, 2 \\ 5 & \text{if } i = 3, \lceil \frac{2k+4}{2} \rceil \; k = 4, 6, .. \\ 6 & \text{if } i = 1, 4k, 4k+2 \\ 9\lceil \frac{k}{2} \rceil - 4 & \text{if } i = 0 \end{cases} \quad (5.58)$$

□

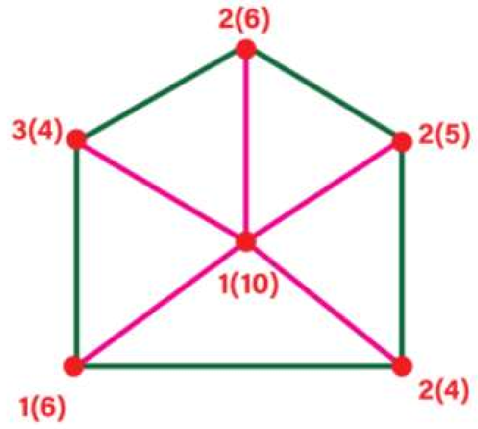

Figure 5.13: Lucky labeling of W_6

Theorem 5.0.0.4. *The Banana Graph B_{mxn} with $n > 1$ admits lucky labeling with*

$$\eta(B_{n,4}) = \begin{cases} 1, & if\, n > 2 \\ 2, & n=2 \end{cases} \tag{5.59}$$

$$\eta(B_{n,5}) = \begin{cases} 1, & if\, n > 2 \\ 2, & n=2 \end{cases} \tag{5.60}$$

$$\eta(B_{n,m}) = \begin{cases} 1, & if\, n > 2, m > 3 \\ 2, & n=2, m > 3 \end{cases} \tag{5.61}$$

Proof. **case(i)** When $n > 2, m > 3$, Label all the vertices of the graph as 1 . In this way we have $s(u) \neq s(v)$ for every pair of adjacent vertices of $\mathrm{B}_{\mathrm{n,m}}$ where $u, v \in V(B_{\mathrm{m,n}})$

w_1 **case(ii)**

Suppose $\eta(B_{\mathrm{n,m}}) = 1$, $n = 2, m > 3$ when $n = 2, m > 3$, When n is 2 there are exactly 2 copies of m-star, in this way there are exactly 2 distinct vertices that are connected to the root vertex of m-star will receive the sum of label as 2 which is equal to sum of label of

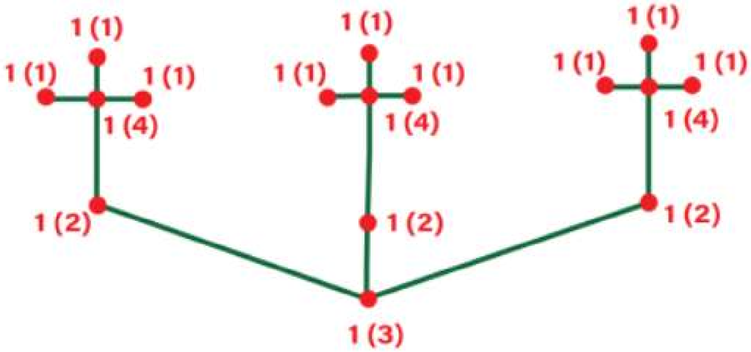

Figure 5.14: Banana Graph $B_{3,5}$

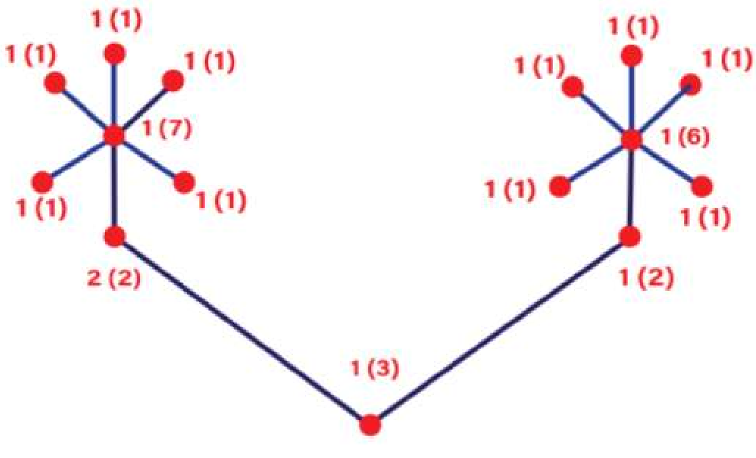

Figure 5.15: Banana Graph $B_{2,7}$

the root vertex, since the root vertex is connected to 1 leaf of each n copies of m star.

Therefore $\eta(B_{n,m}) \neq 1$, $n = 2, m > 3$.

hence $\eta(B_{n,m}) \leq 2$, $n = 2, m > 3$

claim:$\eta(B_{n,m}) = 2$, $n = 2, m > 3$

Label the root vertex (say v) as 1 and label all the vertices except one of the vertex among the two vertices(say w_1 or w_2) that are connected to the root vertex as 1 and label that one vertex that is not labeled as 2 (wlg vertex w_1 is labeled 2). In this way we get $s(w_1) = 2$ and $s(v) = 3$ clearly $S(w_1) \neq s(v)$, similarly $s(w_2) \neq s(v)$.

Now we need to prove that the Sum of labels of all the adjacent vertices of m star copies are also distinct.

Since all the vertices of the two m stars are labeled 1 except □

Theorem 5.0.0.5. *The Mycielskian of Path has lucky Number and for Mycielskian of P_n where $n > 2$ is $\eta(\mu) = 2$*

Proof. let $V(\mu(P_n)) \to \{1, 2, \}$ be defined by

the vertices of the path P_n w_i where $i = 1, 2, .., n$.

The vertices of the copy of P_n as v_i where $i = 1, 2, .., n$.

and the new vertex u that is adjacent to all copies of vertices of P_n as vertex u_i.

case(i)n is even

$$f(u) = 1 \tag{5.62}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i \neq 4k \\ 2 & \text{if } i = 4k \end{cases} \tag{5.63}$$

$$f(w_i) = 1 \tag{5.64}$$

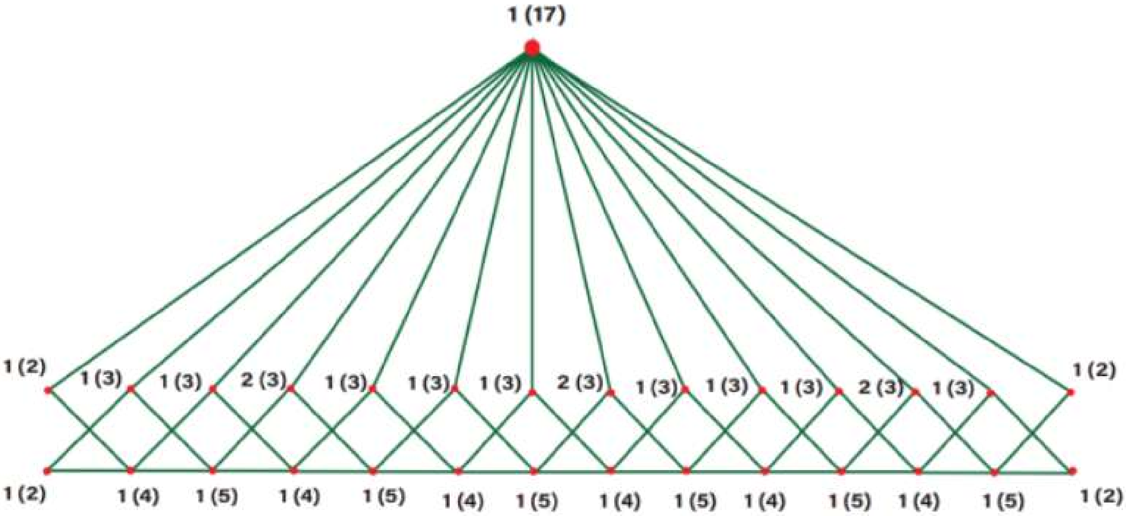

Figure 5.16: lucky labeling of Mycielskian of P_{14}

$$s(u) = 1 \tag{5.65}$$

$$s(v_{\rm i}) = \begin{cases} 2 & \text{if } i = 1, n \\ 3 & \text{if } i \neq 1, n \end{cases} \tag{5.66}$$

$$s(w_{\rm i}) = \begin{cases} 2 & \text{if } i = 1, n \\ 4 & \text{if } i = 2k \\ 5 & \text{if } i = 2k+1 \end{cases} \tag{5.67}$$

case(ii) When n is odd

$$f(u) = 1$$

$$f(v_{\rm i}) = \begin{cases} 1 & \text{if } i =\neq 3, 4k+1 \\ 2 & \text{if } i = 3, 4k+1 \end{cases} \tag{5.68}$$

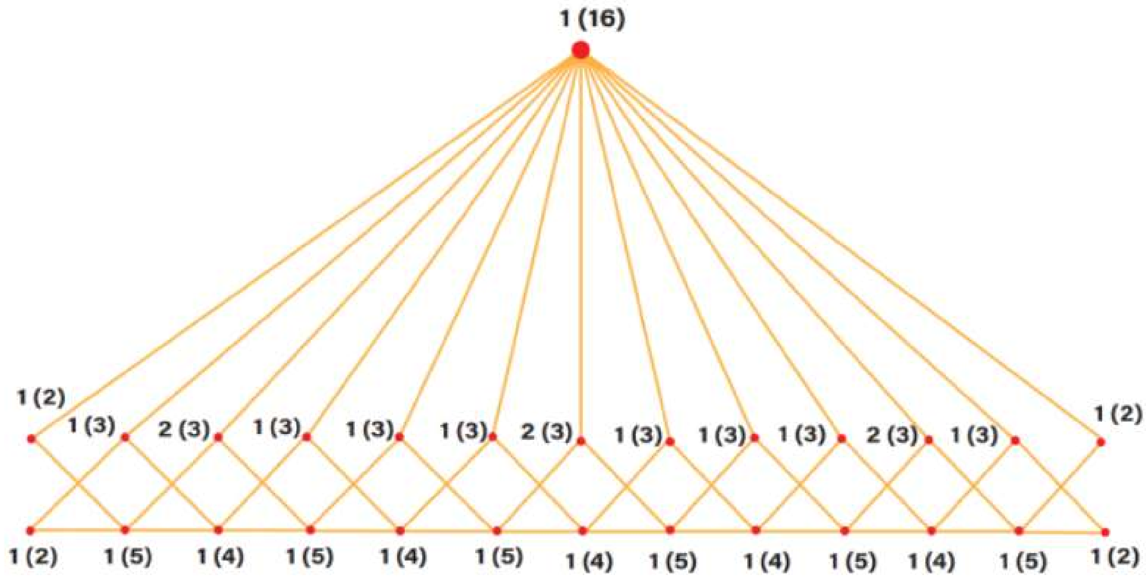

Figure 5.17: lucky labeling of Mycielskian of P_{13}

$$f(w_i) = 1 \tag{5.69}$$

$$s(u) = n + 3 \tag{5.70}$$

$$s(v_i) = \begin{cases} 2 & \text{if } i = 1, n \\ 3 & \text{if } i \neq 1, n \end{cases} \tag{5.71}$$

$$s(w_i) = \begin{cases} 2 & \text{if } i = 1, n \\ 4 & \text{if } i = 2k+1 \\ 5 & \text{if } i = 2k \end{cases} \tag{5.72}$$

□

Theorem 5.0.0.6. *The Mycielskian of Cycle has lucky Number and for Mycielskian of C_n where $n > 2$ is $\eta(\mu) = 2$*

Proof. let $V(\mu(P_n)) \to \{1,2,\}$ be defined by

the vertices of the path P_n , w_i where $i = 1,2,..,n$.

The vertices of the copy of P_n as v_i where $i = 1,2,..,n$.

and the new vertex u that is adjacent to all copies of vertices of P_n as vertex u_i.

case(i) n is even

$f(u) = 1$

$f(v) = 1$

$$f(w_i) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.73}$$

$s(u) = n$

$$s(v_i) = \begin{cases} 5 & i \text{ odd} \\ 3 & i \text{ even} \end{cases} \tag{5.74}$$

$$s(w_i) = \begin{cases} 6 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.75}$$

case(i)n is odd

$$f(u) = 1 \tag{5.76}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 2 & \text{if } i \neq 1,2,n \\ 3 & \text{if } i = 2,n \end{cases} \tag{5.77}$$

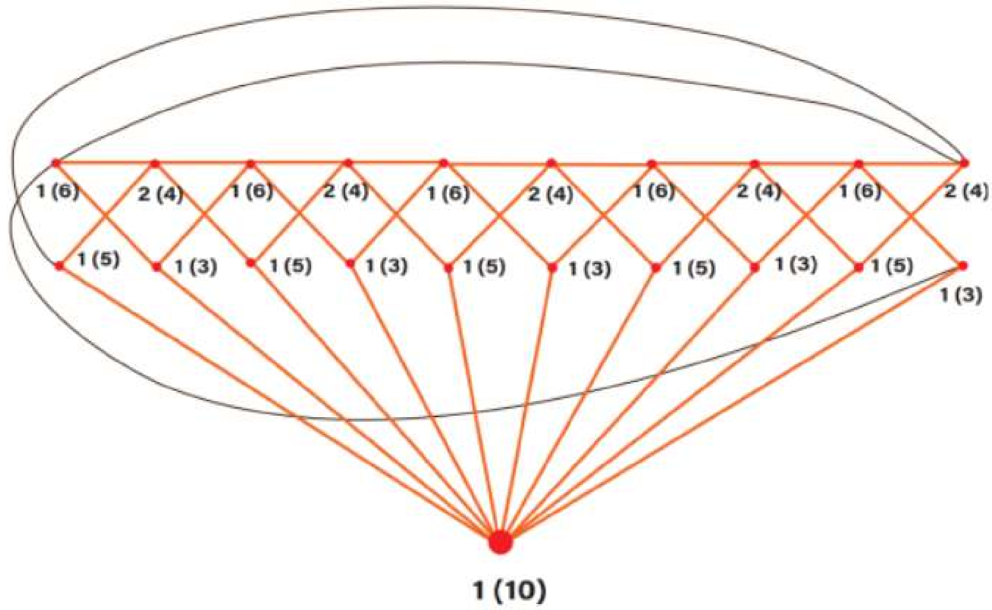

Figure 5.18: lucky labeling of Mycielskian of C_{10}

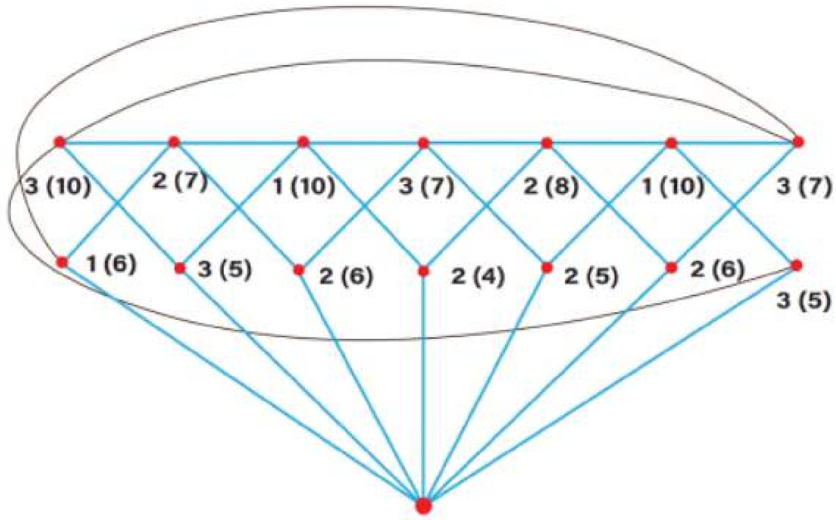

Figure 5.19: lucky labeling of Mycielskian of C_7

$$f(w_{\text{i}}) = \begin{cases} 1 & \text{if } i = 3k \\ 2 & \text{if } i = 2, 3k+2 \\ 3 & \text{if } i = 3k+1 \end{cases} \tag{5.78}$$

$$s(u) = 2n+1 \tag{5.79}$$

$$s(v_{\text{i}}) = \begin{cases} 5 & i \text{ odd} \\ 3 & i \text{ even} \end{cases} \tag{5.80}$$

$$s(w_{\text{i}}) = \begin{cases} 6 & i \text{ odd} \\ 4 & i \text{ even} \end{cases} \tag{5.81}$$

□

Theorem 5.0.0.7. *The cycle Graph C_n with $n > 2$ admits a lucky labeling with lucky labeling number* $\eta(C_n) = \begin{cases} 2, & n \text{ even} \\ 3, & n \text{ odd} \end{cases}$

Proof. Let $f : V(C_{\text{n}}) \to \{1, 2, 3\}$ be defined by

case(i):n is even

$$f(u_{\text{i}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases} \tag{5.82}$$

$$s(u_{\text{i}}) = \begin{cases} 2 & i \text{ even} \\ 4 & i \text{ odd} \end{cases} \tag{5.83}$$

Therefore C_{n} with $n > 2$ is lucky graph with $\eta(C_{\text{n}}) = 2$ for even n.

case(ii):n is odd $k \in N$

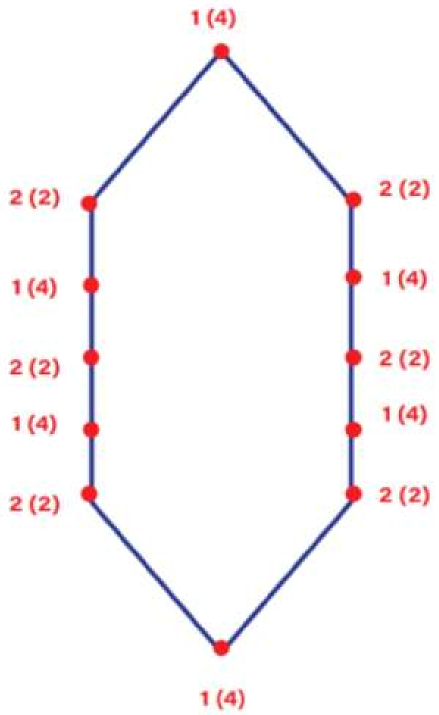

Figure 5.20: Lucky labeling of Cycle C_{12}

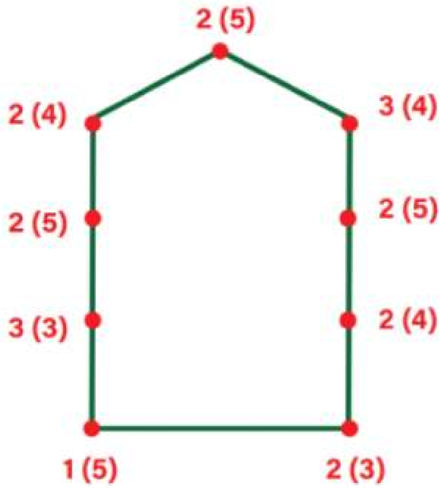

Figure 5.21: Lucky labeling of Cycle C_9

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 2 & i < 1, i = 4k, 4k-1, 4k-2 \\ 3 & i = 4k+1 \end{cases} \tag{5.84}$$

$$s(u_i) = \begin{cases} 3 & i \text{ even} \\ 4 & i = 2k+1 \\ 5 & i = 1, 4k, 4k+2 \end{cases} \tag{5.85}$$

□

Theorem 5.0.0.8. *The lucky number of a Lollipop graph $L_{m,n}$ is $\eta(L_{m,n}) = n-1$*

Proof. Let the Vertex set of the complete graph K_n joined to path P_n be given by u_i such that $V(K_n) = \{u_1, u_2, ..., u_n\}$ where $i = 1, 2, .., m$ and that of P_n be given by v_i such that $V(P_n) = \{v_1, v_2, ..., v_n\}$ where $j = 1, 2, ..., n$.

Let the vertex that is common to both P_n and K_n be w_j.

Without loss of generality let w_k=u_n

Let $f : (L_{\mathrm{m,n}}) \to \{1, 2, .., n-1\}$ be defined by

$$f(u_{\mathrm{i}}) = \begin{cases} 1 & i = 1 \\ 2 & i = 2 \\ \vdots & \\ m-1 & i = m-1 \end{cases}$$

$$f(u_{\mathrm{i}}) = \begin{cases} 1 & i \text{ odd} \\ 2 & i \text{ even} \end{cases}$$

When n is odd, label the vertices of P_n as follows:

$$f(v_{\mathrm{j}}) = \begin{cases} 1 & i \text{ even} \\ 2 & i \text{ odd} \end{cases}$$

When n is even, label the vertices of P_n as follows:

$$f(v_{\mathrm{j}}) = \begin{cases} 1 & i \neq n, n-1, i \text{ even} \\ 2 & i = n, n-1, i \text{ odd} \end{cases}$$

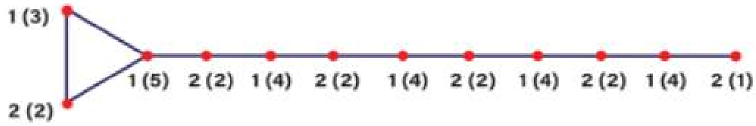

Figure 5.22: Lucky labeling of $L_{3,9}$

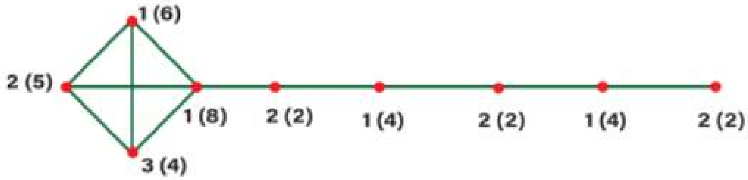

Figure 5.23: Lucky labeling of $L_{4,5}$

$$f(w_j) = m - 1$$

□

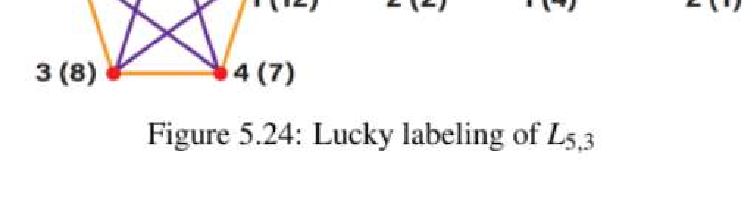

Figure 5.24: Lucky labeling of $L_{5,3}$

Chapter 6

CONCLUSION

6.1 Lucky numbers of few classes of graphs

The following table summarizes the lucky number η of Graph classes obtained in our studies.

Table 6.1: Lucky number of graph classes

Graph Class	**Notation**	$\eta(G)$
Triangular Snake	TS_n	2
Double triangular snake	DTS_n	2
Alternate triangular snake	ATS_n	2
Double alternate triangular snake	$DATS_n$	3
Quadrilateral snake	QS_n	2
Double quadrilateral snake	DQS_n	2
Alternate quadrilateral snake	AQS_n	2
Double alternate quadrilateral snake	$DAQS_n$	2
mxn dimensional bloom graph	B_{mn}	2
Complete graph	k_n	n

6.2 Proper lucky numbers of few classes of graphs

The following table summarizes the proper lucky number $\eta_p(G)$ of Graph classes obtained in our studies.

Table 6.2: Proper lucky number of graphs

Graph Class	**Notation**	η_p
Triangular Snake	TS_n	3
Double triangular snake	DTS_n	3
Alternate triangular snake	ATS_n	4
Double alternate triangular snake	$DATS_n$	3
Quadrilateral snake	QS_n	3
Double quadrilateral snake	DQS_n	2 when n odd
Double quadrilateral snake	DQS_n	3 when n even
mn dimensional bloom, $m,n \geq 3$	B_{mn}	3
mn Mesh	M_{mn}	2
mn extended mesh	EX_{mn}	4
mn enhanced mesh	EN_{mn}	3
Ladder	L_n	2
Open ladder	OL_n	2
Slanting ladder	SL_n	2
Triangular ladder	TL_n	2
Open triangular ladder	OTL_n	3
Diagonal ladder	DL_n	4
Open diagonal ladder	ODL_n	4

6.3 d-lucky numbers of few classes of graphs

The following table summarizes the d-lucky number $\eta_{dl}(G)$ of Graph classes obtained in our studies.

Table 6.3: d-lucky number of graphs

Graph Class	Notation	η_{dl}
n-dimensional butterfly network	$BF(n)$	2
Mesh	M_{mxn}	2
n-dimensional benes network	$BB(n)$	2
Extended triplicate star of a graph	$ETG(K_{1,N})$	3
r-level hypertree	HT_r	2
XTree	XT_n	2

6.4 Some results of Theorems proposed by us

The following table summarizes the lucky number $\eta(G)$, $\eta_p(G)$ of Graph classes obtained by us.

Table 6.4: Lucky number of graph classes

Graph Class	Notation	η
Butterfly	$BF(n)$	2 for $n \neq 2$
Wheel	W_n	2 for n odd
Wheel	W_n	3 for n even
Wheel	W_n	4 for $n = 4$
Banana	B_{mxn}	1 for $n > 2, m > 3$
Banana	B_{mxn}	2 for $n = 2, m > 3$
Mycielskian of Path	$\mu(P_n)$	2 for $n > 2$
Mycielskian of Cycle	$\mu(C_n)$	2 for $n > 2$
Cycle	C_n	2 for n even
Cycle	C_n	3 for n odd

Table 6.5: Proper lucky number of graphs

Graph Class	**Notation**	η_p
Butterfly	$BF(n)$	3

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