STUDY AND ANALYSIS OF MATHEMATICAL MODELS FOR COVID-19 PANDEMIC

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by

Ms. SWIZLE GOMES

22P0410035

ABC ID : 791 - 589 - 331 - 348

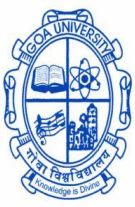
201908108

Under the Supervision of

Dr. MRIDINI GAWAS

School of Physical & Applied Sciences

Mathematics Discipline



GOA UNIVERSITY

APRIL 2024

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Study and Analysis of Mathematical Models for COVID-19 Pandemic" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Mridini Gawas and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.

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Signature: Gomes

Student Name: Swizle Gomes

Seat no: 22P0410035

Date: 29/04/2024

Place: GOA UNIVERSITY

COMPLETION CERTIFICATE

This is to certify that the dissertation report "Study and Analysis of Mathematical Models for COVID-19 Pandemic" is a bonafide work carried out by Ms. Swizle Gomes under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences , Goa University.

Signarure : _____

Supervisor : Dr. Mridini Gawas

Date: 09/05/2024

School of Physical and Applied Sciences Goa University

Signature of HoD of the Dept

Date: 10/5/2024

Place: Goa University

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " STUDY AND ANALYSIS OF MATHEMATICAL MODELS FOR COVID-19 PANDEMIC." This survey is divided into five chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

The Introductory stage of this Project report is based on overview of mathematical Modelling, COVID-19 disease models, Aim and objectives .

SECOND CHAPTER:

This chapter deals with the SEIRS model. The positivity, boundedness , Existence of solution is discussed. Equilibrium Points and their stabilty analysis is done. Basic Reproduction Number is also found.

THIRD CHAPTER:

In this chapter we have introduced an Isolation class Q. Here we study the SEIQR model.The positivity, boundedness , Existence of solution is discussed. Equilibrium Points and their stabilty analysis is done. Basic Reproduction Number is also found. The importance of an Isolation class is shown in this paper.

FOURTH CHAPTER:

This chapter deals with the SEIQRD model which is a slight modification of the model in Chapter 2. Here we include the Recovered Class R for our dynamical analysis. The positivity, boundedness , Existence of solution is discussed. Equilibrium Points and their stabilty analysis is done. Basic Reproduction Number is also found.

FIFTH CHAPTER.

In this chapter we have given some concluding reamarks based on the papers we have reviewed.

ACKNOWLEDGEMENTS

I would like to express my gratitude to all those who gave me the opportunity to complete this dissertation. First and foremost, i would like to thank my Mentor, Dr. Mridini Gawas, who was a continual source of inspiration. She pushed me to think imaginatively and urged me to do this dissertation without hesitation. Her vast knowledge, extensive experience, and professional competence in Mathematical Modelling, Ordinary and Partial Differential Equations enabled me to successfully accomplish this project. This endeavour would not have been possible without her help and supervision.

I would also like to thank the Programme Director of Mathematics Discipline Dr. M.Kunhanandan, Mr. Brandon Fernandes and all the faculty members for their continual support and guidance. I would also express my sincere gratitude to the Dean of School of Physical & Applied Sciences, Dr. Ramesh Pai for providing the facilities required for successful completion of the Dissertation. Lastly, I would like to thank my parents for their help and support.

ABSTRACT

This dissertation project deals with the model formulation and analysis of three COVID-19 pandemic mathematical models. Many compartment models have been formulated to study the spread of COVID-19 disease. In this project, we study the SEIRS, SEIQR and SEIQRD models. The positivity, boundedness, and existence of the solutions of the model are proved. The Disease-free equilibrium point and endemic equilibrium points are identified. Local Stability of disease free Equilibrium point is checked with the help of Next generation matrix. Global stability of endemic equilibrium point is proved using the Concept of Liapunove function. The Basic Reproduction Number is computed. If basic reproduction number is less than one, then number of cases decrease over time and eventually the disease dies out, and if the basic reproduction number is equal to one, then the number of cases are stable. On the other hand, if the basic reproduction number is greater than one, then the number of cases increase over time.

Keywords: COVID-19 Pandemic, Stability Analysis, Next Generation Matrix, Basic Reproduction Number.

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Chapter 1

INTRODUCTION

1.1 Background

A differential equation is a mathematical equation that involves an unknown function and one or more of its derivatives with respect to an independent variable. The equation expresses a relationship between the function and its rates of change, reflecting how the function evolves or behaves over the given variable.

System of differential equation:

A system of differential equations involves multiple equations, each describing the rate of change of one or more dependent variables with respect to an independent variable. These systems are commonly used to model complex relationships where the behavior of one variable is dependent on the behavior of others.

The general form of a system of a n first-order ordinary differential equations (ODEs) is often written as:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, t) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, t) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, t)
\end{aligned}$$

Here, $x_1, x_2, \ldots, x_n$ are the dependent variables, t is the independent variable (often representing time), and $f_1, f_2, \ldots, f_n$ are functions defining the rates of change of the corresponding variables.

Mathematical Modelling:

Mathematical modeling is a way to represent real-world phenomena using mathematical equations and formulas. It allows us to simulate and understand complex systems, such as the spread of diseases, climate patterns, or economic trends. By inputting different variables and parameters, we can predict how the system will behave under different conditions. Mathematical modeling helps us make informed decisions, test hypotheses, and explore various scenarios without having to rely solely on real-world experiments. It's a powerful tool that combines math and science to gain insights and make predictions.

Examples: Population Growth Model, Radio-active Decay Model, Diffusion Model and so on.

Epidemiology:

Epidemiology is the study of how diseases spread and impact populations. It involves analyzing patterns, causes, and effects of diseases in order to understand and control their occurrence. Epidemiologists gather and analyze data to identify risk factors, track the progression of diseases, and develop strategies for prevention and control. It's a fascinating field that plays a crucial role in public health.

COVID-19:

COVID-19, short for "coronavirus disease 2019," is a highly contagious respiratory illness caused by the SARS-CoV-2 virus. It first emerged in late 2019 in Wuhan, China. The virus quickly spread globally, leading to a pandemic. The pandemic has had a profound impact on the world, causing widespread illness, loss of lives, disruptions to economies, travel restrictions, and changes in daily life. It has highlighted the importance of public health measures and the need for global collaboration in fighting infectious diseases.

1.2 Model Formulation and Analysis

Formulation of a mathematical model:

The formulation of a mathematical model using differential equations involves expressing the relationships between variables in a system in terms of differential equations

Positivity, Boundedness and Existence of Solution :

The Existence, positivity and Boundedness of the solution of the model is shown to clarify the model is biologically meaningful and mathematically well posed. A Model is mathememathically well posed if it has a solution, the solution is unique and the solution's behaviour changes continously with initial conditions.

Reproduction number:

The reproduction number, often denoted as R_0 , is a crucial epidemiological concept used to measure the transmission potential of an infectious disease within a population. Specifically, R_0 represents the average number of secondary infections produced by one infected individual in a completely susceptible population.

Equilibrium points:

The equilibrium points represent the states where the system is at rest, as the rates of change are zero at those points. Analyzing the stability and behavior of the system around these equilibrium points is crucial for understanding its dynamics.We investigate the following two equilibrium points:

- **Endemic Equilibrium :**

  The endemic equilibrium represents a stable state in the population where the disease persists at a non-zero level. In this equilibrium, there is a balance between the rates of infection and recovery, leading to a constant, non-zero prevalence of the disease.

- **Disease-Free Equilibrium :**

  The disease-free equilibrium represents a state in the population where no individuals are infected with the disease. At this equilibrium point, all compartments related to the disease (such as susceptible, infected, and recovered) have constant values, and the spread of the disease is not occurring.

Stability analysis:

Stability analysis helps to understand whether small perturbations (changes) from an equilibrium point lead to convergence (stable behavior) or divergence (unstable behavior) over time. There are two main types of stability: local stability and global stability.

- **Local stability:**

  Local stability focuses on the behavior of solutions in the immediate vicinity of a specific equilibrium point. It examines how small perturbations from that equilibrium point evolve over time.

- **Global stability:**

  Global stability considers the behavior of the entire system over its entire state space. It examines whether all trajectories in the system, regardless of initial conditions, converge to a specific equilibrium point..

Lyapunov's Function and Stability Theory:

Lyapunov's Stability Theory has three theorems, namely:

Stability Theorem in Lyapunov Sense, Asymtotic Stability Theorem and Lyapunov Instability Theorem.

Theorem 1.2.0.1. *Stability Analysis based on Lyapunov function*
If in the given domain, the function $F(x)$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of the system is negative semi-definite, i.e., $\dot{F}(x) \leq 0$, then $F(x)$ is said to be a Lyapunov function. The point for which this function exists is said to be stable. The stability is Asymptotic Global Stable if $\dot{F}(x) < 0$.

1.3 COVID-19 Mathematical Model

Corona virus disease 2019 (COVID-19) is an infectious disease that can cause illnesses ranging from the common cold to much more severe illnesses like SARS, MERS, and COVID-19. Severe acute respiratory syndrome corona virus 2 (SARS Cov-2)[5, 14], commonly known as Novel Corona virus (nCoV), is a single, positive-stranded, RNA virus that belongs to Nidoviral type, which are responsible for the Current COVID-19 Pandemic.[24, 23] The novel corona virus (nCoV) or COVID-19 may show signs of fever, cough, breathing difficulties, organ failures or even death of whole society[21]. It can be transmitted from person to person even before any actual signs appeared, which is difficult to prevent and control. According to WHO report, the virus that causes COVID-19 is mainly transmitted through droplets generated when an infected person coughs, sneezes, or speaks.[5] These droplets are too heavy to hang in the air. They quickly fall on floors or surfaces. You can be infected by breathing in the virus if you are

within 1 meter of a person who has COVID-19, or by touching a contaminated surface and then touching your eyes, nose or mouth before washing your hands[6, 4, 16]. There is no specific medicine to prevent or treat corona virus disease (COVID-19). People may need supportive care to help them breathe. If you have mild symptoms, stay at home until you have recovered. You can relieve your symptoms if you:

- rest and sleep
- keep warm
- drink plenty of liquids
- use a room humidifier or take a hot shower to help ease a sore throat and cough

People with COVID-19 develop signs and symptoms, including mild respiratory symptoms and fever, on an average of 5-6 days after infection (mean incubation period 5-6 days, range 1-14 days).

The importance of mathematical modeling in epidemic forecasting is emphasized, ranging from historical outbreaks like cholera to contemporary challenges such as AIDS, COVID-19 and Ebola. The ultimate goal is to refine disease transmission models for better forecasting, preparedness, and intervention strategies to address infectious disease threats effectively. Researchers all round the world have been trying to know how the disease spreads and find out effective ways control the outbreak. Many Compartment models have been formulated to study these outbreaks. Compartments like Susceptible, Exposed, Infected, Recovered, Dead, Quarantined, Hospitalized, Vaccinated etc. are used to form different interesting models.[22, 15]

The ongoing Covid-19 pandemic has similarly caused widespread devastation, affecting social, economic, and health structures globally.Measures such as school closures, travel restrictions, lockdowns, and social distancing have been implemented to curb the virus's spread. Efforts to combat Covid-19 include the development of vaccines, although the virus's ability to mutate has raised concerns. Researchers have employed mathematical models to analyze the disease's dynamics and propose containment strategies. Studies by different mathematicians explore various mathematical models, considering factors like isolation, transmission dynamics, fractional differential equations, super-spreaders, lockdown impact, age groups, hospitalization, vaccination drives and social distancing .These studies aim to understand and predict COVID-19 dynamics, stability, including peak values, infection rates, recovery rates, and case fatality rates[19].

1.4 Aim and Objectives

1.4.1 Aim

To Study and Analyse 3 Mathematical Models for COVID-19 Pandemic.

1. SEIRS Model
2. SEIQR Model
3. SEIQRD Model

1.4.2 Objective of study

Study formulation of mathematical model

To study how we can formulate different models in linear and non linear system of differential equation taking different compartments as variables and the changes that are happening with time.

Positivity and Boundedness of solution

To show that the formulated Model is epidemiologically / biologically meaningful.

Existence of Solution

To show that the Model is mathematically well-posed.

Finding Equilibrium points of the system

Helps in studying stability of the system.

Finding Reproduction number

Help in guiding our understanding of disease transmission and aiding in the design and evaluation of public health interventions.

Stability Analysis

Studying stability of the system whether the system is locally stable and globally stable or not. We use different methods to find stability at the equilibrium points. Methods like Lyapunov function method, Castilo-Chavez method and so on are used.

Chapter 2

REVIEW OF SEIRS MODEL

2.1 Introduction

The main purpose of this article is to formulate and to make Mathematical model analysis that describes the disease transmission dynamics of COVID-19 based on different literature reviews. The paper will create better understanding of the current corona virus pandemic. The SEIRS model is discussed here. SEIRS: Susceptible - Exposed - Infected - Recovered - Susceptible Model. The susceptibles become infected on contact with Infected people. initially they are put in the exposed class (no symptoms) and once they start showing symptoms they are moved to the Infected class [5]. The Infected class is assumed to recover from the disease. Once immunized recovered these individuals may lose their immunity and become susceptible again. Now Let us study the SEIRS Model[14, 24, 23, 6].

PAPER:

Name:**Mathematical Epidemiology Model Analysis on the Dynamics of COVID-19 Pandemic**

Journal:**American Journal of Applied Mathematics** (Published by Science Publishing Group)

Authors: Abayneh Fentie Bezabih, Geremew Kenassa Edessa, Purnachandra Rao Koya

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This paper is organized as follows:

In section 2, **Mathematical model formulation:** Model assumptions, description of variables and parameters, Model diagram and Model equations are presented.

In section 3, **Mathematical Analysis of Model:** Positivity, Boundedness and Existence of solution, Equilibrium points and Basic Reproduction number are Discussed.

In section 4, **Stability Analysis of Equilibrium points:** Next Generation matrix, Local Stability of disease free equilibrium point (LSDFEP), Global Stability of endemic equilibrium point (GSEEP) will be presented.

2.2 Mathematical Model Formulation

In the present study SEIRS model of COVID-19 is Constructed as follows. The total populations are divided into four classes:

1. Susceptible class denoted by **S** (contains population which are capable of becoming infected)

2. Exposed class denoted by **E** (consists of populations being infected but not infectious and waiting for a short period time is called latency period.)

3. Infected class denoted by **I** (consists of population which are infected with COVID-19 and are also infectious)

4. Recovered class denoted by **R** (consists of recovered class from infectious disease COVID-19.)

2.2.1 Assumptions

- The size of total population is assumed to be constant,
  $N(t) = S(t) + E(t) + I(t) + R(t)$

- Both the number of births and death are may not be equal and populations are well mixed.

- Susceptible class are recruited into the compartment $S(t)$ at a constant rate Λ

- The Exposed class has short incubation period and are not yet infective but move to infective class at rate β
- Susceptible class are infected when they come into contact with COVID-19 patient and the disease transmitted according to bilinear interaction rate $\alpha\lambda(t)$ where, $\lambda(t) = I(t)$ which is force of infection.
- Recovered class revert to the susceptible class after losing their immunity at a rate ρ
- All types of population suffer natural mortality at a rate μ .
- All types of population suffer die due to Covid-19 Pandemic at a rate δ
- All parameters in the model are positive.

2.2.2 Model Diagram

The Model Diagram is shown in figure 1:

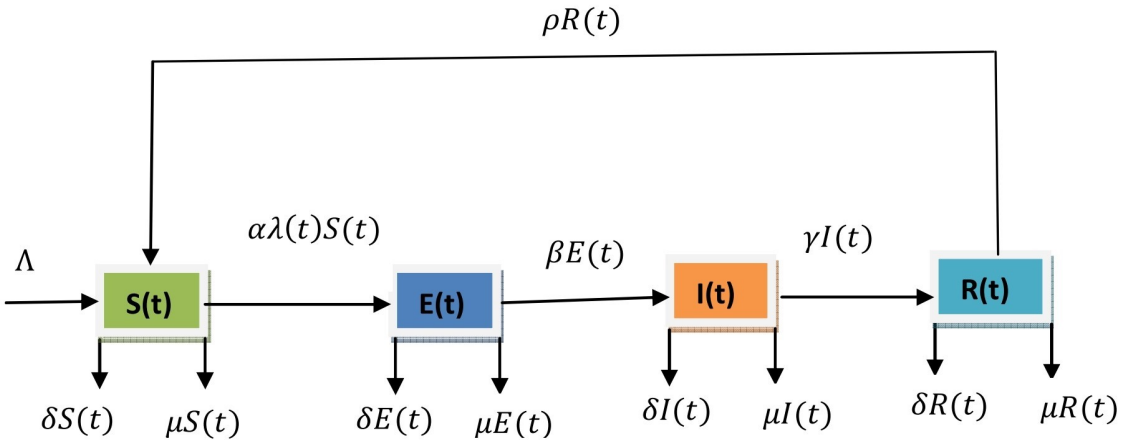

Figure 2.1: Model Diagram

Parameter	Description
Λ	Constant Influx Rate (Rate at which new susceptibles are recruited or enter the Susceptible Compartment)
α	Infection Rate or Contact Rate (rate at which COVID-19 patients transfer from Compartment S to E)
β	Latency Transfer Rate (rate at which COVID-19 patients transfer from Compartment E to I)
γ	Recovery Rate (rate at which COVID-19 patients recover)
ρ	Loss of Immunity or Re-infection Rate (rate at which recovered COVID-19 patients transfer from Compartment R to S)
δ	Death Rate due to infection of COVID-19
μ	Natural Death Rate

Table 2.1: Parameters and Description

Variable	Description
S(t)	Number of Susceptible Individuals at time t
E(t)	Number of Exposed Individuals at time t (infected but not infectious)
I(t)	Number of Infected Individuals at time t (infectious)
R(t)	Number of Recovered Individuals at time t (removed or immune)

Table 2.2: Variables and Description

2.2.3 Model Equations

$$\frac{dS}{dt} = \Lambda + \rho R(t) - \alpha S(t)I(t) - \delta S(t) - \mu S(t) \tag{2.1}$$

$$\frac{dE}{dt} = \alpha S(t)I(t) - \beta E(t) - \delta E(t) - \mu E(t) \tag{2.2}$$

$$\frac{dI}{dt} = \beta E(t) - \gamma I(t) - \delta I(t) - \mu I(t) \tag{2.3}$$

$$\frac{dR}{dt} = \gamma I(t) - \rho R(t) - \delta R(t) - \mu R(t) \tag{2.4}$$

with initial conditions, $S(0) > 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0, \lambda(t) = I(t)$ which is force of infection.

2.3 Model Analysis

In this section mathematical model analysis part is presented.

The analysis consists of the following features:

(i) Positivity of solutions the model

(ii) Boundedness of solutions of the model

(iii) Existence of solutions of the model

(iv) Equilibrium points of the model: Disease free equilibrium points, endemic equilibrium points

(v) Basic Reproduction number

(vi) Stability analysis of equilibrium points: Local stability of disease free equilibrium point and Global stability of endemic equilibrium point.

2.3.1 Positivity of solutions

In order to show that the model is biologically valid, it is required to prove that the solutions of the system of ordinary differential equations are positive and bounded for all time t [4]

Theorem 2.3.1.1 (Positivity)**.** *Solutions of the model equations together with initial conditions* $S(0) > 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0$ *are always positive. That is, the model variables* $S(t), E(t), I(t), R(t)$ *are positive for all* t *and will remain in* $\mathbb{R}^4_+$

Proof: Positivity of the model variables is shown separately for each of the model variables,$S(t), E(t), I(t), \& R(t)$.

Positivity of $S(t)$:

The model equation given by $\frac{dS}{dt} = \Lambda + \rho R(t) - \alpha S(t) I(t) - \delta S(t) - \mu S(t)$ can be expressed without loss of generality, after eliminating the positive terms $(\Lambda + \rho R(t))$ which

are appearing on the right hand side, as an inequality as

$\frac{dS}{dt} \geq -(\alpha I + \delta + \mu)S(t)$.

Using variables separable method and on applying integration,

$\int \frac{dS}{S} \geq \int -(\alpha I + \delta + \mu)dt$

$\implies \ln(S) \geq -(\alpha I + \delta + \mu)t + C_1$,where C_1 is constant of integration.

$\implies \ln(S) \geq -(\alpha I + \delta + \mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $S(t) \geq e^{-(\alpha I+\delta+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $S(t) \geq 0$

Positivity of $E(t)$:

The model equation given by $\frac{dE}{dt} = \alpha S(t)I(t) - \beta E(t) - \delta E(t) - \mu E(t)$ can be expressed without loss of generality, after eliminating the positive terms $(\alpha S(t)I(t))$ which are appearing on the right hand side, as an inequality as

$\frac{dE}{dt} \geq -(\beta + \delta + \mu)E(t)$.

Using variables separable method and on applying integration,

$\int \frac{dE}{E} \geq \int -(\beta + \delta + \mu)dt$

$\implies \ln(E) \geq -(\beta + \delta + \mu)t + C_2$,where C_2 is constant of integration.

$\implies \ln(E) \geq -(\beta + \delta + \mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be

obtained as $E(t) \geq e^{-(\beta+\delta+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $E(t) \geq 0$

Positivity of $I(t)$:

The model equation given by $\frac{dI}{dt} = \beta E(t) - \gamma I(t) - \delta I(t) - \mu I(t)$ can be expressed without loss of generality, after eliminating the positive terms $(\beta E(t))$ which are appearing on the right hand side, as an inequality as

$\frac{dI}{dt} \geq -(\gamma + \delta + \mu)S(t)$.

Using variables separable method and on applying integration,

$\int \frac{dI}{I} \geq \int -(\gamma + \delta + \mu)dt$

$\implies \ln(I) \geq -(\gamma + \delta + \mu)t + C_3$,where C_3 is constant of integration.

$\implies \ln(I) \geq -(\gamma + \delta + \mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $I(t) \geq e^{-(\gamma+\delta+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $I(t) \geq 0$

Positivity of $R(t)$:

The model equation given by $\frac{dR}{dt} = \gamma I(t) - \rho R(t) - \delta R(t) - \mu R(t)$ can be expressed without loss of generality, after eliminating the positive terms $(\gamma I(t))$ which are appearing on

the right hand side, as an inequality as

$\frac{dR}{dt} \geq -(\rho+\delta+\mu)R(t)$.

Using variables separable method and on applying integration,

$\int \frac{dR}{R} \geq \int -(\rho+\delta+\mu)dt$

$\implies \ln(R) \geq -(\rho+\delta+\mu)t + C_4$,where C_4 is constant of integration.

$\implies \ln(R) \geq -(\rho+\delta+\mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $R(t) \geq e^{-(\rho+\delta+\mu)t}$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $R(t) \geq 0$

2.3.2 Boundedness of solution

Theorem 2.3.2.1 (Boundedness)**.** *The positive solutions of the system of model equations are bounded. That is, the model variables $S(t), E(t), I(t)\&R(t)$ are bounded for all t*

Proof: Recall that each population size is bounded if and only if the total population size is bounded. Hence, in the present case it is sufficient to prove that the total population size

$$N = S(t) + E(t) + I(t) + R(t) \quad \text{is bounded for all } t \tag{2.5}$$

. It can be shown that all feasible solutions are uniformly bounded in a proper subset

$\Omega \in \mathbb{R}^4_+$ where the feasible region Ω is given by

$$\Omega = (S,E,I,R) \in \mathbb{R}^4_+; N \leq (\frac{\Lambda}{\delta+\mu}) \tag{2.6}$$

It is clear that the derivative of total population with respect to time t is given by

$$\frac{dN}{dt} = [\frac{dS}{dt}] + [\frac{dE}{dt}] + [\frac{dI}{dt}] + [\frac{dR}{dt}] \tag{2.7}$$

.

Then summation of all the four model equations as follows:

$\frac{dN}{dt} = [\Lambda + \rho R - \alpha SI - \delta S - \mu S] + [\alpha SI - \beta E - \delta E - \mu E] + [\beta E - \gamma I - \delta I - \mu I] + [\gamma I - \rho R - \delta R - \mu R]$

$\implies \frac{dN}{dt} = [\Lambda + \cancel{\rho R} - \cancel{\alpha SI} - \delta S - \mu S] + [\cancel{\alpha SI} - \cancel{\beta E} - \delta E - \mu E] + [\cancel{\beta E} - \cancel{\gamma I} - \delta I - \mu I] + [\cancel{\gamma I} - \cancel{\rho R} - \delta R - \mu R]$

which simplifies to

$$\frac{dN}{dt} = \Lambda - (\delta+\mu)(S+E+I+R) \tag{2.8}$$

$$\Rightarrow \frac{dN}{dt} = \Lambda - (\delta+\mu)N(t) \tag{2.9}$$

Now, $\Lambda - (\delta+\mu)N(t) \geq 0$ if $\Lambda \geq (\delta+\mu)N(t)$

which is

$$\frac{\Lambda}{(\delta+\mu)} \geq N(t) = S+E+I+R \geq 0 \tag{2.10}$$

$\because S(t),E(t),I(t),R(t) \geq 0$ (from Theorem 2.3.1.1) .

Thus, it can be concluded that $N(t)$ is bounded as it is shown that

$$0 \leq N(t) \leq (\frac{\Lambda}{(\delta+\mu)})$$

Therefore, $(\frac{\Lambda}{\delta+\mu})$ is an upper bound of $N(t)$.

Hence, feasible solution of the system of model equations remains in the region Ω which is positively invariant set.

Thus, the system is biologically meaningful and mathematically well posed in the domain Ω. It is sufficient to consider the dynamics of the populations represented by the model system in that domain.
Therefore, it can be summarized that the model variables $S(t), E(t), I(t), \& R(t)$ are bounded for all t.

2.3.3 Existence of solution

Theorem 2.3.3.1 (Existence)**.** *Solutions of the model equations together with the initial conditions,* $S(0) > 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0$ *exist in* $\mathbb{R}^4_+$ *i.e. the model variables* $S(t), E(t), I(t), \& R(t)$ *exist for all* t.

Proof: Let the system of equations arranged as follows:

$f_1 = \Lambda + \rho R - \alpha SI - (\delta + \mu)S$

$f_2 = \alpha SI - (\beta + \delta + \mu)E$

$f_3 = \beta E - (\gamma + \delta + \mu)I$

$f_4 = \gamma I - (\rho + \delta + \mu)R$

According to Derrick and Grossman Theorem [11], let us now define the feasible region Ω that has been discussed under primarily results boundedness of the solutions,

$$\Omega = (S,E,I,R) \in \mathbb{R}^4_+; N \leq (\frac{\Lambda}{\delta+\mu})$$

Then model equations have a unique solution if $(\frac{\partial f_i}{\partial x_j}), i,j = 1,2,3,4$ are continuous and bounded in Ω. Here,$x_1 = S, x_2 = E, x_3 = I, x_4 = R$, The continuity and the boundedness are shown as follows:

Partial Differentiation,

for f_1,

$$\frac{\partial f_1}{S} = (-\alpha I - \delta - \mu) \quad \Longrightarrow |\frac{\partial f_1}{S}| = |-(\alpha I + \delta + \mu)| < \infty$$

$$\frac{\partial f_1}{E} = 0 \quad \Longrightarrow |\frac{\partial f_1}{E}| = 0 < \infty$$

$$\frac{\partial f_1}{I} = (-\alpha S) \quad \Longrightarrow |\frac{\partial f_1}{I}| = |-\alpha S| < \infty$$

$$\frac{\partial f_1}{R} = \rho \quad \Longrightarrow |\frac{\partial f_1}{R}| = |\rho| < \infty$$

for f_2,

$$\frac{\partial f_2}{S} = (\alpha I) \quad \Longrightarrow |\frac{\partial f_2}{S}| = |\alpha I| < \infty$$

$$\frac{\partial f_2}{E} = -\beta - \delta - \mu \quad \Longrightarrow |\frac{\partial f_2}{E}| = |-(\beta + \delta + \mu)| < \infty$$

$$\frac{\partial f_2}{I} = (\alpha S) \quad \Longrightarrow |\frac{\partial f_2}{I}| = |\alpha S| < \infty$$

$$\frac{\partial f_2}{R} = 0 \quad \Longrightarrow |\frac{\partial f_2}{R}| = 0 < \infty$$

for f_3,

$$\frac{\partial f_3}{S} = 0 \quad \Longrightarrow |\frac{\partial f_3}{S}| = 0 < \infty$$

$$\frac{\partial f_3}{E} = \beta \quad \Longrightarrow |\frac{\partial f_3}{E}| = |\beta| < \infty$$

$$\frac{\partial f_3}{I} = -\gamma - \delta - \mu \quad \Longrightarrow |\frac{\partial f_3}{I}| = |-(\gamma + \delta + \mu)| < \infty$$

$$\frac{\partial f_3}{R} = 0 \quad \Longrightarrow |\frac{\partial f_3}{R}| = 0 < \infty$$

for f_4,

$$\frac{\partial f_4}{S} = 0 \quad \Longrightarrow |\frac{\partial f_4}{S}| = 0 < \infty$$

$$\frac{\partial f_4}{E} = 0 \quad \Longrightarrow |\frac{\partial f_4}{E}| = 0 < \infty$$

$$\frac{\partial f_4}{I} = \gamma \quad \Longrightarrow |\frac{\partial f_4}{I}| = |\gamma| < \infty$$

$$\frac{\partial f_4}{R} = -\rho - \delta - \mu \quad \Longrightarrow |\frac{\partial f_4}{R}| = |-(\rho + \delta + \mu)| < \infty$$

Thus, all the partial derivatives , $(\frac{\partial f_i}{\partial x_j}), i, j = 1,2,3,4$ exist, are continuous and bounded in Ω. Hence, by Derrick and Groosman theorem, a solution for the model exists and is unique.

2.3.4 Equilibrium Points

Disease Free Equilibrium Point

Disease Free Equilibrium Points are steady state solutions where there is **no disease** in the population. In the absence of the disease this implies that $E(t) = I(t) = R(t) = 0$ and

the right hand side of the model is equal to zero. Thus $\Lambda-(\delta+\mu)S=0$ which implies $S=\frac{\Lambda}{(\delta+\mu)}$. Thus, the disease-free equilibrium point of the model equation is given by,

$$E(S,E,I,R)=(\frac{\Lambda}{(\delta+\mu)},0,0,0)$$

Endemic Equilibrium Point

Endemic Equilibrium Point $E^*(S^*,E^*,I^*,R^*)$ in the feasible region is a steady state solution where the **disease persists in the population**. The endemic equilibrium point is obtained by setting rates of changes of variables with respect to time in model equations to zero. That is, setting

$$\frac{dS}{dt}=\frac{dE}{dt}=\frac{dI}{dt}=\frac{dR}{dt}=0 \tag{2.11}$$

The model equations can be written as the system of non linear equations

$$\Lambda+\rho R-\alpha SI-aS=0 \tag{2.12}$$

$$\alpha SI-bE=0 \tag{2.13}$$

$$\beta E-cI=0 \tag{2.14}$$

$$\gamma I-dR=0 \tag{2.15}$$

Where, $a=\delta+\mu, b=\beta+\delta+\mu, c=\gamma+\delta+\mu, d=\rho+\delta+\mu$

Solving these equations will give expression for I & R in terms of variable E as follows: i.e;

$$\beta E-cI=0 \tag{2.16}$$

$$\implies cI = \beta E$$

$$\implies I = (\frac{\beta}{c})E \tag{2.17}$$

substitute $I = (\frac{\beta}{c})E$ in eq (2.15)

$$\gamma(\frac{\beta}{c})E - dR = 0$$

$$dR = \gamma(\frac{\beta}{c})E$$

$$R = (\frac{\gamma}{d})I = (\frac{\gamma}{d})(\frac{\beta}{c})E \tag{2.18}$$

This expression could be re-written as

$$R = (\frac{\beta\gamma}{dc})E \tag{2.19}$$

Now substitute in eq (2.13), so as to solve E which results

$$\alpha S(\frac{\beta}{c})E - bE = 0$$

This can be arranged as

$$\alpha S((\frac{\beta}{c}) - b)E = 0 \tag{2.20}$$

However, E does not vanish, since the disease is assumed endemic and it is a computation of non zero equilibrium point of the system. i.e, $E \neq 0$

Thus the only meaningful solution is $\alpha S(\frac{\beta}{c}) - b = 0$

$$\implies \alpha S(\frac{\beta}{c}) = b$$

$$\implies \alpha S\beta = bc$$

then after rearranging the terms, solution is given by the expression

$$S^* = \frac{[bc]}{[\alpha\beta]} \tag{2.21}$$

Then substituting equations (2.19), (2.21) into (2.12) we get

$$\Lambda + \rho(\frac{\beta\gamma}{dc})E - \alpha(\frac{bc}{\alpha\beta}) - a(\frac{bc}{\alpha\beta}) = 0 \qquad here(I = 1)$$

after some algebraic simplifications an expression for E^* can be obtained as

$$E^* = \frac{[dc(bc(\alpha + a) - \alpha\beta\Lambda)]}{\alpha\beta^2\gamma\rho} \tag{2.22}$$

Finally, substitution of E^* will give expressions for I^* & R^* in terms of parameters

$$I^* = \frac{[d(bc(\alpha + a) - \alpha\beta\Lambda)]}{\alpha\beta\gamma\rho} \tag{2.23}$$

$$R^* = \frac{[bc(\alpha + a) - \alpha\beta\Lambda]}{\alpha\beta\rho} \tag{2.24}$$

Therefore the endemic equilibrium points computed above is given by

$$E^*(S^*, E^*, I^*, R^*) = (\frac{[bc]}{[\alpha\beta]}, \frac{[dc(bc(\alpha + a) - \alpha\beta\Lambda)]}{\alpha\beta^2\gamma\rho}, \frac{[d(bc(\alpha + a) - \alpha\beta\Lambda)]}{\alpha\beta\gamma\rho}, \frac{[bc(\alpha + a) - \alpha\beta\Lambda]}{\alpha\beta\rho}) \tag{2.25}$$

2.3.5 Reproduction Number

The basic reproduction number represent the average number of new infections generated by each infected person[4, 16, 12] .

Higher value of R_0 implies fast disease transmission rate.

Smaller values of R_0 implies slow disease transmission rate .

There are three options for the values of R_0

1. $R_0 < 1$ means the number of new cases will decrease over time and eventually the outbreak will end on its own.
2. $R_0 = 1$ means the cases are stable.
3. $R_0 > 1$ means the outbreak is self-sustaining unless effective control measures are implemented.

To derive the general Reproduction number for the formulated model of Covid-19 under the discussion of primary results.

$$S^* \geq \frac{\Lambda}{a} \iff \frac{bc}{\alpha\beta} \geq \frac{\Lambda}{a}$$

Without losing original generality dividing both sides of the inequality by $\frac{bc}{\alpha\beta}$ yields

$$1 \geq \frac{(\alpha\beta\Lambda)}{(abc)} = R_0$$

where the letters notation a, b, c, d are given by

$a = \delta + \mu, b = \beta + \delta + \mu, c = \gamma + \delta + \mu, d = \rho + \delta + \mu$

and hence, the basic Reproduction Number of the model would be

$$R_0 = \frac{(\alpha\beta\Lambda)}{[(\delta+\mu)(\beta+\delta+\mu)(\gamma+\delta+\mu)]} \tag{2.26}$$

2.4 Stability Analysis

In absence of the infectious disease, the model equations have a unique disease free steady state E_0. It is shown that DFEP of model is given by $E_0 = (\frac{\Lambda}{(\delta+\mu)}, 0, 0, 0)$.

Now local stability analysis of DFEP is presented in the following theorem and proved with the help of next generation matrix.

Theorem 2.4.0.1 (Local Stability of Disease-free equilibrium points (LSDFEP))**.** . *The model equations are locally asymptotically stable at disease free equilibrium point (DFEP)* E_0

Proof: Consider the right hand side expressions of the equations as functions to compute Jacobian matrix.

$$\frac{dS}{dt} = \Lambda + \rho R - \alpha SI - aS \equiv f(S,E,I,R) \tag{2.27}$$

$$\frac{dE}{dt} = \alpha SI - bE \equiv g(S,E,I,R) \tag{2.28}$$

$$\frac{dI}{dt} = \beta E - cI \equiv h(S,E,I,R) \tag{2.29}$$

$$\frac{dR}{dt} = \gamma I - dR \equiv k(S,E,I,R) \tag{2.30}$$

where $a = \delta + \mu, b = \beta + \delta + \mu, c = \gamma + \delta + \mu, d = \rho + \delta + \mu$

Compute the Jacobian matrix of functions (f,g,h,k) with respect to (S , E, I, R) is given by

$$J(S,E,I,R) = \begin{bmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial E} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial E} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial R} \\ \frac{\partial h}{\partial S} & \frac{\partial h}{\partial E} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial R} \\ \frac{\partial k}{\partial S} & \frac{\partial k}{\partial E} & \frac{\partial k}{\partial I} & \frac{\partial k}{\partial R} \end{bmatrix}$$

$$J(S,E,I,R) = \begin{bmatrix} -\alpha I - a & 0 & -\alpha S & \rho \\ \alpha I & -b & \alpha S & 0 \\ 0 & \beta & -c & 0 \\ 0 & 0 & \gamma & -d \end{bmatrix}$$

At $E_0 = (\frac{\Lambda}{(\delta+\mu)}, 0, 0, 0)$ we get,

$$J(\frac{\Lambda}{(\delta+\mu)}, 0, 0, 0) = \begin{bmatrix} -a & 0 & -\alpha(\frac{\Lambda}{a}) & \rho \\ I & -b & \alpha(\frac{\Lambda}{a}) & 0 \\ 0 & \beta & -c & 0 \\ 0 & 0 & \gamma & -d \end{bmatrix}$$

Then the eigen values of $J(E_0)$ are computed from characteristic equation

$$| J(E_0) - \lambda I^d | = 0.$$

$$\Longrightarrow \begin{vmatrix} -a-\lambda & 0 & -\alpha\frac{\Lambda}{a} & \rho \\ 0 & -b-\lambda & \alpha\frac{\Lambda}{a} & 0 \\ 0 & \beta & -c-\lambda & 0 \\ 0 & 0 & \gamma & -d-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow (-a-\lambda)\begin{vmatrix} -b-\lambda & \alpha\frac{\Lambda}{a} & 0 \\ \beta & -c-\lambda & 0 \\ 0 & \gamma & -d-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow (-a-\lambda)(-b-\lambda)\begin{Vmatrix} (c-\lambda) & 0 \\ \gamma & (-d-\lambda) \end{Vmatrix} = 0$$

$$\Longrightarrow (-a-\lambda)(-b-\lambda)(-c-\lambda)(-d-\lambda) = 0$$

$$\Longrightarrow -a-\lambda = 0 \Longrightarrow \lambda_1 = -a$$

$$\& -b-\lambda = 0 \Longrightarrow \lambda_2 = -b$$

$$\& -c-\lambda = 0 \Longrightarrow \lambda_3 = -c$$

$$\& -d-\lambda = 0 \Longrightarrow \lambda_4 = -d$$

Thus the four eigen values are $\lambda_1 = -a, \quad \lambda_2 = -b, \quad \lambda_3 = -c, \quad \lambda_4 = -d$

Therefore, it is concluded that the Local Stability of Disease Free Equilibrium point of the system of differential equations is **locally asymptotically stable** because all the

eigen values are negative.

The Global stability Analysis of endemic equilibrium point $E^*(S^*,E^*,I^*,R^*)$ is stated in the following Theorem and proved by taking appropriate Liapunove function.[4, 16]

Theorem 2.4.0.2 (Global Stability of endemic equilibrium point (GSEEP))**.** *The endemic equilibrium point $E^*(S^*,E^*,I^*,R^*)$ is globally asymptotically stable.*

Proof: Let

$$L(S,E,I,R) = m_1\frac{(S-S^*)^2}{2} + m_2\frac{(E-E^*)^2}{2} + m_3\frac{(I-I^*)^2}{2} + m_4\frac{(R-R^*)^2}{2} \tag{2.31}$$

differentiate with respect to t

$$\frac{dL}{dt} = m_1(S-S^*)\frac{dS}{dt} + m_2(E-E^*)\frac{dE}{dt} + m_3(I-I^*)\frac{dI}{dt} + m_4(R-R^*)\frac{dR}{dt} \tag{2.32}$$

Substitute the model equations

$\frac{dL}{dt} = m_1(S-S^*)(\Lambda+\rho R-\alpha SI-aS) + m_2(E-E^*)(\alpha SI-bE) + m_3(I-I^*)(\beta E-cI) + m_4(R-R^*)(\gamma I-dR)$

Take out S,E,I,R and put as change

$\frac{dL}{dt} = m_1(S-S^*)(S-S^*)[(\frac{\Lambda+\rho R}{S}) - \alpha I - a] + m_2(E-E^*)(E-E^*)[(\frac{\alpha SI}{E}) - b] + m_3(I-I^*)(I-I^*)[(\frac{\beta E}{I}) - c] + m_4(R-R^*)(R-R^*)[(\frac{\gamma I}{R}) - d]$

By rearranging and take out negative sign from the bracket it could be otained as

$\frac{dL}{dt} = -m_1(S-S^*)^2[-(\frac{\Lambda+\rho R}{S})+\alpha I+a]-m_2(E-E^*)^2[-(\frac{\alpha SI}{E})+b]-m_3(I-I^*)^2[-(\frac{\beta E}{I})+$
$c]-m_4(R-R^*)^2[-(\frac{\gamma I}{R})+d]$

Thus it is possible to set m_1, m_2, m_3, m_4 as non-negative integers such that $\frac{dL}{dt} \leq 0$ and endemic equilibrium point is globally stable.

2.5 Conclusions

In this Paper, *SEIRS* mathematical model describing the dynamics of COVID-19 is formulated and analyzed. The model is developed based on biologically reasonable assumptions. The mathematical analysis has shown that if basic reproduction number is less than one, then number of cases decrease over time and eventually the disease die out, and if the basic reproduction number is equals to one, then cases are stable. On the other hand, if the basic reproduction number is greater than one then the number of cases increase over time gets worse, and the disease continue to spread more rapidly.
Moreover, Existence, Positivity and Boundedness of the solution of the model is shown to clarify the model is biologically meaningful and mathematically well posed. Stability analysis of the model is checked by computing the Jacobian matrix and its eigen values and the global stability are proved by taking appropriate liapunove function.

Chapter 3

REVIEW OF SEIQR MODEL

3.1 Introduction

Mathematical models are useful to understand the behavior of an infection when it enters a community and investigate under which conditions it will be wiped out or continued. Currently, COVID-19 is of great concern to researches, governments, and all people because of the high rate of the infection spread and the significant number of deaths that occurred. This paper deals with the SEIQR model i.e, Susceptible - Exposed - Infected - Quarantined - Recovered Model[25]. This paper helps us to under the Importance of Isolating or quaranining the COVID-19 patients. By Isolation, futher transmission can be stopped to a certain extent. Isolation can be done in the house(for the ones having mild symptoms) and in the hospitals (for the ones having severe symptoms) . Now let us study the SEIQR Model.[20, 3, 21]

PAPER:

Name:**Mathematical Model for Coronavirus Disease 2019 (COVID-19) Containing Isolation Class**

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3.2 Mathematical Model Formulation

3.2.1 Assumptions

1. The total population is divided into five Compartments, namely,
    - Susceptible Compartment S
    - Exposed Compartment E
    - Infected Compartment I
    - Isolated / Quarantined Compartment Q
    - Recovered Compartment R
2. Human to Human contact is the potential cause of outbreaks of COVID-19.
3. Interaction between the Exposed population and Infected population with the susceptible population leads to rise in the number of cases.

4. Exposed and Infected make contact with susceptible individuals in the same rate.

5. The Infected individuals and the Exposed individuals (i.e, individuals showing no symptoms apparently but have the disease in weak form inside their bodies) must be sent to Isolated Class in different rates.

3.2.2 Model Diagram

The Model Diagram is shown in figure :

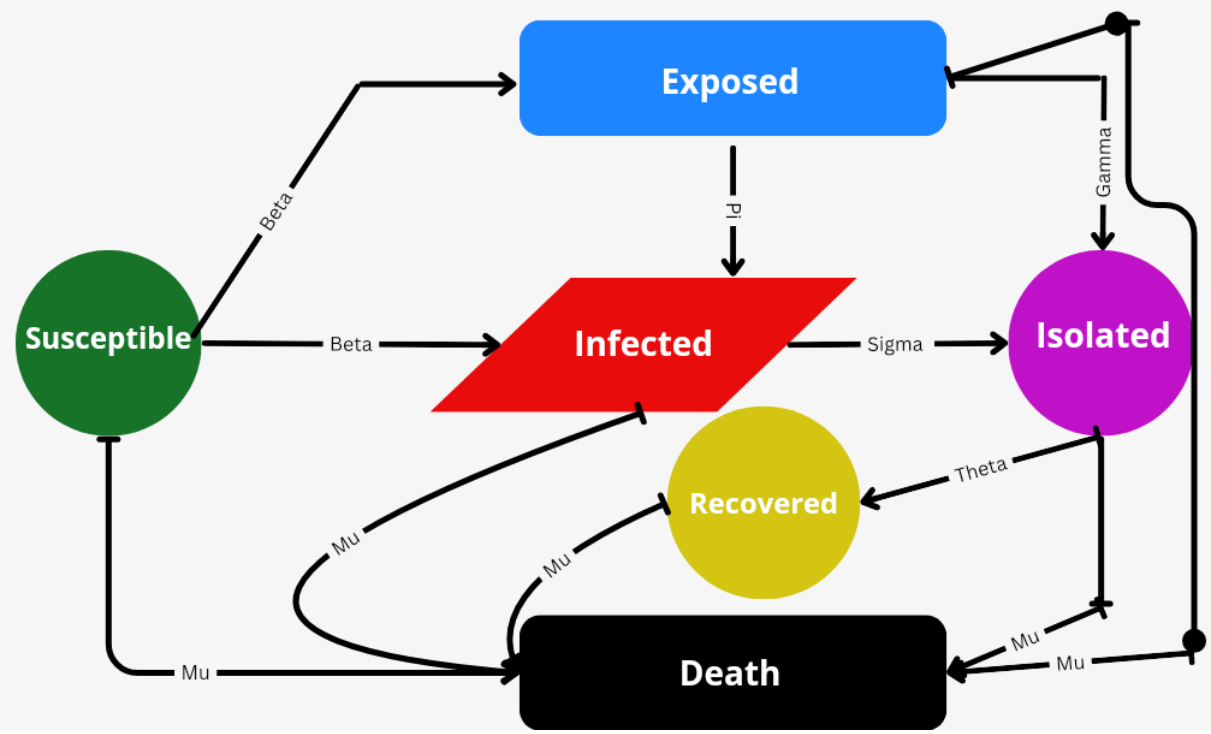

Figure 3.1: Model Diagram

Symbols	Description
S	Susceptible Population
E	Exposed Population
I	Infected Population
Q	Isolated Population
R	Recovered Population
β	rate at which COVID-19 patients transfer from Compartment S to E & I
π	rate at which COVID-19 patients transfer from Compartment E to I
γ	rate at which COVID-19 patients transfer from Compartment E to Q
σ	rate at which COVID-19 patients transfer from Compartment I to Q
θ	rate at which COVID-19 patients transfer from Compartment Q to R
μ	Natural Death Rate

Table 3.1: Symbols and Description

3.2.3 Model Equations

$$\frac{dS(t)}{dt} = A - \mu S(t) - \beta(N)S(t)(E(t)+I(t)) \tag{3.1}$$

$$\frac{dE(t)}{dt} = \beta(N)S(t)(E(t)+I(t)) - \pi E(t) - (\mu+\gamma)E(t) \tag{3.2}$$

$$\frac{dI(t)}{dt} = \pi E(t) - (\sigma+\mu)I(t) \tag{3.3}$$

$$\frac{dQ(t)}{dt} = \gamma E(t) + \sigma I(t) - (\theta+\mu)Q(t) \tag{3.4}$$

$$\frac{dR(t)}{dt} = \theta Q(t) - \mu R(t) \tag{3.5}$$

As the first four equations are independent of $R(t)$, so omit without generality the last equation for $R(t)$ and the modified system becomes

$$\frac{dS(t)}{dt} = A - \mu S(t) - \beta(N)S(t)(E(t)+I(t))$$

$$\frac{dE(t)}{dt} = \beta(N)S(t)(E(t)+I(t)) - \pi E(t) - (\mu+\gamma)E(t)$$

$$\frac{dI(t)}{dt} = \pi E(t) - (\sigma+\mu)I(t)$$

$$\frac{dQ(t)}{dt} = \gamma E(t) + \sigma I(t) - (\theta+\mu)Q(t)$$

For system, let $N = \frac{A}{\mu}, s = \frac{S}{N}, e = \frac{E}{N}, i = \frac{I}{N}$, and $q = \frac{Q}{N}$, and rescale the system to get the normalized form.

$$\frac{ds}{dt} = \mu - \mu s - \beta N s(e+i) \tag{3.6}$$

$$\frac{de}{dt} = \beta N s(e+i) - (\pi+\mu+\gamma)e \tag{3.7}$$

$$\frac{di}{dt} = \pi e - (\sigma+\mu)i \tag{3.8}$$

$$\frac{dq}{dt} = \gamma e + \sigma i - (\theta+\mu)q \tag{3.9}$$

with the initial conditions, $s(0) = s_0 \geq 0$, $e(0) = e_0 \geq 0$, $i(0) = i_0 \geq 0$, $q(0) = q_0 \geq 0$

3.3 Model Analysis

3.3.1 Positivity of solution

Theorem 3.3.1.1. *Under the initial conditions, all the solutions* (s,e,i,q) *of the system remain non-negative for* $t \geq 0$

Proof: Positivity of the model variables is shown separately for each of the model variables,$s, e, i, \& q$.

Positivity of s :

The model equation given by $\frac{ds}{dt} = \mu - \mu s - \beta Ns(e+i)$ can be expressed without loss of generality, after eliminating the positive terms (μ) which are appearing on the right hand side, as an inequality as

$\frac{ds}{dt} \geq -(\mu + \beta Ns(e+i))s.$

Using variables separable method and on applying integration,

$\int \frac{ds}{s} \geq \int -(\mu + \beta Ns(e+i))dt$

$\implies \ln(s) \geq -(\mu + \beta Ns(e+i))t + k_1$,where k_1 is constant of integration.

$\implies \ln(s) \geq -(\mu + \beta Ns(e+i))t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $s \geq exp(-(\mu + \beta Ns(e+i))t)$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $s \geq 0$

Positivity of e :

The model equation given by $\frac{de}{dt} = \beta Ns(e+i) - (\pi + \mu + \gamma)e$ can be expressed without loss of generality, after eliminating the positive terms $(\beta Ns(e+i))$ which are appearing on the right hand side, as an inequality as

$\frac{de}{dt} \geq -(\pi + \mu + \gamma)e.$

Using variables separable method and on applying integration,

$\int \frac{de}{e} \geq \int -(\pi+\mu+\gamma)dt$

$\implies \ln(e) \geq -(\pi+\mu+\gamma)t + k_2$,where k_2 is constant of integration.

$\implies \ln(e) \geq -(\pi+\mu+\gamma)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $e \geq exp(-(\pi+\mu+\gamma)t)$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $e \geq 0$

Positivity of i :

The model equation given by $\frac{di}{dt} = \pi e - (\sigma+\mu)i$ can be expressed without loss of generality, after eliminating the positive terms (πe) which are appearing on the right hand side, as an inequality as

$\frac{di}{dt} \geq -(\sigma+\mu)i.$

Using variables separable method and on applying integration,

$\int \frac{di}{i} \geq \int -(\sigma+\mu)dt$

$\implies \ln(i) \geq -(\sigma+\mu)t + k_3$,where k_3 is constant of integration.

$\implies \ln(i) \geq -(\sigma+\mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $i \geq exp(-(\sigma+\mu)t)$.

Recall that an exponential function is always non-negative irrespective of the sign of the

exponent, hence it can be concluded that $i \geq 0$

Positivity of q :

The model equation given by $\frac{dq}{dt} = \gamma e + \sigma i - (\theta + \mu)q$ can be expressed without loss of generality, after eliminating the positive terms $(\gamma e + \sigma i)$ which are appearing on the right hand side, as an inequality as

$\frac{dq}{dt} \geq -(\theta + \mu)q.$

Using variables separable method and on applying integration,

$\int \frac{dq}{q} \geq \int -(\theta + \mu)dt$

$\implies \ln(q) \geq -(\theta + \mu)t + k_4$,where k_4 is constant of integration.

$\implies \ln(q) \geq -(\theta + \mu)t$

taking anti-log on both sides, the solution of the foregoing differential inequality can be obtained as $q \geq exp(-(\theta + \mu)t)$.

Recall that an exponential function is always non-negative irrespective of the sign of the exponent, hence it can be concluded that $q \geq 0$

Therefore we have $s \geq 0$, $e \geq 0$, $i \geq 0$ & $q \geq 0$

3.3.2 Boundedness of solution

Consider the total population

$N'(t) = S(t) + E(t) + I(t) + Q(t)$

$\frac{dN'(t)}{dt} = \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dQ(t)}{dt}$

$\frac{dN'(t)}{dt} = \mu - \mu s - \beta Ns(e+i) + \beta Ns(e+i) - (\pi+\mu+\gamma)e + \pi e - (\sigma+\mu)i + \gamma e + \sigma i -$

$(\theta+\mu)q$

$\frac{dN'(t)}{dt} = \mu - \mu s - \cancel{\beta Ns(e+i)} + \cancel{\beta Ns(e+i)} - \cancel{\pi e} - \mu e - \cancel{\gamma e} + \cancel{\pi e} - \cancel{\sigma i} - \mu i + \cancel{\gamma e} + \cancel{\sigma i} -$

$(\theta+\mu)q$

$\frac{dN'(t)}{dt} = \mu - \mu s - \mu e - \mu i - (\theta+\mu)q$

$\frac{dN'(t)}{dt} = \mu - \mu(s+e+i+q) - \theta q$

$\frac{dN'(t)}{dt} = \mu - \mu N' - \theta q$

$\frac{dN'(t)}{dt} \geq \mu - \mu N'$

$\frac{dN'(t)}{dt} \leq \mu - \mu N'$

Now solve using variable separable method

$\frac{dN'(t)}{(\mu-\mu N')} \leq dt$

Integrate, $\int \frac{dN'(t)}{(\mu-\mu N')} \leq \int dt$

$\ln \frac{\mu-\mu N'}{-\mu} \leq t + c$

$-\ln(\mu - \mu N') \leq \mu t + \mu c$

$\ln(\mu - \mu N')^{-1} \leq \mu t + \mu c$

take anti-log on both sides,

$(\mu - \mu N')^{-1} \leq e^{(\mu t + \mu c)}$

$\implies \frac{1}{(\mu-\mu N')} \leq Ce^{\mu t}$

$\implies \mu - \mu N' \geq \frac{1}{Ce^{\mu t}}$

$\implies \mu - \mu N' \geq \frac{e^{-\mu t}}{C}$

$\implies -\mu N' \geq -\mu + \frac{e^{-\mu t}}{C}$

$$\implies \mu N' \leq \mu - \frac{e^{-\mu t}}{C}$$

$$\implies N' \leq \frac{\mu}{\mu} - \frac{e^{-\mu t}}{C\mu}$$

$$\implies N' \leq \frac{\cancel{\mu}}{\cancel{\mu}} - \frac{e^{-\mu t}}{C\mu}$$

$$\implies N' \leq 1 - \frac{e^{-\mu t}}{C\mu}$$

$$\implies N'(t) \leq 1 - C_1 e^{-\mu t} \qquad \text{where } C_1 = \frac{1}{C\mu}$$

At $t = 0, N'(t) = N'(0)$

$$\implies N'(0) \leq 1 - C_1 e^{-\mu(0)}$$

$$\implies N'(0) \leq 1 - C_1$$

$$\implies N'(0) - 1 \leq -C_1$$

$$\implies N'(0) - 1 \leq -C_1 \leq C_1$$

$$\implies N'(0) - 1 \leq C_1$$

$$\implies -(N'(0) - 1) \geq -C_1$$

$$\implies -(N'(0) - 1)e^{-\mu t} \geq -C_1 e^{-\mu t}$$

$$\implies 1 - (N'(0) - 1)e^{-\mu t} \geq 1 - C_1 e^{-\mu t}$$

$$\implies N'(t) \leq 1 - C_1 e^{-\mu t} \leq 1 - (N'(0) - 1)e^{-\mu t}$$

$$\implies N'(t) \leq 1 - (N'(0) - 1)e^{-\mu t}$$

$$\implies N'(t) \leq 1 + (1 - N'(0))e^{-\mu t}$$

It is clear that,

$$\lim_{t \to \infty} N'(t) \leq 1$$

and thus $N'(t)$ is bounded with $N'(t) \leq 1$.

Hence, we can see that the feasible region of model is

$$\Omega = ((S,E,I,Q) \in \mathbb{R}^4_+ : N' = S+E+I+Q \leq 1) \tag{3.10}$$

which is positively invariant region.

3.3.3 Existence of solution

The existence and uniqueness of solution of model can be proved by Derrick and Groosman theorem [11].

The model subject to non-negative initial values has a unique solution in Ω for all $t \geq 0$

The right hand side of the model can be written as follows:

$$g_1 = \mu - \mu s - \beta N s(e+i) \tag{3.11}$$

$$g_2 = \beta N s(e+i) - (\pi+\mu+\gamma)e \tag{3.12}$$

$$g_3 = \pi e - (\sigma+\mu)i \tag{3.13}$$

$$g_4 = \gamma e + \sigma i - (\theta+\mu)q \tag{3.14}$$

Suppose that $x_1 = s, x_2 = e, x_3 = i, x_4 = q$.

partially differentiate g_1, g_2, g_3, g_4 with respect to x_1, x_2, x_3, x_4

$\frac{\partial g_1}{x_1} = (-\mu - \beta N(e+i)) \quad \Longrightarrow |\frac{\partial f_1}{x_1}| = |--\mu - \beta N(e+i)| < \infty$

$\frac{\partial g_1}{x_2} = -\beta Ns \quad \Longrightarrow |\frac{\partial g_1}{x_2}| = |-\beta Ns|$

$\frac{\partial g_1}{x_3} = (-\beta N) \quad \Longrightarrow |\frac{\partial f_1}{x_3}| = |-\beta N| < \infty$

$\frac{\partial g_1}{x_4} = 0 \quad \Longrightarrow |\frac{\partial g_1}{x_4}| = 0$

$\frac{\partial g_2}{x_1} = \beta Ne + \beta Ni \quad \Longrightarrow |\frac{\partial g_2}{x_1}| = |\beta Ne + \beta Ni|$

$\frac{\partial g_2}{x_2} = \beta Ns - \pi - \mu - \gamma \quad \Longrightarrow |\frac{\partial g_2}{x_2}| = |\beta Ns - \pi - \mu - \gamma|$

$\frac{\partial g_2}{x_3} = \beta Ns \quad \Longrightarrow |\frac{\partial g_2}{x_3}| = |\beta Ns|$

$\frac{\partial g_2}{x_4} = 0 \quad \Longrightarrow |\frac{\partial g_2}{x_4}| = 0$

$\frac{\partial g_3}{x_1} = 0 \quad \Longrightarrow |\frac{\partial g_3}{x_1}| = 0$

$\frac{\partial g_3}{x_2} = \pi \quad \Longrightarrow |\frac{\partial g_3}{x_2}| = |\pi|$

$\frac{\partial g_3}{x_3} = -(\sigma + \mu) \quad \Longrightarrow |\frac{\partial g_3}{x_3}| = |-(\sigma + \mu)|$

$\frac{\partial g_3}{x_4} = 0 \quad \Longrightarrow |\frac{\partial g_3}{x_4}| = 0$

$\frac{\partial g_4}{x_1} = 0 \quad \Longrightarrow |\frac{\partial g_4}{x_1}| = 0$

$\frac{\partial g_4}{x_2} = \gamma \quad \Longrightarrow |\frac{\partial g_4}{x_2}| = |\gamma|$

$\frac{\partial g_4}{x_3} = \sigma \quad \Longrightarrow |\frac{\partial g_4}{x_3}| = |\sigma|$

$\frac{\partial g_4}{x_4} = -(\theta + \mu) \quad \Longrightarrow |\frac{\partial g_4}{x_4}| = |-(\theta + \mu)|$

Then, it can be shown that $\frac{\partial g_i}{\partial x_j}$ is continuous and $|\frac{\partial g_i}{\partial x_j}| < \infty$ for all $i, j = 1, 2, 3, 4$

$\therefore$ Based on Derrick and Groosman theorem the system satisfies Lipchitz's condition.

Hence the model has a unique solution.

3.3.4 Equilibrium points

Disease Free Equilibrium Point

As the name suggests, Disease Free Equilibrium Point or Virus Free Equilibrium means that $e = 0$, $i = 0$, $q = 0$

$$\frac{ds}{dt} = 0$$

$$\implies \mu - \mu s - \beta N s(e+i) = 0$$

Substitute $e = 0, i = 0, q = 0$ in the above equation

$$\mu - \mu s - \beta N s(0+0) = 0$$

$$\mu - \mu s - \beta N s(0) = 0$$

$$\mu - \mu s = 0$$

$$\mu s = \mu$$

$$\cancel{\mu} s = \cancel{\mu}$$

$$\implies s = 1$$

Therefore the Disease Free Equilibrium Point is

$$P_0 = (1,0,0,0) \tag{3.15}$$

3.3.5 Reproduction Number

Now we find the Basic Reproduction Number R_0 . Consider the following Matrices[13]

$$F = \begin{pmatrix} \beta Ns(e+i) \\ 0 \end{pmatrix} \tag{3.16}$$

$$V = \begin{pmatrix} \pi e + \mu e + \gamma e \\ \pi e - (\sigma + \mu) i \end{pmatrix} \tag{3.17}$$

Now we Calculate Jacobian of F and V at $P_0 = (1,0,0,0)$

$$F = \begin{pmatrix} \beta N & \beta N \\ 0 & 0 \end{pmatrix} \tag{3.18}$$

$$V = \begin{pmatrix} \pi + \mu + \gamma & 0 \\ -\pi & \sigma + \mu \end{pmatrix} \tag{3.19}$$

Now let us calculate V^{-1}

$$V^{-1} = \frac{1}{|V|}(adj(V))$$

$$|V| = \begin{vmatrix} \pi + \mu + \gamma & 0 \\ -\pi & \sigma + \mu \end{vmatrix}$$

$$\implies |V| = (\pi + \mu + \gamma)(\sigma + \mu)$$

$$adj(V) = \begin{pmatrix} \sigma+\mu & 0 \\ \pi & \pi+\mu+\gamma \end{pmatrix}$$

$$\therefore V^{-1} = \frac{1}{(\pi+\mu+\gamma)(\sigma+\mu)} \begin{pmatrix} \sigma+\mu & 0 \\ \pi & \pi+\mu+\gamma \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{\sigma+\mu}{(\pi+\mu+\gamma)(\sigma+\mu)} & 0 \\ \frac{\pi}{(\pi+\mu+\gamma)(\sigma+\mu)} & \frac{\pi+\mu+\gamma}{(\pi+\mu+\gamma)(\sigma+\mu)} \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{\cancel{(\sigma+\mu)}}{(\pi+\mu+\gamma)\cancel{(\sigma+\mu)}} & 0 \\ \frac{\pi}{(\pi+\mu+\gamma)(\sigma+\mu)} & \frac{\cancel{(\pi+\mu+\gamma)}}{\cancel{(\pi+\mu+\gamma)}(\sigma+\mu)} \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\pi+\mu+\gamma)} & 0 \\ \frac{\pi}{(\pi+\mu+\gamma)(\sigma+\mu)} & \frac{1}{(\sigma+\mu)} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} \beta N & \beta N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\pi+\mu+\gamma)} & 0 \\ \frac{\pi}{(\pi+\mu+\gamma)(\sigma+\mu)} & \frac{1}{(\sigma+\mu)} \end{pmatrix}$$

Now we use Matrix Multiplication and simplify,

$$FV^{-1} = \begin{pmatrix} \frac{\beta N}{(\pi+\mu+\gamma)} + \frac{\beta\pi N}{(\pi+\mu+\gamma)(\sigma+\mu)} & \frac{\beta N}{\sigma+\mu} \\ 0 & 0 \end{pmatrix}$$

Next find Eigen Values,

$$|FV^{-1} - \lambda I| = 0$$

$$\implies \begin{vmatrix} (\frac{\beta N}{(\pi+\mu+\gamma)} + \frac{\beta\pi N}{(\pi+\mu+\gamma)(\sigma+\mu)}) - \lambda & \frac{\beta N}{\sigma+\mu} \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\implies ((\frac{\beta N}{(\pi+\mu+\gamma)}+\frac{\beta\pi N}{(\pi+\mu+\gamma)(\sigma+\mu)})-\lambda)(-\lambda)-0=0$$

$$\implies \lambda_1=0 \quad \& \quad \lambda_2=(\frac{\beta N}{(\pi+\mu+\gamma)}+\frac{\beta\pi N}{(\pi+\mu+\gamma)(\sigma+\mu)})$$

To Compute R_0 we have to find the Spectral radius of the next generation matrix so we find the Eigen Values,

(Spectral Radius is the maximum of all the eigenvalues of next-generation matrix.)

$$\therefore R_0=\rho(FV^{-1})=max(\lambda_1,\lambda_2)$$

$$R_0=(\frac{\beta N}{(\pi+\mu+\gamma)}+\frac{\beta\pi N}{(\pi+\mu+\gamma)(\sigma+\mu)})$$

Simplify

$$R_0=\frac{\beta N}{(\pi+\mu+\gamma)}(1+\frac{\pi}{\sigma+\mu})$$

Hence we get,

$$R_0=\beta N\frac{\sigma+\mu+\pi}{(\pi+\mu+\gamma)(\sigma+\mu)} \tag{3.20}$$

3.4 Stability Analysis

Theorem 3.4.0.1. *The system is locally stable related to disease-free equilibrium point P_0, $R_0<1$ and unstable if $R_0>1$.*

Proof: For local Stability at P_0, the Jacobian of the normalized system is

$$J = \begin{pmatrix} -\mu & -\beta N & -\beta N & 0 \\ 0 & -\beta N - \pi - \mu - \gamma & \beta N & 0 \\ 0 & \pi & -(\sigma + \mu) & 0 \\ 0 & \gamma & \sigma & -(\theta + \mu) \end{pmatrix} \tag{3.21}$$

Using Block Matrix Technique to find eigen values,

Block 1:

$$-\mu - \lambda = 0$$

$$\implies \lambda_1 = -\mu < 0$$

Block 2:

$$-(\theta + \mu) - \lambda = 0$$

$$\implies \lambda_2 = -(\theta + \mu) < 0$$

Block 3:

$$J_R = \begin{pmatrix} -\beta N - \pi - \mu - \gamma & \beta N \\ \pi & -(\sigma + \mu) \end{pmatrix}$$

$$Trace(J_R) = (-\beta N - \pi - \mu) + (-\sigma - \mu)$$

$$Trace(J_R) = -\beta N - \pi - \mu - \sigma - \mu$$

$$Trace(J_R) = -\beta N - \pi - \sigma - 2\mu \tag{3.22}$$

$$Det(J_R) = \begin{vmatrix} -\beta N - \pi - \mu - \gamma & \beta N \\ \pi & -(\sigma + \mu) \end{vmatrix}$$

$$Det(J_R) = (-\beta N - \pi - \mu - \gamma)(-\sigma - \mu) - (\beta N)(\pi)$$

Finding Eigen values using the Trace and Determinant of J_R

$$\lambda^2 - Trace(J_R)\lambda + Det(J_R) = 0$$

$$i.e,\ \ \lambda^2 - (-\beta N - \pi - \sigma - 2\mu)\lambda + (-\beta N - \pi - \mu - \gamma)(-\sigma - \mu) - (\beta N)(\pi) = 0$$

Using the quadratic formula to find roots (here, λ)

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e,

$$\lambda = \frac{-(-\beta N - \pi - \sigma - 2\mu) \pm \sqrt{(-\beta N - \pi - \sigma - 2\mu)^2 - 4(1)((-\beta N - \pi - \mu - \gamma)(-\sigma - \mu) - (\beta N)}}{2(1)}$$

$$\lambda = \frac{-(-\beta N - \pi - \sigma - 2\mu) \pm \sqrt{(-\beta N - \pi - \sigma - 2\mu)^2 - 4(1)((-\beta N - \pi - \mu - \gamma)(-\sigma - \mu) - (\beta N)}}{2(1)} \tag{3.23}$$

The Eigen Values $\lambda_3 < 0, \lambda_4 < 0$ if $R_0 < 1$.

Since all the eigen values are negative, the system is locally Stable.

It is unstable if $R_0 > 1$

Endemic Equilibrium Point or Positive Virus

There exists a unique positive virus equilibrium point $P^* = (s^*, e^*, i^*, q^*)$ for system , if $R_0 > 1$ [18]

Let the RHS of the model equations be equal to 0

$$\mu - \mu s - \beta N s(e+i) = 0$$

$$\beta N s(e+i) - (\pi + \mu + \gamma)e = 0$$

$$\pi e - (\sigma + \mu)i = 0$$

$$\gamma e + \sigma i - (\theta + \mu)q = 0$$

After solving this system we get

$$s^* = \frac{1}{R_0} \tag{3.24}$$

$$e^* = \frac{(\sigma + \mu)}{\pi} i^* \tag{3.25}$$

$$i^* = \frac{\pi\mu(R_0 - 1)}{\beta N(\pi + \sigma + \mu)} \tag{3.26}$$

$$q^* = \frac{\gamma(\sigma + \mu) + \pi\sigma}{\pi(\theta + \mu)} i^* \tag{3.27}$$

notice that from the value of i^* , it is obvious that all the values of s^*, e^*, q^* are all positive if $R_0 > 1$

Theorem 3.4.0.2. *If $R_0 < 1$, then the system is globally stable.*

Proof: For the proof of this theorem, first, we construct the Lyapunov function L as follows,

$$L = (e - e^*) + \frac{\beta N}{\sigma + \mu}(i - i^*) \tag{3.28}$$

Differentiating the above equation with respect to time and keeping the reality in mind that $R_0 < 1$ and $0 < s < 1$, we obtained

$$\frac{dL}{dt} = \frac{de}{dt} + \frac{\beta N}{\sigma + \mu}\frac{di}{dt} \tag{3.29}$$

substitute for $\frac{de}{dt}$ and $\frac{di}{dt}$

$$\frac{dL}{dt} = \beta Nse + \beta Nsi - (\pi + \mu + \gamma)e + \frac{\beta N}{\sigma + \mu}(\pi e) + \frac{\beta N}{\sigma + \mu}(\sigma + \mu)i \tag{3.30}$$

$$\frac{dL}{dt} = \beta Nse + \beta Nsi - (\pi + \mu + \gamma)e + \frac{\beta N}{\sigma + \mu}(\pi e) + \frac{\beta N}{\cancel{\sigma + \mu}}\cancel{(\sigma + \mu)}i \tag{3.31}$$

$$\frac{dL}{dt} = \beta Nse + \beta Nsi - (\pi + \mu + \gamma)e + \frac{\beta N}{\sigma + \mu}(\pi e) + \beta Ni \tag{3.32}$$

$$\leq \beta Ne - (\pi + \mu + \gamma)e + \frac{\beta N}{\sigma + \mu}(\pi e) \tag{3.33}$$

$$= (R_0 - 1)e \tag{3.34}$$

Therefore, if $R_0 < 1$, then $\frac{dL}{dt} < 0$, which implies that the system is globally stable for $R_0 < 1$

3.5 Conclusion

In this work, we presented that isolation of the infected human overall can reduce the risk of future COVID-19 spread. This SEIQR model shows that the coronavirus spreads through contact and describes how fast something changes by counting the number of people who are infected and the likelihood of new infections. Those new infections are what induce the epidemic. For this reason, we think that this research may lead to better guessing of the spread of this pandemic in the future. This paper is devoted to implement the coronavirus mathematical model containing isolation class. The reproductive number-related stability is discussed, which showed the impact of interaction of infected people to susceptible population and proved graphically and analytically that if we control this contact rate, the control of the current disease is possible, otherwise. State and territory governments have different restrictions in place for public gatherings. Therefore, citizens need to follow the directions from time to time to minimize the health risk. The more the isolation, the lesser will be the transmission.

Chapter 4

REVIEW OF SEIQRD MODEL

4.1 Introduction

In this article we learn the about the Model Formulation and Model Dynamics of another COVID-19 model.[9] Currently COVID-19 is attracting the attention of various Mathematicians, researchers, Scholars government and general public due to its high rate of disease transmisson and high number of deaths[22, 15]. As yet no medicine is found to cure this disease. Many Mathematical Models are formulated to understand the disease and take proper preventive measures. The SIR model is one of the simplest mathematical model that has been formulated to study any disease. Several investigations of COVID-19 SIR models have been studied in [17, 2]. Many Compartment Models have been formed and modified to give accurate results and predictions.[19, 10, 8]. Now let us study the SEIQRD model.

Name: **A SEIQRD Epidemic Model to Study the Dynamics of COVID-19 Disease**

Journal: **Communication in Mathematical Biology and Neuroscience** (Published by SCIK Publishing Corporation)

Authors: Isnani Darti, Trisilowati, Maya Rayungsari, Raqqasyi Rahmatullah Musafir, Agus Suryanto

https://doi.org/10.28919/cmbn/7822

4.2 Model Formulation

4.2.1 Assumptions

1. The total population is divided into 6 compartments, namely,
    - Susceptible Class **S**
    - Exposed Class **E**
    - Infected Class **I**
    - Quarantined Class **Q**
    - Recovered Class **R**
    - Death Class **D**
2. COVID-19 disease has Latency or Incubation Period. Hence the Exposed Class **E** is included. Those people who have been infected but have not exhibited any

disease symptoms and cannot transmit the disease are included in the Exposed Class.

3. Quarantined Class **Q** cannot spread the disease to others as they have lost contact with the susceptible people.

4. COVID-19 transmission occurs only when there is contact between susceptible people and infected people with transmission rate being standard incident rate.

5. Deaths due to COVID-19 disease is considered in this model.

6. Recovery is seen in both infected and quarantined population.

4.2.2 Model Diagram

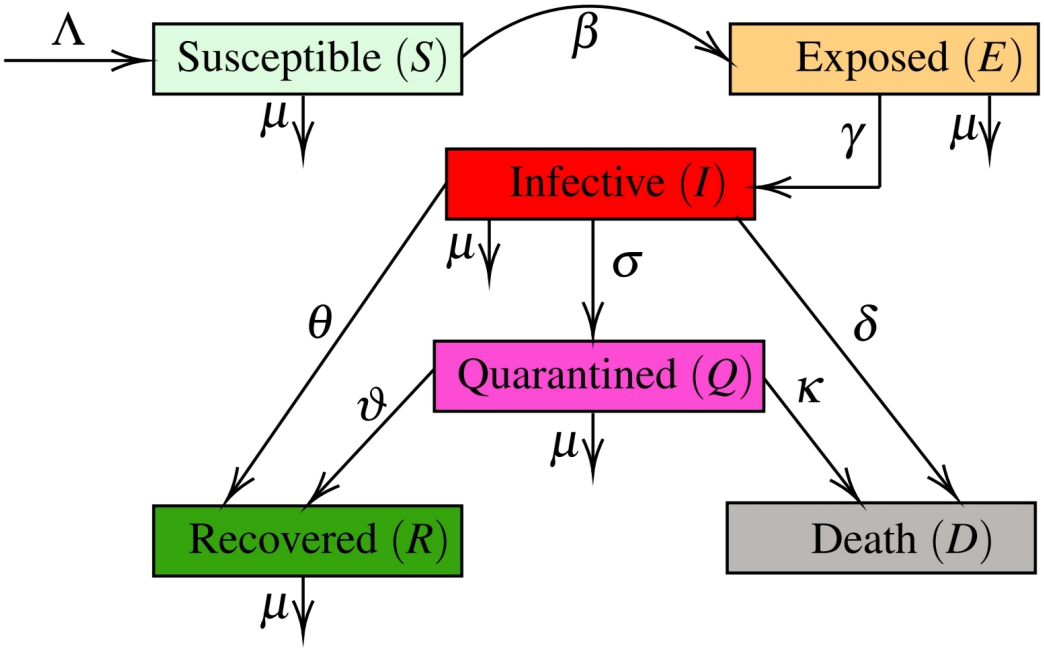

Figure 4.1: Model Diagram

4.2.3 Model Equations

The Model Equations are

$$\frac{dS(t)}{dt} = \Lambda - \beta \frac{S(t)I(t)}{N(t)} - \mu S(t) \tag{4.1}$$

$$\frac{dE(t)}{dt} = \beta \frac{S(t)I(t)}{N(t)} - (\gamma + \mu)E(t) \tag{4.2}$$

$$\frac{dI(t)}{dt} = \gamma E(t) - (\sigma + \theta + \delta + \mu)I(t) \tag{4.3}$$

$$\frac{dQ(t)}{dt} = \sigma I(t) - (\nu + \kappa + \mu)Q(t) \tag{4.4}$$

$$\frac{dR(t)}{dt} = \theta I(t) + \nu Q(t) - \mu R(t) \tag{4.5}$$

$$\frac{dD(t)}{dt} = \delta I(t) + \kappa Q(t) \tag{4.6}$$

with initial conditions $S(0) > 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0$

Parameters	Description
Λ	Recruitment Rate
β	Infection Rate
μ	Natural Death Rate
γ	Incubation Rate
σ	Quarantine Rate
θ	Recovery Rate of I
δ	Death Rate of I induced by disease
ν	Recovery Rate of Q
κ	Death Rate of Q induced by disease

4.3 Model Analysis

4.3.1 Positivity and Boundedness

In this section,the Non-negativity (Positivity) and Boundedness of solution of model is proved to show that the model is epidemiologically meaningful.[5, 18]

Theorem 4.3.1.1. *All solutions of model subject to non-negative initial values are non-negative and ultimately bounded.*

Proof: First let us show the non-negativity of the solutions of the model.

Non-negativity of S :

The model equation given by $\frac{dS}{dt} = \Lambda - \beta \frac{S(t)I(t)}{N(t)} - \mu S(t)$ can be expressed without loss of generality, after eliminating the positive term Λ which is appearing on the right hand side, as an inequality as

$\frac{dS}{dt} \geq -(\beta \frac{I}{N} + \mu)S.$

Use variables separable method

$\frac{dS}{S} \geq -(\beta \frac{I}{N} + \mu)dt$

Integrate,

$\int \frac{dS}{S} \geq \int -(\beta \frac{I}{N} + \mu)dt$

$\implies \ln(S) \geq -(\beta \frac{I}{N} + \mu)t + M_1$,where M_1 is constant of integration.

$\implies \ln(S) \geq -(\beta \frac{I}{N} + \mu)t$

take anti-log on both sides, the solution of the foregoing differential inequality will be obtained as $S(t) \geq e^{-(\beta \frac{I}{N} + \mu)t}$.

Now recall that an exponential function is always non-negative irrespective of the sign of the exponent.

Hence it can be concluded that $S(t) \geq 0$

Non-negativity of E :

The model equation given by $\frac{dE}{dt} = \beta \frac{S(t)I(t)}{N(t)} - (\gamma + \mu)E(t)$ can be expressed without loss

of generality, after eliminating the positive term $\beta\frac{S(t)I(t)}{N(t)}$ which is appearing on the right hand side, as an inequality as

$\frac{dE}{dt} \geq -(\gamma+\mu)E.$

Use variables separable method

$\frac{dE}{E} \geq -(\gamma+\mu)dt$

Integrate,

$\int \frac{dE}{E} \geq \int -(\gamma+\mu)dt$

$\implies \ln(E) \geq -(\gamma+\mu)t + M_2$,where M_2 is constant of integration.

$\implies \ln(E) \geq -(\gamma+\mu)t$

take anti-log on both sides, the solution of the foregoing differential inequality will be obtained as $E(t) \geq e^{-(\gamma+\mu)t}$.

Now recall that an exponential function is always non-negative irrespective of the sign of the exponent.

Hence it can be concluded that $E(t) \geq 0$

Non-negativity of I :

The model equation given by $\frac{dI}{dt} = \gamma E(t) - (\sigma+\theta+\delta+\mu)I(t)$ can be expressed without loss of generality, after eliminating the positive term $\gamma E(t)$ which is appearing on the right hand side, as an inequality as

$\frac{dI}{dt} \geq -(\sigma+\theta+\delta+\mu)I.$

Use variables separable method

$\frac{dI}{I} \geq -(\sigma+\theta+\delta+\mu)dt$

Integrate,

$\int \frac{dI}{I} \geq \int -(\sigma+\theta+\delta+\mu)dt$

$\implies \ln(I) \geq -(\sigma+\theta+\delta+\mu)t + M_3$,where M_3 is constant of integration.

$\implies \ln(I) \geq -(\sigma+\theta+\delta+\mu)t$

take anti-log on both sides, the solution of the foregoing differential inequality will be obtained as $I(t) \geq e^{-(\sigma+\theta+\delta+\mu)t}$.

Now recall that an exponential function is always non-negative irrespective of the sign of the exponent.

Hence it can be concluded that $I(t) \geq 0$

Non-negativity of Q :

The model equation given by $\frac{dQ}{dt} = \sigma I(t) - (\nu+\kappa+\mu)Q(t)$ can be expressed without loss of generality, after eliminating the positive term $\sigma I(t)$ which is appearing on the right hand side, as an inequality as

$\frac{dQ}{dt} \geq -(\nu+\kappa+\mu)Q.$

Use variables separable method

$\frac{dQ}{Q} \geq -(\nu+\kappa+\mu)dt$

Integrate,

$\int \frac{dQ}{Q} \geq \int -(\nu+\kappa+\mu)dt$

$\implies \ln(Q) \geq -(\nu+\kappa+\mu)t + M_4$,where M_4 is constant of integration.

$\implies \ln(Q) \geq -(\nu+\kappa+\mu)t$

take anti-log on both sides, the solution of the foregoing differential inequality will be obtained as $Q(t) \geq e^{-(\nu+\kappa+\mu)t}$.

Now recall that an exponential function is always non-negative irrespective of the sign of the exponent.

Hence it can be concluded that $Q(t) \geq 0$

Non-negativity of R :

The model equation given by $\frac{dR}{dt} = \theta I(t) + \nu Q(t) - \mu R(t)$ can be expressed without loss of generality, after eliminating the positive term $\theta I(t) + \nu Q(t)$ which is appearing on the right hand side, as an inequality as

$\frac{dR}{dt} \geq -\mu R.$

Use variables separable method

$\frac{dR}{R} \geq -\mu dt$

Integrate,

$\int \frac{dR}{R} \geq \int -\mu dt$

$\implies \ln(R) \geq -\mu t + M_5$,where M_5 is constant of integration.

$\implies \ln(R) \geq -\mu t$

take anti-log on both sides, the solution of the foregoing differential inequality will be obtained as $R(t) \geq e^{-\mu t}$.

Now recall that an exponential function is always non-negative irrespective of the sign of the exponent.

Hence it can be concluded that $R(t) \geq 0$

Therefore, $S \geq 0$, $E \geq 0$, $I \geq 0$, $Q \geq 0$ & $R \geq 0$,

Now let us prove Boundedness

NOTE: Total Population is generally defined as the number of living humans. So therefore the total population is obtained by adding up all sub-populations in the model except the sub-population D

Consider the total population

$$N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$$

$$\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} + \frac{dQ(t)}{dt} + \frac{dR(t)}{dt}$$

$$\frac{dN(t)}{dt} = (\Lambda - \beta\frac{SI}{N} - \mu S) + (\beta\frac{SI}{N} - (\gamma+\mu)E) + (\gamma E - (\sigma+\theta+\delta+\mu)I) + (\sigma I - (\nu + \kappa + \mu)Q) + (\theta I + \nu Q - \mu R)$$

$$\frac{dN(t)}{dt} = \Lambda - \cancel{\beta\frac{SI}{N}} - \mu S + \cancel{\beta\frac{SI}{N}} - \cancel{\gamma E} + \mu E + \cancel{\gamma E} - \cancel{\sigma I} - \cancel{\theta I} - \delta I - \mu I + \cancel{\sigma I} - \cancel{\nu Q} - \kappa Q - \mu Q + \cancel{\theta I} + \cancel{\nu Q} - \mu R$$

$$\frac{dN(t)}{dt} = \Lambda - \mu(S+E+I+Q+R) - \delta I - \kappa Q$$

$$\frac{dN(t)}{dt} = \Lambda - \mu N - \delta I - \kappa Q \qquad \because N = S+E+I+Q+R$$

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N$$

Now solve using variable separable method

$$\frac{dN(t)}{(\Lambda-\mu N)} \leq dt$$

Integrate, $\int \frac{dN(t)}{(\Lambda-\mu N)} \leq \int dt$

$$\ln\frac{\Lambda-\mu N}{-\mu} \leq t + c$$

$$-\ln(\Lambda - \mu N) \leq \mu t + \mu c$$

$$\ln(\Lambda - \mu N)^{-1} \leq \mu t + \mu c$$

take anti-log on both sides,

$(\Lambda - \mu N)^{-1} \leq e^{(\mu t + \mu c)}$

$\implies \frac{1}{(\Lambda - \mu N)} \leq Ce^{\mu t}$

$\implies \Lambda - \mu N \geq \frac{1}{Ce^{\mu t}}$

$\implies \Lambda - \mu N \geq \frac{e^{-\mu t}}{C}$

$\implies -\mu N \geq -\Lambda + \frac{e^{-\mu t}}{C}$

$\implies \mu N \leq \Lambda - \frac{e^{-\mu t}}{C}$

$\implies N \leq \frac{\Lambda}{\mu} - \frac{e^{-\mu t}}{C\mu}$

$\implies N(t) \leq \frac{\Lambda}{\mu} - C_1 e^{-\mu t}$ where $C_1 = \frac{1}{C\mu}$

At $t = 0, N(t) = N(0)$

$\implies N(0) \leq \frac{\Lambda}{\mu} - C_1 e^{-\mu(0)}$

$\implies N(0) \leq \frac{\Lambda}{\mu} - C_1$

$\implies N(0) - \frac{\Lambda}{\mu} \leq -C_1$

$\implies N(0) - \frac{\Lambda}{\mu} \leq -C_1 \leq C_1$

$\implies N(0) - \frac{\Lambda}{\mu} \leq C_1$

$\implies -(N(0) - \frac{\Lambda}{\mu}) \geq -C_1$

$\implies -(N(0) - \frac{\Lambda}{\mu})e^{-\mu t} \geq -C_1 e^{-\mu t}$

$\implies \frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})e^{-\mu t} \geq \frac{\Lambda}{\mu} - C_1 e^{-\mu t}$

$\implies N(t) \leq \frac{\Lambda}{\mu} - C_1 e^{-\mu t} \leq \frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})e^{-\mu t}$

$\implies N(t) \leq \frac{\Lambda}{\mu} - (N(0) - \frac{\Lambda}{\mu})e^{-\mu t}$

$\implies N(t) \leq \frac{\Lambda}{\mu} + (\frac{\Lambda}{\mu} - N(0))e^{-\mu t}$

It is clear that,

$\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$

and thus $N(t)$ is bounded with $N(t) \leq \frac{\Lambda}{\mu}$.

Hence, we can see that the feasible region of model is

$$\Omega = ((S,E,I,Q,R) \in \mathbb{R}^5_+ \cup \vec{0} : N = S+E+I+Q+R \leq \frac{\Lambda}{\mu}) \tag{4.7}$$

which is positively invariant region.

4.3.2 Existence And Uniqueness of Solution

The existence and uniqueness of solution of model can be proved using Derrick and Groosman theorem[11],which states that if Lipchitz's condition as in Definition 1 is satisfied, then the solution of the model exists and is unique.

Definition 1: f in system satisfies Lipchitz's condition in $\Omega \subseteq \mathbb{R}^5_+$ if there is a positive con stant k such as

$$||\vec{f}(\vec{X_1}) - \vec{f}(\vec{X_2})|| < k||\vec{X_1} - \vec{X_2}|| \quad \forall \vec{X_1}, \vec{X_2} \in \Omega \tag{4.8}$$

The following theorem guarantees the existence and uniqueness of solution of model .

Theorem 4.3.2.1. *The model subject to non-negative initial values has a unique solution in* Ω *for all* $t \geq 0$.

Proof: The right hand side of the model can be written as

$$f_1 = \Lambda - \beta \frac{S(t)I(t)}{N(t)} - \mu S(t) \tag{4.9}$$

$$f_2 = \beta \frac{S(t)I(t)}{N(t)} - (\gamma + \mu)E(t) \tag{4.10}$$

$$f_3 = \gamma E(t) - (\sigma + \theta + \delta + \mu)I(t) \tag{4.11}$$

$$f_4 = \sigma I(t) - (\nu + \kappa + \mu)Q(t) \tag{4.12}$$

$$f_5 = \theta I(t) + \nu Q(t) - \mu R(t) \tag{4.13}$$

$$f_6 = \delta I(t) + \kappa Q(t) \tag{4.14}$$

Suppose that $x_1 = S, x_2 = E, x_3 = I, x_4 = Q \ \ \& \ \ x_5 = R$.

partially differentiate $f_1, f_2, f_3, f_4, f_5, f_6$ with respect to x_1, x_2, x_3, x_4, x_5

$\frac{\partial f_1}{x_1} = (-\frac{\beta I}{N} - \mu) \quad \Longrightarrow |\frac{\partial f_1}{x_1}| = |-\frac{\beta I}{N} - \mu| < \infty$

$\frac{\partial f_1}{x_2} = 0 \quad \Longrightarrow |\frac{\partial f_1}{x_2}| = 0$

$\frac{\partial f_1}{x_3} = (-\beta \frac{S}{N}) \quad \Longrightarrow |\frac{\partial f_1}{x_3}| = |-\beta \frac{S}{N}| < \infty$

$\frac{\partial f_1}{x_4} = 0 \quad \Longrightarrow |\frac{\partial f_1}{x_4}| = 0 < \infty$

$\frac{\partial f_1}{x_5} = 0 \quad \Longrightarrow |\frac{\partial f_1}{x_5}| = 0 < \infty$

$\frac{\partial f_2}{x_1} = (\frac{\beta I}{N}) \quad \Longrightarrow |\frac{\partial f_2}{x_1}| = |\frac{\beta I}{N}| < \infty$

$\frac{\partial f_2}{x_2} = -(\gamma + \mu) \quad \Longrightarrow |\frac{\partial f_2}{x_2}| = |-(\gamma + \mu)|$

$\frac{\partial f_2}{x_3} = (\beta \frac{S}{N}) \quad \Longrightarrow |\frac{\partial f_2}{x_3}| = |\beta \frac{S}{N}| < \infty$

$\frac{\partial f_2}{x_4} = 0 \quad \Longrightarrow |\frac{\partial f_2}{x_4}| = 0 < \infty$

$\frac{\partial f_2}{x_5} = 0 \quad \Longrightarrow |\frac{\partial f_2}{x_5}| = 0 < \infty$

$\frac{\partial f_3}{x_1} = 0 \quad \Longrightarrow |\frac{\partial f_3}{x_1}| = 0 < \infty$

$\frac{\partial f_3}{x_2} = \gamma \quad \Longrightarrow |\frac{\partial f_3}{x_2}| = |\gamma|$

$\frac{\partial f_3}{x_3} = -(\sigma + \theta + \delta + \mu) \quad \Longrightarrow |\frac{\partial f_3}{x_3}| = |-(\sigma + \theta + \delta + \mu)| < \infty$

$\frac{\partial f_3}{x_4} = 0 \quad \Longrightarrow |\frac{\partial f_3}{x_4}| = 0 < \infty$

$\frac{\partial f_3}{x_5} = 0 \quad \Longrightarrow |\frac{\partial f_3}{x_5}| = 0 < \infty$

$\frac{\partial f_4}{x_1} = 0 \quad \Longrightarrow |\frac{\partial f_4}{x_1}| = 0 < \infty$

$\frac{\partial f_4}{x_2} = 0 \quad \Longrightarrow |\frac{\partial f_4}{x_2}| = 0$

$\frac{\partial f_4}{x_3} = \sigma \quad \Longrightarrow |\frac{\partial f_4}{x_3}| = |\sigma| < \infty$

$\frac{\partial f_4}{x_4} = -(\nu + \kappa + \mu) \quad \Longrightarrow |\frac{\partial f_4}{x_4}| = |-(\nu + \kappa + \mu)| < \infty$

$\frac{\partial f_4}{x_5} = 0 \quad \Longrightarrow |\frac{\partial f_4}{x_5}| = 0 < \infty$

$\frac{\partial f_5}{x_1} = 0 \quad \Longrightarrow |\frac{\partial f_5}{x_1}| = 0 < \infty$

$\frac{\partial f_5}{x_2} = 0 \quad \Longrightarrow |\frac{\partial f_5}{x_2}| = 0$

$\frac{\partial f_5}{x_3} = \theta \quad \Longrightarrow |\frac{\partial f_5}{x_3}| = |\theta| < \infty$

$\frac{\partial f_5}{x_4} = \nu \quad \Longrightarrow |\frac{\partial f_5}{x_4}| = |\nu| < \infty$

$\frac{\partial f_5}{x_5} = -\mu \quad \Longrightarrow |\frac{\partial f_5}{x_5}| = |-\mu| < \infty$

$\frac{\partial f_6}{x_1} = 0 \quad \Longrightarrow |\frac{\partial f_6}{x_1}| = 0 < \infty$

$\frac{\partial f_6}{x_2} = 0 \quad \Longrightarrow |\frac{\partial f_6}{x_2}| = 0$

$\frac{\partial f_6}{x_3} = \delta \quad \Longrightarrow |\frac{\partial f_6}{x_3}| = |\delta| < \infty$

$\frac{\partial f_6}{x_4} = \kappa \quad \Longrightarrow |\frac{\partial f_6}{x_4}| = |\kappa| < \infty$

$\frac{\partial f_6}{x_5} = 0 \quad \Longrightarrow |\frac{\partial f_6}{x_5}| = 0 < \infty$

Then, it can be shown that $\frac{\partial f_i}{\partial x_j}$ is continuous and $|\frac{\partial f_i}{\partial x_j}| < \infty$ for all $i = 1,2,...,6$ & $j = 1,2,...,5$

$\therefore$ Based on Derrick and Groosman theorem the system satisfies Lipchitz's condition.

Hence the model has a unique solution.

4.3.3 Equilibruim Points

Let $X = (S,E,I,Q,R)^T$

Set

$$\frac{DX(t)}{dt} = \vec{0} \qquad ,\vec{0} = (0,0,0,0,0)$$

we get

$$\frac{dI}{dt} = 0 \tag{4.15}$$

$$\Longrightarrow \gamma E(t) - (\sigma + \theta + \delta + \mu)I(t) = 0$$

$$\Longrightarrow E(t) = \frac{(\sigma + \theta + \delta + \mu)}{\gamma} I(t) \tag{4.16}$$

$$\frac{dQ}{dt} = 0$$

$$\Longrightarrow \sigma I(t) - (\nu + \kappa + \mu)Q(t) = 0$$

$$\Longrightarrow \sigma I(t) = (\nu + \kappa + \mu)Q(t)$$

$$\implies Q(t) = \frac{\sigma}{(\nu+\kappa+\mu)} I(t) \tag{4.17}$$

$$\frac{dR}{dt} = 0$$

$$\implies \theta I(t) + \nu Q(t) - \mu R(t) = 0$$

$$\implies \mu R(t) = \theta I(t) + \nu Q(t)$$

$$R(t) = \frac{\theta I(t) + \nu Q(t)}{\mu} \tag{4.18}$$

$$R(t) = \frac{\theta I(t) + \nu(\frac{\sigma}{(\nu+\kappa+\mu)}) I(t)}{\mu}$$

$$R(t) = \frac{(\theta + \nu(\frac{\sigma}{(\nu+\kappa+\mu)})) I(t)}{\mu} \tag{4.19}$$

$$\frac{dE}{dt} = 0$$

$$\implies \frac{\beta S(t) I(t)}{N} - (\gamma+\mu) E(t) = 0$$

Substitute eq (4.16) in above equation

$$\implies \frac{\beta S(t) I(t)}{N} - (\gamma+\mu)(\frac{(\sigma+\theta+\delta+\mu)}{\gamma}) I(t) = 0$$

$$\implies (\frac{\beta S(t)}{N} - (\gamma+\mu)\frac{(\sigma+\theta+\delta+\mu)}{\gamma}) I(t) = 0$$

$$\implies I = 0 \vee (\frac{\beta S(t)}{N} - (\gamma+\mu)\frac{(\sigma+\theta+\delta+\mu)}{\gamma}) = 0$$

If $I = 0$, then $E = 0, Q = 0, R = 0$

Also,

$$\frac{dN}{dt} = 0$$

$$\implies \Lambda - \mu N - \delta I - \kappa Q = 0$$

$$(\because I = 0 \implies Q = 0)$$

$$\implies \Lambda - \mu N = 0$$

$$\Lambda = \mu N$$

$$N = \frac{\Lambda}{\mu}$$

and

$$N = S + E + I + Q + R$$

$$\implies \frac{\Lambda}{\mu} = S + 0 + 0 + 0 + 0$$

$$\therefore S = \frac{\Lambda}{\mu}$$

If $I \neq 0$

$$(\frac{\beta S(t)}{N} - (\gamma + \mu)\frac{(\sigma + \theta + \delta + \mu)}{\gamma}) = 0$$

$$S = \frac{(\gamma + \mu)(\sigma + \theta + \delta + \mu)}{\beta\gamma} N$$

Now,

$$\frac{dN}{dt} = 0$$

$$\implies \Lambda - \mu N - \delta I - \kappa Q = 0$$

$$\implies \mu N = \Lambda - \delta I - \kappa Q$$

$$\implies N = \frac{\Lambda - \delta I - \kappa Q}{\mu}$$

$$\implies N = \frac{\Lambda - \delta I - \kappa(\frac{\sigma}{(\nu+\kappa+\mu)} I(t))}{\mu}$$

$$\implies N = \frac{\Lambda - (\delta - \kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))I(t)}{\mu}$$

Substitute S in $\frac{dS}{dt} = 0$

i.e, Substitute S in $\Lambda - \frac{\beta SI}{N} - \mu S = 0$

$$i.e, \Lambda - \frac{\beta I}{N}(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} N) - \mu(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} N) = 0$$

$$\implies \Lambda - \frac{\cancel{\beta} I}{\cancel{N}}(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\cancel{\beta}\gamma} \cancel{N}) - \mu(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} N) = 0$$

$$\implies \Lambda - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma})I - \mu(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} N) = 0$$

$$\implies \Lambda - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma})I - \mu(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} \frac{(\Lambda - (\delta - \kappa(\frac{\sigma}{(\nu+\kappa+\mu)})))I(t)}{\mu}) = 0$$

$$\implies \Lambda - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma})I - \mu(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma} \frac{(\Lambda - (\delta - \kappa(\frac{\sigma}{(\nu+\kappa+\mu)})))I(t)}{\cancel{\mu}}) = 0$$

$$\implies \Lambda - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma})I - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\Lambda-(\delta-\kappa(\frac{\sigma}{(\nu+\kappa+\mu)})))}{\beta\gamma}I(t)) = 0$$

$$\implies \Lambda - (\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma})I - \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda}{\beta\gamma} + \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))}{\beta\gamma} = 0$$

$$\implies I = \frac{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda}{\beta\gamma} - \Lambda}{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))}{\beta\gamma} - \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma}}$$

$$\implies I = \frac{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda-\beta\gamma\Lambda}{\beta\gamma}}{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))-(\gamma+\mu)(\sigma+\theta+\delta+\mu)\beta}{\beta\gamma}}$$

$$\implies I = \frac{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda-\beta\gamma\Lambda}{\cancel{\beta\gamma}}}{\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))-(\gamma+\mu)(\sigma+\theta+\delta+\mu)\beta}{\cancel{\beta\gamma}}}$$

$$\implies I = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda-\beta\gamma\Lambda}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))-(\gamma+\mu)(\sigma+\theta+\delta+\mu)\beta}$$

Denote I to be I^*

$$I^* = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda-\beta\gamma\Lambda}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))-(\gamma+\mu)(\sigma+\theta+\delta+\mu)\beta}$$

Now substitute I^* in eq (4.16), (4.17), (4.18) to get S^*, E^*, Q^*, R^*

$$S = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma}N$$

$$\implies S = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\beta\gamma}\left(\frac{\Lambda-(\delta-\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))I(t)}{\mu}\right.$$

$$\implies S = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda}{\beta\gamma\mu} - \left(\frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+\kappa(\frac{\sigma}{(\nu+\kappa+\mu)}))I^*}{\beta\gamma\mu}\right.$$

denote S as S^*

$$S^* = \frac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\Lambda-(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\delta+(\frac{\kappa\sigma}{(\nu+\kappa+\mu)}))I^*}{\beta\gamma\mu}$$

$$E^* = \frac{(\sigma+\theta+\delta+\mu)I^*}{\gamma}$$

$$Q^* = \frac{\sigma}{\nu+\kappa+\mu}I^*$$

$$R^* = \frac{\theta I^*+\nu Q^*}{\mu}$$

$\therefore$ The Disease free equilibrium point is

$$E^0 = (S^0,0,0,0,0) \qquad with \quad S^0 = \frac{\Lambda}{\mu}$$

and the Endemic Equilibrium point is

$$E^* = (S^*,E^*,I^*,Q^*,R^*) \qquad \text{with } S^*,E^*,I^*,Q^*,R^* \text{are given as above}$$

4.3.4 Basic Reproduction Number

One of important epidemiology metric is the basic reproduction number (R^0), which measures the contagiousness or transmibility of infectious agents[4, 1].
The Reproduction Number can be determined by the next generation matrix method. For that aim, consider

$$Z = (E, I, Q)^T$$

then we have

$$\frac{dZ}{dt} = \mathscr{F}(Z) - \mathscr{V}(Z)$$

where

$$\mathscr{F}(Z) = \begin{pmatrix} \frac{\beta SI}{N} \\ 0 \\ 0 \end{pmatrix} \qquad and \qquad \mathscr{V}(Z) = \begin{pmatrix} (\gamma+\mu)E \\ -\gamma E + (\sigma+\theta+\delta+\mu)I \\ -\sigma I + (\nu+\kappa+\mu)Q \end{pmatrix}$$

The Jacobian Matrices of $\mathscr{F}$ and $\mathscr{V}$ evaluated at E^0 are respectively given by F and V as follows:

$$F = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad and \qquad V = \begin{pmatrix} \gamma+\mu & 0 & 0 \\ -\gamma & \sigma+\theta+\delta+\mu & 0 \\ 0 & -\gamma & \nu+\kappa+\mu \end{pmatrix}$$

Now we use the next generation. The next generation matrix is given by FV^{-1}
First let us find V^{-1}

$$V^{-1} = \frac{1}{|V|}(adj(V))$$

$$|V| = \begin{vmatrix} \gamma+\mu & 0 & 0 \\ -\gamma & \sigma+\theta+\delta+\mu & 0 \\ 0 & -\gamma & \nu+\kappa+\mu \end{vmatrix}$$

$$\implies |V| = (\gamma+\mu)((\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)-0)-0+0$$

$$\implies |V| = (\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)$$

Now finding adjoint of V

$$Adj(V) = \begin{vmatrix} +|A_{11}| & -|A_{12}| & +|A_{13}| \\ -|A_{21}| & +|A_{22}| & -|A_{23}| \\ +|A_{31}| & -|A_{32}| & +|A_{33}| \end{vmatrix}^T$$

Let us calculate the minors

$$+|A_{11}| = (\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)$$

$$-|A_{12}| = -(-\gamma)(\nu+\kappa+\mu) = (\gamma)(\nu+\kappa+\mu)$$

$$+|A_{13}| = \sigma\delta$$

$$-|A_{21}| = 0$$

$$+|A_{22}| = (\gamma+\mu)(\nu+\kappa+\mu)$$

$$-|A_{23}| = -(-\sigma)(\gamma+\mu) = (\sigma)(\gamma+\mu)$$

$$+|A_{31}| = 0$$

$$-|A_{32}| = 0$$

$$+|A_{33}| = (\gamma+\mu)(\nu+\kappa+\mu)$$

therefore,

$$Adj(V) = \begin{vmatrix} (\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu) & (\gamma)(\nu+\kappa+\mu) & \sigma\delta \\ 0 & (\gamma+\mu)(\nu+\kappa+\mu) & (\sigma)(\gamma+\mu) \\ 0 & 0 & (\gamma+\mu)(\nu+\kappa+\mu) \end{vmatrix}^T$$

$$i.e, Adj(V) = \begin{vmatrix} (\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu) & 0 & 0 \\ (\gamma)(\nu+\kappa+\mu) & (\gamma+\mu)(\nu+\kappa+\mu) & 0 \\ \sigma\delta & (\sigma)(\gamma+\mu) & (\gamma+\mu)(\nu+\kappa+\mu) \end{vmatrix}$$

Now, $V^{-1} = \frac{1}{|V|}(AdjV)$

$$V^{-1} = \frac{1}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} \begin{pmatrix} (\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu) & 0 & 0 \\ (\gamma)(\nu+\kappa+\mu) & (\gamma+\mu)(\nu+\kappa+\mu) & 0 \\ \sigma\delta & (\sigma)(\gamma+\mu) & (\gamma+\mu)(\nu+\kappa+\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{0}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{0}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} \\ \frac{(\gamma)(\nu+\kappa+\mu)}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{(\gamma+\mu)(\nu+\kappa+\mu)}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{0}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} \\ \frac{\sigma\delta}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{(\sigma)(\gamma+\mu)}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{(\gamma+\mu)(\nu+\kappa+\mu)}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cancel{(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)}}{\cancel{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)}} & 0 & 0 \\ \frac{(\gamma)\cancel{(\nu+\kappa+\mu)}}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)\cancel{(\nu+\kappa+\mu)}} & \frac{\cancel{(\gamma+\mu)(\nu+\kappa+\mu)}}{\cancel{(\gamma+\mu)}(\sigma+\theta+\delta+\mu)\cancel{(\nu+\kappa+\mu)}} & 0 \\ \frac{\sigma\delta}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{(\sigma)\cancel{(\gamma+\mu)}}{\cancel{(\gamma+\mu)}(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{\cancel{(\gamma+\mu)(\nu+\kappa+\mu)}}{\cancel{(\gamma+\mu)}(\sigma+\theta+\delta+\mu)\cancel{(\nu+\kappa+\mu)}} \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\gamma+\mu)} & 0 & 0 \\ \frac{\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} & \frac{1}{(\sigma+\theta+\delta+\mu)} & 0 \\ \frac{\sigma\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} & \frac{\sigma}{(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{1}{(\nu+\kappa+\mu)} \end{pmatrix}$$

Now let us calculate FV^{-1}

$$FV^{-1} = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\gamma+\mu)} & 0 & 0 \\ \frac{\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} & \frac{1}{(\sigma+\theta+\delta+\mu)} & 0 \\ \frac{\sigma\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} & \frac{\sigma}{(\sigma+\theta+\delta+\mu)(\nu+\kappa+\mu)} & \frac{1}{(\nu+\kappa+\mu)} \end{pmatrix}$$

Now we use Matrix Multiplication and get

$$FV^{-1} = \begin{pmatrix} \frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} & \frac{\beta}{(\sigma+\theta+\delta+\mu)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To Compute R_0 we have to find the Spectral radius of the next generation matrix so we find the Eigen Values,

(Spectral Radius is the maximum of all the eigenvalues of next-generation matrix.)

$$|FV^{-1} - \lambda I| = 0$$

$$\implies \begin{vmatrix} \frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} - \lambda & \frac{\beta}{(\sigma+\theta+\delta+\mu)} & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\implies (\frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} - \lambda)(\lambda^2) = 0$$

$$\implies \lambda_1 = \lambda_2 = 0 \quad \& \quad \lambda_3 = (\frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)})$$

$$\therefore R_0 = \rho(FV^{-1}) = max(\lambda_1, \lambda_2, \lambda_3)$$

$$\implies R_0 = \lambda_3 = (\frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)})$$

Therefore

$$R_0 = \frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}$$

We observe that R_0 is inversely proportional to the infected sub-population θ and the quarantine rate σ.

The larger the value of θ or σ, the smaller the R_0 value.

We see that I^* can be written as

$$I^* = \frac{\Lambda(1-R_0)}{\gamma + \frac{\kappa\sigma}{\nu+\kappa+\mu} - \beta}$$

$$\text{If } R_0 > 1, \text{ then } \frac{\beta\gamma}{(\gamma+\mu)(\sigma+\theta+\delta+\mu)} > 1$$

$$\implies \beta > \tfrac{(\gamma+\mu)(\sigma+\theta+\delta+\mu)}{\gamma} > (\sigma+\theta+\delta+\mu) > \delta+\sigma > \delta+\sigma\tfrac{\kappa}{\nu+\kappa+\mu}$$

$$\therefore \beta > \delta + \sigma\frac{\kappa}{\nu+\kappa+\mu}$$

Thus the Endemic Equilibrium I^* exists if $R_0 > 1$ since I^* is positive when $R_0 > 1$. I^* does not exist if $R_0 < 1$ as I^* is negative which is not possible.

4.4 Stability Analysis

Local Stability of Disease Free Equilibrium Point[1]

Theorem 4.4.0.1. *The Disease Free Equilibrium Point (DFEP) E^0 is locally Asymptotically stable in domain Ω if $R_0 < 1$*

Proof. The Jacobian matrix of system ** evaluated at E^0 is given by

$$J(S,E,I,Q,R) = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial E} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial Q} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial E} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial Q} & \frac{\partial g}{\partial R} \\ \frac{\partial h}{\partial S} & \frac{\partial h}{\partial E} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial Q} & \frac{\partial h}{\partial R} \\ \frac{\partial i}{\partial S} & \frac{\partial i}{\partial E} & \frac{\partial i}{\partial I} & \frac{\partial i}{\partial Q} & \frac{\partial i}{\partial R} \\ \frac{\partial j}{\partial S} & \frac{\partial j}{\partial E} & \frac{\partial j}{\partial I} & \frac{\partial j}{\partial Q} & \frac{\partial j}{\partial R} \end{pmatrix}$$

□

where ,

$$f = \Lambda - \beta \frac{SI}{N} - \mu S$$

$$g = \beta \frac{SI}{N} - (\gamma + \mu)E$$

$$h = \gamma E - (\sigma + \theta + \delta + \mu)I$$

$$i = \sigma I - (\nu + \kappa + \mu)Q$$

$$j = \theta I + \nu Q - \mu R$$

$$J(S,E,I,Q,R) = \begin{pmatrix} -\frac{\beta I}{N} - \mu & 0 & -\frac{\beta S}{N} & 0 & 0 \\ \frac{\beta I}{N} & -(\gamma+\mu) & \frac{\beta S}{N} & 0 & 0 \\ 0 & \gamma & -(\sigma+\theta+\delta+\mu) & 0 & 0 \\ 0 & 0 & \sigma & -(\nu+\kappa+\mu) & 0 \\ 0 & 0 & \theta & \nu & -\mu \end{pmatrix}$$

At $E^0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$,

$$J(\tfrac{\Lambda}{\mu},0,0,0,0) = \begin{pmatrix} -\mu & 0 & -\frac{\beta(\frac{\Lambda}{\mu})}{N} & 0 & 0 \\ 0 & -(\gamma+\mu) & \frac{\beta(\frac{\Lambda}{\mu})}{N} & 0 & 0 \\ 0 & \gamma & -(\sigma+\theta+\delta+\mu) & 0 & 0 \\ 0 & 0 & \sigma & -(\nu+\kappa+\mu) & 0 \\ 0 & 0 & \theta & \nu & -\mu \end{pmatrix}$$

We know that at E^0 , $E(t) = I(t) = Q(t) = R(t) = 0$

$\therefore N = S+E+I+Q+R = \frac{\Lambda}{\mu} + 0+0+0+0 = \frac{\Lambda}{\mu}$

$$\therefore J(E^0) = \begin{pmatrix} -\mu & 0 & -\beta & 0 & 0 \\ 0 & -(\gamma+\mu) & \beta & 0 & 0 \\ 0 & \gamma & -(\sigma+\theta+\delta+\mu) & 0 & 0 \\ 0 & 0 & \sigma & -(\nu+\kappa+\mu) & 0 \\ 0 & 0 & \theta & \nu & -\mu \end{pmatrix}$$

Next we find the Eigen values using Block Matrix Technique

Block 1:

$$\begin{vmatrix} -(\nu+\kappa+\mu)-\lambda & 0 \\ \nu & -\mu-\lambda \end{vmatrix} = 0$$

$$\implies (-(\nu+\kappa+\mu)-\lambda)(-\mu-\lambda)-0=0$$

$$\implies (-(\nu+\kappa+\mu)-\lambda)(-\mu-\lambda)=0$$

$$\implies (-(\nu+\kappa+\mu)-\lambda)=0 \ , \ (-\mu-\lambda)=0$$

$$\implies \lambda_1 = -(\nu+\kappa+\mu) < 0,$$

$$\lambda_2 = -\mu < 0$$

Block 2:

$$-\mu-\lambda=0$$

$$\implies \lambda_3 = -\mu < 0$$

The remaining Eigen values λ_4 and λ_5 are the eigen values of the following matrix:

$$J_L = \begin{pmatrix} -(\gamma+\mu) & \beta \\ \gamma & -(\sigma+\theta+\mu+\delta) \end{pmatrix}$$

$$Trace(J_L) = -(\gamma+\mu)+(-(\sigma+\theta+\mu+\delta))$$

$$\implies Trace(J_L) = -\gamma-\mu-\sigma-\theta-\mu-\delta$$

$$\implies Trace(J_L) = -(\gamma+\sigma+\theta+\mu+\delta+2\mu) < 0$$

$$Det(J_L) = (\gamma+\mu)(\sigma+\theta+\mu+\delta) - \beta\gamma$$

$$Det(J_L) = 1 - \frac{\beta\delta}{(\gamma+\mu)(\sigma+\theta+\mu+\delta)} = 1 - R_0$$

which implies that $Det(J_L) > 0$ if $R_0 < 1$

Thus if $R_0 < 1$, then the real parts of λ_4 and λ_5 are negative. (i.e, $-\frac{1}{2}(**)$)

$\therefore \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are negative eigen values.

Consequently, the Disease Free Equilibrium Point E^0 is locally Asymptotically stable if $R_0 < 1$

Next we discuss the global asymptotic stability of Disease Free Equilibrium Point and Endemic Equilibrium Point.

Theorem 4.4.0.2. *The Disease Free Equilibrium Point E^0 is globally Asymptotically stable if $R_0 < 1$*

Proof: We prove this Theorem by following the method of Castilo-Chavez et.al published in his paper [7]

First we rewrite the model as follows:

$$\frac{dY}{dt} = F_1(Y,Z) = \begin{pmatrix} \Lambda - \beta\frac{SI}{N} - \mu S \\ \theta I + \nu Q - \mu R \end{pmatrix}$$

$$\frac{dZ}{dt} = F_2(Y,Z) = \begin{pmatrix} \beta\frac{SI}{N} - (\gamma+\mu)E \\ \gamma E - (\sigma+\theta+\delta+\mu)I \\ \sigma I - (\nu+\kappa+\mu)Q \end{pmatrix}, \quad F_2(Y,\vec{0}) = \vec{0}$$

where,

$Y = (S,R) \in \mathbb{R}^2_+$ indicates the number of non-infected individuals.

$Z = (E,I,Q) \in \mathbb{R}^3_+$ indicates the number of infected individuals.

let $E^0 = (Y^0,\vec{0})$ with $Y^0 = (\frac{\Lambda}{\mu},0)$.

E^0 is globally asymptotically Stable if $R_0 < 1$ and the following conditions hold:

(H1:) Y^0 is globally asymptotically stable for system $\frac{dY}{dt} = F_1(Y,\vec{0})$

(H2:) $F_2(Y,\vec{0}) = \vec{0}$ and $F_2(Y,Z) = CZ - \hat{F}_2(Y,Z)$ where $\hat{F}_2(Y,Z) \geq 0$ for any$(Y,Z) \in \Omega$ and C is the Jacobian Matrix $(\frac{dF_2}{dz})$ evaluated at E^0

We notice that

$$C = \begin{pmatrix} \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I} & \frac{\partial F_2}{\partial Q} \end{pmatrix}$$

$$i.e, C = \begin{pmatrix} -(\gamma+\mu) & \beta & 0 \\ \gamma & -(\sigma+\theta+\delta+\mu) & 0 \\ 0 & \sigma & -(\nu+\kappa+\mu) \end{pmatrix}$$

and $\hat{F}_2(Y,Z)$ is calculated as follows:

$$F_2(Y,Z) = CZ - \hat{F}_2(Y,Z)$$

$$\Longrightarrow \hat{F}_2(Y,Z) = CZ - F_2(Y,Z)$$

$$\hat{F}_2(Y,Z) = \begin{pmatrix} -(\gamma+\mu) & \beta & 0 \\ \gamma & -(\sigma+\theta+\delta+\mu) & 0 \\ 0 & \sigma & -(\nu+\kappa+\mu) \end{pmatrix} \begin{pmatrix} E \\ I \\ Q \end{pmatrix} - F_2(Y,Z)$$

$$= \begin{pmatrix} -(\gamma+\mu)E+\beta I+0 \\ \gamma E-(\sigma+\theta+\delta+\mu)I+0 \\ 0+\sigma I-(\nu+\kappa+\mu)Q \end{pmatrix} - \begin{pmatrix} \frac{\beta SI}{N}-(\gamma+\mu)E \\ \gamma E-(\sigma+\theta+\delta+\mu)I \\ \sigma I-(\nu+\kappa+\mu)Q \end{pmatrix}$$

$$= \begin{pmatrix} \beta I-\frac{\beta SI}{N} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\beta I}{N}(N-S) \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\beta I}{N}(\frac{\Lambda}{\mu}-S) \\ 0 \\ 0 \end{pmatrix}$$

It is clearly seen that elements of $\hat{F}_2(Y,Z)$ are non-negative, hence (H2) is satisfied.

Next we consider,

$$\frac{dY}{dt} = F_1(Y,\vec{0}) = \begin{pmatrix} \Lambda-\frac{\beta S(0)}{N}-\mu S \\ \theta(0)+\nu(0)-\mu R \end{pmatrix}$$

$$\begin{pmatrix} \frac{dS}{dt} \\ \frac{dR}{dt} \end{pmatrix} = \begin{pmatrix} \Lambda-\mu S \\ -\mu R \end{pmatrix}$$

$$i.e, \frac{dS}{dt} = \Lambda - \mu S \quad and \quad \frac{dR}{dt} = -\mu R$$

now solve using variable separable method,

solving the equation:

$$\frac{dS}{dt} = \Lambda - \mu S \tag{4.20}$$

$$\frac{dS}{\Lambda - \mu S} = dt$$

$$\int \frac{dS}{\Lambda - \mu S} = \int dt$$

$$-\frac{\Lambda - \mu S}{\mu} = t + c$$

$$-\ln(\Lambda - \mu S) = \mu t + \mu c$$

$$\ln(\Lambda - \mu S) = -\mu t - \mu c$$

Taking anti-log on both sides,

$$\Lambda - \mu S = c_1 e^{-\mu t}$$

$$-\mu S = -\Lambda + c_1 e^{-\mu t}$$

$$\mu S = \Lambda - c_1 e^{-\mu t}$$

$$S = \frac{\Lambda}{\mu} - \frac{c_1}{\mu} e^{-\mu t}$$

$$S(t) = \frac{\Lambda}{\mu} - c_2 e^{-\mu t}$$

At $t = 0$

$$S(0) = \frac{\Lambda}{\mu} - c_2$$

$$c_2 = \frac{\Lambda}{\mu} - S(0)$$

$$\implies S(t) = \frac{\Lambda}{\mu} - (\frac{\Lambda}{\mu} - S(0))e^{-\mu t}$$

$$\implies S(t) = \frac{\Lambda}{\mu} + (S(0) - \frac{\Lambda}{\mu})e^{-\mu t} \tag{4.21}$$

Now solving the Equation:

$$\frac{dR}{dt} = -\mu R \tag{4.22}$$

$$\frac{dR}{-\mu R} = dt$$

$$\int \frac{dR}{-\mu R} = \int dt$$

$$-\ln \frac{-\mu R}{\mu} = t + k$$

$$\ln \frac{-\mu R}{\mu} = -t - k$$

$$\ln(-\mu R) = -\mu t - \mu k$$

Taking Anti-log on both sides,

$$-\mu R = k_1 e^{-\mu t}$$

$$\mu R = -k_1 e^{-\mu t}$$

$$R = -\frac{k_1}{\mu} e^{-\mu t}$$

$$R(t) = K_2 e^{-\mu t}$$

At $t = 0$

$$R(0) = k_2$$

$$\implies R(t) = R(0)e^{-\mu t} \tag{4.23}$$

Hence we get,

$$Y(t) = \begin{pmatrix} S(t) \\ R(t) \end{pmatrix} = \begin{pmatrix} \frac{\Lambda}{\mu} + (S(0) - \frac{\Lambda}{\mu})e^{-\mu t} \\ R(0)e^{-\mu t} \end{pmatrix} \tag{4.24}$$

It is observed that as $t \longrightarrow \infty$,

$$S(t) \longrightarrow \frac{\Lambda}{\mu} \quad \& \quad R(t) \longrightarrow 0$$

This shows that Y^0 is globally Asymptotically Stable. Hence (H1) is satisfied.

Therefore the Disease Free Equilibrium Point E^0 is globally Asymptotically Stable in domain Ω.

4.5 Conclusion

The SEIQRD model describing the spread of COVID-19 disease using the standard incidence rate has been developed in this paper. The model consists of susceptible (S), exposed (E), infected (I), quarantined (Q), recovered (R), and death caused by the COVID-19 disease(D) sub-populations. The existence, uniqueness, positivity, and boundedness of solution have been proven, showing that the proposed model is biologically feasible.

The model has two equilibrium points, namely the disease-free equilibrium point and the endemic equilibrium point. Using the next generation matrix method, we have

determined the basic reproduction number. The disease-free equilibrium point always exists and it is locally and globally asymptotically stable if the basic reproduction number is less than unity. If the endemic equilibrium point exists, i.e. when the basic reproduction number is greater than unity, then it is always globally asymptotically stable. Furthermore, from the basic reproduction number formula and numerical simulation results, the basic reproduction number can be reduced by increasing the rate of recovery or quarantine of the infected sub-population. This shows that COVID-19 disease can be controlled by treating infected individuals or by quarantining them.

Chapter 5

CONCLUSION

5.1 Conclusion

In this paper, we first formulated the model. Compartments were included to modify the existing models. This modifications made the model more accurate in its predictions and results. Next we Analyzed the models . The models were found to be biologically meaningful and mathematically well posed since we were able to prove the positivity, boundedness and existence of the solutions. Further, Equilibrium points were calculated to study the behaviour or stability of the system of model equations. Two equilibrium points each were found in chapter 2,3 and 4. The two equilibrium points are the DFEP and EE points. We find equilibrium points to study the stability of the model. We find the local stability and global stability of the model by using different methods like Lyapunpov function, Castillo -Chavez [**Chavez**] method and so on. Lastly the Basic

Reproduction Number is calculated to check on the disease transmission rate.

5.2 Further Scope

By taking Data from the Hospitals and related institutions, we can numerically simulate and find accurate results according to the data used. this might be helpful to make proper predictions to control the outbreak in a particular area.

The Models discussed in this paper can be further modified by adding more compartments like Symptomatic, Asymptomatic, Hospitalized, Vaccinated and so on. This will help us to formulate a better accurate model.

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