Permutable Subgroups

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by

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DECLARATION BY STUDENT

I hereby declare that the data presented in this Dissertation report entitled, "Permutable Subgroups" is based on the results of investigations carried out by me in the Mathematics Discipline at the School of Physical & Applied Sciences, Goa University under the Supervision of Dr. Manvendra Tamba and the same has not been submitted elsewhere for the award of a degree or diploma by me. Further, I understand that Goa University will not be responsible for the correctness of observations / experimental or other findings given the dissertation.
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COMPLETION CERTIFICATE

This is to certify that the dissertation report " Permutable Subgroups" is a bonafide work carried out by Ms. Veda Kush Naik under my supervision in partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics in the Discipline Mathematics at the School of Physical & Applied Sciences, Goa University.

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PREFACE

This Project Report has been prepared in partial fulfilment of the requirement for the Subject: MAT - 651 Discipline Specific Dissertation of the programme M.Sc. in Mathematics in the academic year 2023-2024.

The topic assigned for the research report is: " Permutable Subgroups." This survey is divided into four chapters. Each chapter has its own relevance and importance. The chapters are divided and defined in a logical, systematic and scientific manner to cover every nook and corner of the topic.

FIRST CHAPTER :

This chapter consists of all the preliminary results which will be used in this dissertion.

SECOND CHAPTER :

This chapter deals with the paper [8]. The goals of this chapter are twofold. One is to look at the behavior of the collections of permutable subgroups and S-permutable subgroups under the intersection map into a fixed subgroup of a group. The other is to locally analyze the intersection map in connection with T -, PT -, and PST -groups.

THIRD CHAPTER:

This chapter deals with the paper [18]. The main aims of this chapter is to classify the family of all nearly S-permutable subgroups for certain groups and study the direct product of their subgroups. Moreover, we prove that the direct product of certain nearly S-permutable subgroups is necessary nearly S-permutable.

FOURTH CHAPTER:

In this chapter we conjecture that for $G = S_n$ where n $\geq 4, |$ P(H) $| / |$ G $|$

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ABSTRACT

This proposal revolves around Some topics of Group Theory, specifically Permutable subgroups. Permutable subgroups in group theory facilitate the study of group structures, aiding in the analysis of relationships between subgroups and contributing to a deeper understanding of overall group behavior. These papers give an introduction to T-Groups, Subnormal subgroups, Intersection map, Finite groups, Permutable subgroups, Sylow subgroups, Normal subgroups, S-permutable subgroups, Nearly S-permutable subgroups, and Direct product of subgroups.

A subgroup H is said to be S-permutable in G if it permutes with all Sylow subgroups of G. A subgroup H of G is called nearly S-permutable in G if for every prime p such that gcd(p, $|H|$) = 1 and for every subgroup K of G containing H, the normalizer N_K(H) contains some Sylow p-subgroup of K. The main aims of this article is to classify the family of all nearly S-permutable subgroups for certain groups and study the direct product of their subgroups. Moreover, we prove that the direct product of certain nearly S-permutable subgroups is necessary nearly S-permutable.

Papers referred for the dissertation are[8] "The intersection map of subgroups and certain classes of finite groups" by James C. Beidleman (University of Kentucky) and Matthew F. Ragland (Auburn University Montgomery) and [18] "On the direct product of Nearly S-permutable subgroups" by Bilal N. Al-Hasanat, Awni F. Al-Dababseh, Baheej R. Al-Shuraifeen (Al Hussein Bin Talal University) and Khaled A. Al-Sharo (Al al-Bayt University).

Keywords: T-Groups; Subnormal; Intersection map; Finite groups; Permutable Sylow subgroups; normal subgroup; subnormal subgroup; S-permutable subgroup; nearly S-permutable subgroup; direct product of sub-groups

Table of contents

Notations and Abbreviations

$SP(G)$	Sylow Permutable subgroup of G
$NSP(G)$	Nearly Sylow Permutable sybgroup of G
$\#SP(G)$	Number of Sylow Permutable subgroup of G
$\#NSP(G)$	Number of Nearly Sylow Permutable subgroup of G
H $\leq$ G	H is a subgroup of G
H $\lhd$ G	H is a normal subgroup of G
$\mid$ G : H $\mid$	Index of subgroup H in the group G
$\mathscr{C}G$	Center of G
S_n	Symmetric group of degree n
D_n	Dihedral group of order n

Chapter 1

Preliminaries

In this section we present certain facts and results that are needed to prove some Theorems.The following lemma is a well known result of Kegel's.

Lemma 1.0.1. [8] Let G be a group. If H and K are S-permutable in G, then $H \cap K$ is S-permutable in G.

Lemma 1.0.2. [8] Let L be the nilpotent residual of the solvable group G and let H be any subgroup of G.

1. If G is a PST -group, then LH is S-permutable in G.
2. If G is a PT -group, then LH is permutable in G.

Proof: Let G be a PT -group (PST -group). Using Theorem 2.1.1, we see G/L is an Iwasawa group (nilpotent group). Thus LH/L is permutable (S-permutable) in G/L. Hence LH is permutable (S-permutable) in G.
It is worth mentioning that if G is a T -group with nilpotent residual L, then L H $\lhd$G for any subgroup H of G. This result is not needed in our work, however, it is used in the proof of Theorem 2.1.5.

Definition 1.0.3. A group G is called an SC-group if all its chief factors are simple.

SC-groups were introduced and classified by Robinson [17].

Lemma 1.0.4. [8] Let G be a group such that every normal subgroup of G is permutable sensitive in G. Then G is an SC-group.

Proof: Let M be a minimal normal subgroup of G. Then every normal subgroup of G/M is permutable sensitive in G/M. By induction, G/M is an SC-group, and thus it is enough to show M is a simple chief factor.

First assume that M is an elementary abelian p-group for some prime p and let P be a Sylow p-subgroup of G. Let x be a non identity element of M ∩ Z(P). Then $\langle x\rangle$ is a permutable subgroup of M so that there is a permutable subgroup Y of G such that Y ∩ M = $\langle x\rangle$. By Lemma 1.0.1, $\langle x\rangle$ is S-permutable in G. Let Q be a Sylow q-subgroup of G, q $\neq$ p. Then $\langle x\rangle$ Q = Q$\langle x\rangle$ and $\langle x\rangle$ is a subnormal Sylow p-subgroup of Q$\langle x\rangle$. Thus $\langle x\rangle$ is normal in Q$\langle x\rangle$ and O^p(G) normalizes $\langle x\rangle$. Hence $\langle x\rangle$ is normal in G and we deduce M is simple.

Now assume that M is non abelian. Then M = $M_1 \times M_2 \times \cdots \times M_t$ where each M_i is a non abelian simple group. Since M_1 is permutable in M there is a permutable subgroup Y of G such that Y ∩ M = M_1. By Lemma 1.0.1, M_1 is S-permutable in G. Let R be a Sylow r-subgroup of G for some prime r. Then M_1R = RM_1 and $M_1^R = M_1(M_1^R \cap R)$. Thus M_1^R/M_1 is an r-group. But M_1^R is a direct product of non abelian simple groups and so $M_1^R = M_1$. Thus R normalizes M_1 and we deduce M_1 is normal in G. So M = M_1 the proof is complete.

The following concepts and results of Robinson [17] are needed to prove part (2) of Theorem 2.2.3.

Lemma 1.0.5. [17] Let D be the solvable residual of a group G. G is an SC-group if and only if G/D is supersolvable, D/Z(D) is a direct product of G-invariant simple groups, and Z(D) is supersolvably embedded in G (that is, there is a G-admissible series in Z(D) with cyclic factors).

Definition 1.0.6. Let p be a prime.

1. A group G satisfies condition N_p if, for all solvable normal subgroups N of G, the p'-elements of G induce power automorphisms on O_p(G/N).

2. A group G satisfies condition P_p if, for all solvable normal subgroups N of G, each subgroup of O_p(G/N) is permutable in a Sylow p-subgroup of G/N.

Theorem 1.0.7. [17] Let D be the solvable residual of a group G. G is a PT-group if and only if

1. G/D is a solvable PT -group;
2. D/Z(D) = U_1/Z(D)×···×U_k/Z(D) where U_i is normal in G and U_i /Z(D) is simple;
3. if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ where $1 \leq r \leq k$, then G/$U'_{i1}U'_{i2} \ldots U'_{ir}$ satisfies N_p for all $p \in \pi(Z(D))$ and P_p for all $p \in \pi$(D).

Proof: Only the sufficiency of the three conditions is in doubt. So assume that G satisfies the conditions but is not a PT-group, and that of all such groups G has smallest order. Let H be a subnormal subgroup of G which is not permutable.

Case (a): H is insoluble. Then (H ∩ D)Z/Z is non-trivial and subnormal in D/Z. By (2) it must contain some U_i/Z, and therefore $H' \geq (H \cap D)Z' \geq U'_i$. Passing to G/ U'_i, which inherits the hypotheses on G, we conclude that H/U'_i is permutable in G/U'_i, that is, H is permutable in G, a contradiction.

Case (b): H is soluble. Here H is contained in the soluble radical S of G. Put K = Y_∞(S), the limit of the lower central series of S. We claim that $H \cap K \lhd G$. Since G/D is a soluble PT-group, KD/D is abelian by Zacher's theorem. Also $K \cap D \leq Z(K)$ since [D, S] = [D', S] ≤ [D, S, D] = 1. Hence K is nilpotent, and it is enough to show that H ∩K_p⊲ G for all primes p. If $K_p \leq$ Z := Z(D), then [K_p, S] = 1 and [K, S] $\neq$ K. Hence $K_p \nleq Z$ and so $K_p \nleq D$. We can assume that p ∈ π(Z). For otherwise K_p∩ D = 1 and $K_p \simeq K_pD/D \leq y_\infty$(G/D); therefore elements of G induce power automorphisms in K_p and $H \cap K_P$⊲ G.

Since $1 \neq K_pD/D \leq y_\infty$(G/D), which is a Hall subgroup of G/D, we see that p cannot divide $|G/D : y_\infty(G/D)|$. NOW consider G/$C_G(K_P)$; by N_p the p'-elements in this group form a normal subgroup V/$C_G(K_P)$ and G/ V is a p-group. Therefore $y_\infty(G/D) \leq$ V/D and consequently V = G, so that H ∩K_p⊲ G, as required.

Now pass to the group G/H ∩ K and use minimality of order to conclude that H ∩ K = 1. Hence H is nilpotent, and obviously we can suppose it is a p -group. It is enough to show that H $\langle g \rangle = \langle g \rangle$ H where g is either a p-element or a p'-element. Let g be a p'-element. If p ∈ π(Z), the condition N_p implies that H^8 =H. If on the other hand p ∉ π(Z), then H^G∩ Z = 1, so that H^G∩ D = 1 and $H^G \simeq H^GD/D$. Since g induces power automorphisms in O_p (G/D), we again obtain H = H^8.

Finally, suppose that g is a p-element. Let P be a Sylow p-subgroup containing g; then of course H $\leq$ P. If p $\in \pi$(D), we have H $\langle g \rangle = \langle g \rangle$ H by condition P_p. If p $\notin \pi$(D) on the other hand, P $\cap$ D = 1 and P $\simeq$ PD/D, showing that P is modular and H $\langle g \rangle = \langle g \rangle$ H.

Corollary 1.0.8. *[8] X_p^* and C_p^* are subgroup-closed classes.*

Lemma 1.0.9. [7] A group G satisfies X_p if and only if a Sylow p-subgroup P of G is modular and the p'-elements of $N_G(P)$ induce power automorphisms in P.

Proof: Assume that G satisfies X_p. Then a Sylow p-subgroup P of G is clearly modular. Let a $\in$ P and let x be a p'-element of $N_G(P)$. Then $a^{\langle x \rangle}$ = $a^{\langle x \rangle} \cap \langle a \rangle \langle x \rangle = \langle x \rangle$ since P $\cap \langle x \rangle$ = 1. Thus x induces a power automorphism in P. Conversely, these conditions clearly imply that G satisfies X_p.

Lemma 1.0.10. [7] Let N be a Normal Hall subgroup of a G and assume that the following hold:

1. G/N is a PT-group.
2. Every subnormal subgroup of N is normal in G.

Then G is a PT-group.

Proof: Let H be a subnormal subgroup of G. We show that H is permutable. By (2) H $\cap$ N is normal in G and G/H $\cap$ N satisfies (1) and (2). By induction on $|G|$ we can assume that H $\cap$ N = 1. By the Schur–Zassenhaus theorem N has a complement M in G and all complements are conjugate to M. Since ($|H|$, $|N|$) = 1 and H is subnormal, H $\leq$ M. Also note that [H, N] = 1. It is enough to show that H permutes with any subgroup T of G of order p^n where p is a prime and n is a positive integer. If p divides $|N|$, then T $\leq$ N and HT = TH.

Assume that (p, $|N|$) = 1. Then T is contained in some conjugate of M, say M^x, where x $\in$ G. By (1), M^x is a PT-group and H $\leq M^x$, so that TH = HT and the result follows.

Theorem 1.0.11. [1] A group G is a PST group if and only if G is a PST_p group for every prime.

Proof: Clearly every PST-group is a PST_p-group for every prime p. In order to prove the converse statement, let G be a non-PST-group which satisfies PST_p for every prime p.
There exists a subnormal subgroup H of G such that H does not permute with some Sylow subgroup of G. Let us assume H to be chosen of minimal order. It is not difficult to see that H has exactly one maximal normal subgroup; it there would exist two maximal normal subgroups H_1 and H_2 of H, then the minimal choice of H would imply that both H_1 and H_2 would permute with every Sylow subgroup of G. This is not possible since H = H_1H_2.
Let H_0 be this unique maximal normal subgroup of H. Clearly $|H : H_0|$ = p for some prime p. Therefore $O^{p'}(H) = H$ and H is p'-perfect. Therefore H permutes with every Hall p' -subgroup G_p of G since G satisfies the property PST_p.

Theorem 1.0.23:Let p be a prime. A group G is a U_p^*-group if and only if G is a PST_p-group.
Let G be a group and p a prime. We say that G is a U_p^*-group if it is p-supersoluble and all its p-chief factors form a single isomorphism class of G-modules.
Moreover, we can say by applying Theorem 1.0.23 that G is a U_p^*-group.

Lemma 1.0.24: Let p be a prime, and let G be a p-super soluble group. If $O_{p'}$ (G) = 1, then the derived subgroup G of G is a p-group. In particular, G has a unique Sylow p-subgroup.

If $O_{p'}$ (G) = 1, we would have as a consequence of Lemma 1.0.24 that G has a unique Sylow p-subgroup, P, say. Note that there must exist a Hall p'-subgroup $G_{p'}$ of G such that H$G_{p'}$ = G, since otherwise H$G_{p'}$ would be a PST-group by induction (note that the class of PST_p-groups is subgroup-closed). Therefore H would permute with every Sylow r-subgroup of G for r $\neq$ p as well as with P, a contradiction.
Consequently P must be contained in H. Arguing again by induction on $|G|$, it is clear that G/P is a PST-group, and therefore H permutes with TP for every Sylow t-subgroup T of G, where t $\neq$ p.
Since P $\leq$ H, we can conclude in fact that H permutes with every such T. But H permutes with P as well, and thus we reach a contradiction.

Therefore we can assume that $O_{p'}$ (G) $\neq$ 1. Using a similar argument to that used above, we can easily obtain that $Core_G$ (H)is trivial. Let then N be a minimal normal subgroup of G contained in $O_{p'}$(G). The quotient group G/N is a PST-group, by induction.
Therefore it is clear that HN permutes with every Sylow subgroup of G.

Moreover, there must exist a prime r and a Sylow r-subgroup R of G such that (HN)R = G, since otherwise we could see by using induction that H would permute with every Sylow subgroup of G. Note that the index $|G : HN|$ is a power of r. Since H is not a normal subgroup of G, it is easy to see that HN cannot be the whole group G. Clearly, then, for every prime q $\neq$ r and every Sylow q-subgroup Q of G, (HN)Q must be a proper subgroup of G. Consequently H permutes with every Sylow q-subgroup of G, for each q $\neq$ r.
We shall see next that r = p.
If r $\neq$ p, then H permutes with every Sylow p-subgroup of G. Since (HN)R= G and N is a p-group, clearly every Sylow subgroup of H is a Sylow subgroup of G. Let P be a Sylow p-subgroup of H. For every g $\in$ G, we have that H$P_g \leq$ G, and therefore P_g is in fact contained in H. The normal closure PG is thus contained in H, but since $Core_G$ (H) = 1, necessarily P= 1. Then G is a p'-group, and consequently $O^{p'}$(H) = 1 and thus H = 1, a contradiction.
The previous discussion allows us to state that r = p and thus we can write G = (HN)P for any Sylow p-subgroup P of G. Let q be the prime dividing $|N|$ (note that q $\neq$ p) and let $G_{\{p,q\}'}$ be any Hall $\{p,q\}'$-subgroup of G. Clearly, (HN)$G_{\{p,q\}'}$ is a proper subgroup of G, since the index $|G : HN|$ is a power of p. We have then that H permutes with $G_{\{p,q\}'}$. Since every Hall $\{p,q\}'$-subgroup of H is again a Hall $\{p,q\}'$ -subgroup of G, using an argument similar to that above we would conclude that H = 1 if we assumed that $G_{\{p,q\}'} \neq 1$. As a result of these facts, we have that G is a $\{p,q\}$-group.
Let us now consider the nilpotent group H/H^N, where H^N denotes the nilpotent residual of H. The normality of its Sylow p-subgroup implies that, given a Sylow p-subgroup H_p of H, we have that H_pH^N is a normal subgroup of H. Clearly, H/H_pH^N is a q-group, and hence O^q(H) $\leq H_pH^N \leq$ H. But O^q(H)= $O^{p'}$(H)= H, and thus H= H_pH^N.
On the other hand, we have that G is p-supersoluble, and therefore G is a p-nilpotent group. Clearly H' and H^N are p-nilpotent as well. Let H_q be a Sylow q-subgroup of H contained in H^N. Thus H_q is a normal subgroup of H^N. Since H_q is a Sylow q-subgroup of H^N, it is not only normal but also characteristic in H^N. Consequently H_q is a normal subgroup of H.
It follows that G = $O_{p'p}$(G)P. Note finally that H_q is a subnormal subgroup of G. The subgroup $O_{p'p}$(G) can be seen as $O_{p'p}$(G)= $O_{p'}$(G)B, where B is a Sylow p-subgroup of $O_{p'p}$(G), and thus G= $O_{p'}$(G)P. That fact forces $O_{p'}$(G) to be a Sylow q-subgroup of G, and hence G is a p-nilpotent group. We will denote $O_{p'}$(G) as Q. We recall that N is any minimal normal

subgroup of G contained in Q. We have Q = H_qN. The subnormality of H_q in G implies that N normalizes H_q, and thus H_q is a normal subgroup of Q.

We can assume that the Frattini subgroup Φ(Q) of Q is trivial. Certainly, if Φ(Q) $\neq$ 1, we could choose N to be contained in Φ(Q) , and therefore Q would be equal to H_q. We would have in such a case that H_q permutes with P as well as H_p does, and hence the whole H would permute with P,a contradiction. Therefore Q is an abelian group.

We are now in a good position to complete the proof; we can write G = [Q]P, where Q can be seen as a completely reducible P-module, giving the expression G = $[Q_1 \bigoplus Q_2 \bigoplus \ldots \bigoplus Q_r]$P. But since G is a PST_q-group, it is also a U_q^*-group and then the q-chief factors of G are G-isomorphic.

We can conclude that P normalizes each subgroup of Q, and in particular that P normalizes H_q. Therefore both H_p and H_q permute with P, a final contradiction.

Theorem 1.0.12. [5] A p-soluble group is a PST_p group if and only if it satisfies Y_p.

Proof: Assume that G satisfies p. We prove that G is a PST_p-group by induction on $|G|$. Denote $O_{p'}$(G) by A and suppose that A $\neq$ 1. Let H be a p'-perfect subnormal subgroup of G and let B be a Hall p'-subgroup of G. Then A $\leq$ B and B/A is a Hall p'-subgroup of G/A

Since G/A is a PST_p-group, it follows that HA/A permutes with B/A. Consequently H permutes with B and hence G is a PST_p-group. Therefore we may assume that A= $O_{p'}$(G)= 1.

Let N be a minimal normal subgroup of G. Then N is a p-group because G is p-soluble.

If N_0 is a subgroup of N, then N_0 is S-permutable in N_G(N) = G. This means that if Q is a Sylow q-subgroup of G for q $\neq$ p, then N_0 is a Sylow p-subgroup of N_0Q and so Q normalizes N_0.

Therefore O^p(G) normalizes every subgroup of N. Let P be a Sylow p-subgroup of G and let N_1 be a minimal normal subgroup of P contained in N. Then PO^p(G) = G normalizes N1 and so N_1 = N. This means that N is cyclic of order p.

Lemma 1.0.25:Let G be a group.

1. If G has property y_p and A is a normal p-subgroup of G, then G/A has property y_p.
2. If G has property y_p and N is a normal p'-subgroup of G, then G/N has property y_p.

By Lemma 1.0.25, we know that G/N has p. Therefore G/N is a PSTp-group by induction.

Corollary 1.0.13. *[7] Let G be a finite soluble PT-group. Then if H is a subgroup of G, then H is a PT-group.*

Theorem 1.0.14. [7] Let p be the smallest prime divisor of the order of G. Then G has X_p if and only if G is p-nilpotent and Sylow p-subgroups of G are modular.

Theorem 1.0.15. [21] A soluble group G is a PT-group if and only if it has an abelian normal Hall subgroup L of odd order such that G/L is a nilpotent modular group and elements of G induce power automorphisms in L.

Theorem 1.0.16. [15]If G is a finite group, the following are equivalent statements.

1. G is soluble T-group.
2. G is a $T^{'}$-group.
3. G satisfies C_p for all p

Lemma 1.0.17. [14] Let G be a finite p-group. Then G has modular subgroup lattice if and only if each of its sections of order p^3 does. Therefore if G is not an M-group, then there exist subgroups H, K of G with K≤H such that H/K is dihedral of order and or nonabelian of order p^3 and exponent p for p $>$ 2.

Theorem 1.0.18. [16] Let G be a periodic soluble group and let H be S-permutable in G. If H satisfies min-p for all primes p, then H is serial in G.

Theorem 1.0.19. [6] G is a T^{*}-group if and only if G is a PST -group.

Theorem 1.0.20. [5]Let G be a group. Suppose that p is a prime number and that H is an S-permutable p-subgroup of G. If the Sylow p-subgroups of G are Dedekind, then H is normal in G.

Proof: Let A be a subgroup of G and denote T = ,$H\rangle$. Since H is S-permutable in T, H is a subnormal subgroup of T and H is contained in O_pT, which is contained in every Sylow p-subgroup P of T. Therefore T = $\langle H,A\rangle \leq \langle O_pT,A\rangle$= O_p(T)A $\leq$ T. Let A_q be a Sylow q-subgroup of A for a prime q $\neq$ p, and let G_q be a Sylow q-subgroup of G containing A_q. We have that A_q is a Sylow q-subgroup of T, and $A_q = G_q \cap$ T because $A_q \leq G_q \cap$ T. Hence

HA_q = H(G_q ∩ T) = HG_q ∩ T is a subgroup of T. Moreover, $O_p(T)$ ∩ HA_q = H. Therefore H is normalized by A_q. On the other hand, since P is Dedekind, we have that H is normalized by a Sylow p-subgroup A_q of A. Therefore H is normalized by all Sylow subgroups of A. In particular, H is normalized by A. This implies that H is a normal subgroup of G.

Theorem 1.0.21. [5]Let G be a group. Suppose that p is a prime number and that H is an S-permutable p-subgroup of G. If the Sylow p-subgroups of G are modular, then H is permutable in G.

Proof: Let A be a subgroup of G and denote T = $,H\rangle$. Since H is S-permutable in T, H is a subnormal subgroup of T and H is contained in O_pT, which is contained in every Sylow p-subgroup P of T. Therefore T = $\langle H,A\rangle \leq \langle O_pT,A\rangle$= O_p(T)A $\leq$ T. Let A_q be a Sylow q-subgroup of A for a prime q $\neq$ p, and let G_q be a Sylow q-subgroup of G containing A_q. We have that A_q is a Sylow q-subgroup of T, and $A_q = G_q$ ∩ T because $A_q \leq G_q$ ∩ T. Hence HA_q = H(G_q ∩ T) = HG_q ∩ T is a subgroup of T. Moreover, $O_p(T)$ ∩ HA_q = H. Therefore H is normalized by A_q. On the other hand, since P is modular, we have that H permutes with a Sylow p-subgroup A_q of A. Therefore H permutes with all Sylow subgroups of A. In particular, H permutes with A. This implies that H is a permutable subgroup of G.

Theorem 1.0.22. [12]Let G be a group of order p^n and let X be a finite G-set.
Then $|X| = |X_G|$ (mod p)

Where $X_G = \{x \in X|$ xg=x $for\ all\ g \in G\}$
Proof: In the notation of the equation

$$|X| = |X_G| + \sum_{i=s+1}^{r} |x_iG| \tag{1.1}$$

Theorem 1.0.26 : [12]X be a G-set and let $x \in X$. Then $|xG| = (G : G_x)$. we know that $|x_iG| = (G : G_x)$ by Theorem 1.0.26. But $(G : G.)$ divides (G), and consequently p divides thus divides $|x_iG|$ for s + 1 $\leq$ i $\leq$ r. Equation 1.1 then shows $|X| - |X_G|$ is divisible by p, so $|X| \equiv |X_G| \pmod{p}$

Theorem 1.0.27. (Gaschiitz, Schenkman, Carter) Let G be a finite soluble group and denote by L the smallest term of the lower central series of G. If N is any system normalizer in G, then G = NL. If in addition L is abelian, then also N $\cap$ L = 1 and N is a complement of L .

Proof: Form a principal series of G through L by refining 1 $\lhd L \lhd$ G.
Theorem 1.0.29:(P. Hall). If N is a system normalizer of a finite soluble group G, then N covers the central principal factors and avoids the noncentral principal factors of G. Since G /L is nilpotent, principal factors "above" L wil l be central and hence are covered by N (by Theorem 1.0.29). Therefore G =NL. Now assume that L is an abelian. Then it is sufficient to prove that no principal factor of G "below" L is central: for the system normalizer N will avoid such factors and N $\cap$ L will be trivial. We shall accomplish this by induction on $|L| >$1. By the induction hypothesis, it suffices to show that L $\cap \mathscr{C}$ G = 1.
If C = C_G (L), then L $\leq$ C $<$ G since L = [L , G] $\neq$ 1. Hence G/C is nilpotent; we now choose a nontrivial element gC from the center of G/C,noting that [L , [g , G]] = 1. We deduce from this relation and one of the fundamental commutator identities that if a $\in$ L and x $\in$ G, then

$$[a,g]^x = [a^x, g^x] = [a^x, [x, g^{-1}]g] = [a^x, g]$$

Hence the mapping $\theta : L \rightarrow$ L defined by a^{θ}= [a,g] is a nonzero G-endomorphism of L, and $Ker\theta \lhd$ G. Since L $\cap \mathscr{C}$ G $\leq Ker\theta$, we may assume $Ker\theta \neq 1$, so that (L/$Ker\theta$) $\cap \mathscr{C}$ ($Ker\theta$/ G) is trivial by induction. Also L/$Ker\theta \approx^G L^{\theta}$, from which it follows that $L^{\theta} \cap \mathscr{C}$ G = 1. Now 1 $\neq L^{\theta} \lhd$ G and (L/L^{θ}) $\cap \mathscr{C}$ (G/L^{θ}) is trivial by induction. Therefore, L $\cap \mathscr{C}$ G = 1, as required.

Theorem 1.0.28.(Carter) Let G be a finite soluble group of nilpotent length at most 2.Then the system normalizers coincide with the Carter subgroups of G.

Proof: By hypothesis there exists a normal nilpotent subgroup M such that G/M is nilpotent. If A is a system normalizer of G, then A covers every central principal factor by Theorem 1.0.29 we have G = NM . Denote by $p_1, \ldots, p_k$ the distinct prime divisors of |G|. Let N_i and M_i be the unique Sylow p_i' subgroups of the nilpotent groups N and M respectively. Then $Q_i = M_i N_i$ is a Hal l p-subgroup of G since G =NM. Thus $\{Q_1, \ldots, Q_k\}$ is a Sylow system of G. Now $M_i \lhd$ G; hence N normalizes Q_i and N $\leq N_G(Q_i)$ for all i. Since all system normalizers have the same order, being conjugate, it follows that N $=\bigcap_{i=1}^{k} N_G(Qi)$. If g normalizes N, it also normalizes N_i and hence Q_i. Thus g $\in$ N and N is self-normalizing, which means that N is a Carter subgroup. Since system normalizers and Carter subgroups are conjugate, the theorem follows.

Chapter 2

The Intersection Map Of Subgroups And Certain Classes Of Finite Groups

2.1 Introduction

All groups considered are finite. All unexplained notation and terminology is standard and can be found in [20]. A group G is a T -group if normality is transitive; that is if $H \lhd K \lhd G$, then $H \lhd G$. A subgroup H of G is said to be permutable in G if H K = K H for all subgroups K of G. Ore [13] proved that a permutable subgroup is subnormal. A group G is a PT -group if permutability is transitive; that is, if H is permutable in K and K is permutable in G, then H is permutable in G. By Ore's result, G is a PT -group if and only if every subnormal subgroup of G is permutable in G. A subgroup H of G is called Sylow-permutable in G, or S-permutable, if H permutes with every Sylow subgroup of G. Kegel [11] showed that an S-permutable subgroup is subnormal. A group G is said to be a PST -group if S-permutability is a transitive relation in G. Applying Kegel's result, a group G is a PST -group if and only if every subnormal subgroup of G is S-permutable in G. The basic structures of solvable T -, PT -, and PST -groups were established by Gaschütz [10], Zacher [33], and Agrawal [4], respectively, and are presented in the following theorem.

Theorem 2.1.1. Let L be the nilpotent residual of a group G. Then

1. [4]G is a solvable PST -group if and only if L is an abelian Hall

   subgroup of G on which G acts by conjugation as a group of power automorphisms.

2. [21] G is a solvable PT -group if and only if G is a solvable PST -group and G/L is an Iwasawa group.

3. [10] G is a solvable T -group if and only if G is a solvable PST - group and G/L is a Dedekind group.

A group G is called an Iwasawa group if every subgroup of G is permutable in G. If every subgroup of G is normal in G, then G is called a Dedekind group

Definition 2.1.2. Let G be a group and p a prime. Then

1. G is a Y_p-group if, for all p-subgroups H and K of G such that$H \leq K$, H is S-permutable in $N_G(K)$.

2. G is an X_p-group if each subgroup of a Sylow p-subgroup P of G is permutable in $N_G(P)$.

3. G is a C_p-group if each subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$.

Theorem 2.1.3. Let G be a group.

1. [5] G is a solvable PST -group if and only if G is a Y_p-group for all primes p.

2. [7] G is a solvable PT -group if and only if G is an X_p-group for all primes p.

3. [15] G is a solvable T -group if and only if G is a C_p-group for all primes p.

Proof 1): Assume that G is a soluble PST-group.
Theorem 1.0.11: A group G is a PST group if and only if G is a PST_p group for every prime.
Then G is a p-soluble PST_p-group for all prime p by theorem 1.0.11.

Theorem 1.0.12: A p-soluble group is a PST_p group if and only if it satisfies Y_p.

By theorem 1.0.12, it follows that G is a Y_p-group for all primes p.

Conversely, Suppose that G satisfies Y_p-group for all primes p.

Then every subgroup has the same property.

Therefore, if G is a group with least order subject to not being a soluble PST group, then every proper subgroup of G is a soluble PST group.

According to Agarwal's theorem,

Every soluble PST group is supersoluble.

Therefore, either G is supersoluble or G is a minimal non- supersoluble group. In both cases, we have G is soluble.

Since Y_p coincides with PST_p in the p-soluble universe by theorem 1.0.12.

It follows that G is a PST_p group for all p.

Then G is a PST group by theorem 1.0.11.

Proof 2): Corollary 1.0.13: Let G be a finite soluble PT-group. Then if H is a subgroup of G, then H is a PT-group. A soluble PT-group satisfies X_p for all primes p by Corollary 1.0.13.

Conversely, assume that G satisfies X_p for all primes p, and G is of least order subject to not being a soluble PT-group. Let p be the smallest prime divisor of G.

Theorem 1.0.14: Let p be the smallest prime divisor of the order of G. Then G has X_p if and only if G is p-nilpotent and Sylow p-subgroups of G are modular.

By Theorem 1.0.14, G is p- nilpotent and $O_p^{'}$(G) $\neq G$.

Put $K = O_p^{'}$(G), let q be a prime divisor of K, and let Q be a Sylow q-subgroup of G.

Lemma 1.0.9: A group G satisfies X_P if and only if a Sylow p-subgroup P of G is modular and the $p^{'}$-elements of $N_G(P)$ induce power automorphisms in P.

Then Lemma 1.0.9 shows that Q is modular and the $q^{'}$-elements of $N_K(Q)$ induce power automorphisms in Q.

Applying Lemma 1.0.9 again, we see that K satisfies X_P.

It follows from the minimality of G that K is a soluble PT-group, and so G is certainly soluble.

Let L = Y^* (K)

Theorem 1.0.15: A soluble group G is a PT-group if and only if it has an abelian normal Hall subgroup L of odd order such that G/L is a nilpotent modular group and elements of G induce power automorphisms in L.

By Theorem 1.0.15, L is an abelian normal Hall subgroup of K in which K induces power automorphisms.

Let r be a prime divisor of |L |and let R be a Sylow r-subgroup of L.

Then R is a normal Sylow r- subgroup of G.

By X_r the r'-elements of G induce power automorphisms in R.

Hence all the elements of G induce power automorphisms in L.

Suppose that $L \neq 1$.

Then G/L inherits the hypotheises of the theorem and so G/L is a soluble PT-group.

Lemma 1.0.10: Let N be a Normal Hall subgroup of a G and assume that the following hold:

1. G/N is a PT-group.

2. every subnormal subgroup of N is normal in G.

   Then G is a PT-group.

By Lemma 1.0.9, G is a PT-group, a contradiction.

Hence L = 1 and so K is nilpotent.

Finally, let T be a Sylow subgroup of K. Then T is also a Sylow subgroup of G.

As in the previous paragraph, if $T \neq 1$,

then G/T is a PT-group and G induces a group of power automorphisms in T.

Again G is a PT-group by Lemma 1.0.9.

This means that K = 1 so that G is a modular p-group, a final contradiction.

Proof 3): Let T' denote the class of all groups G for which

$H \lhd K \lhd L \leq G$ always implies that $H \leq L$:

in short T' is the largest subclass of T that is closed with respect to forming subgroups.

Now every finite soluble T-group is a T'-group and it is obvious that a finite T'-group satisfies C_p for all p, since every subgroup of a finite p-group is sub-normal.

Theorem 1.0.16: If G is a finite group, the following are equivalent statements.

1. G is soluble T-group.

2. G is a $T^{'}$-group.

3. G satisfies C_p for all p.

   By theorem 1.0.16, converse of the theorem is implied.

Definition 2.1.4. A subgroup H of a group G is said to be normal sensitive in G if the map $N \to H \cap N$ sends the lattice of normal subgroups of G onto the lattice of normal subgroups of H, that is, if $\{L|L \lhd H\} = \{H \cap N|N \lhd G\}$.

Theorem 2.1.5. Every subgroup of G is normal sensitive in G if and only if G is a solvable T -group.

2.2 The intersection map and the classes T , PT , and PST

First let us define the analogues of normal sensitivity for permutability and S-permutability.

Definition 2.2.1. A subgroup H of a group G is said to be

1. permutable sensitive in G if the following holds:
   $\{$N |N is permutable in H$\}$ = $\{$H $\cap W|$ W is permutable in G$\}$.

2. S-permutable sensitive in G if the following holds:
   $\{$N |N is S-permutable in H$\}$ = $\{$H $\cap W|$ W is S-permutable in G$\}$.

The collection of S-permutable subgroups of a group G is a sublattice of the lattice of a subnormal subgroups [11] of G so that a subgroup H of G is S-permutable sensitive if the map W $\to$ H $\cap$ W sends the lattice of S-permutable subgroups W of G onto the lattice of S-permutable subgroups $H \cap W$ of H. Although the collection of permutable subgroups of a group G is a subset of the lattice of subnormal subgroups of G, they need not be a sublattice, as the example (found through the use of GAP [16]) below illustrates. Thus the intersection map, in the case of permutable subgroups in G, need not be a lattice mapping.
One purpose here is to establish the following result which is the analogue of Theorem 2.1.5 for solvable PT - and PST -groups.

Theorem 2.2.2. Let G be a group.

1. G is a solvable PST -group if and only if every subgroup of G is S-permutable sensitive in G..
2. G is a solvable PT -group if and only if every subgroup of G is permutable sensitive in G.

Proof 2): Suppose every subgroup of G is permutable sensitive in G.
Further, suppose G is minimal with respect to not being aPT -group. Let $K \leq H \leq G$. Then K is permutable sensitive in G. So if L is permutable in K then $L = K \cap N$ where N is permutable in G. Now N ∩ H is permutable in H and $L = K \cap N = K \cap H \cap N$.
So K is permutable sensitive in H. Thus, by minimality, we can assume every proper subgroup of G is a solvable PT -group.
Hence every proper subgroup of G is supersolvable. By a result of Huppert (10.3.4 of [20]),that is every maximal subgroup of a finite group G is super-soluble, then G is soluble. G must be solvable.
Let N◁ G with K/N permutable in $H/N \leq G/N$. Then K is permutable in H and so $K = L \cap H$ with L permutable in G. But then $K/N = L \cap H/N$ with L/N permutable in G/N. So we can assume every proper factor group of G is a PT -group. Suppose G is a p-group. Since G is not a PT -group, G is not an Iwasawa group.
Lemma 1.0.17: Let G be a finite p-group. Then G has modular subgroup lattice if and only if each of its sections of order p^3 does. Therefore if G is not an M-group, then there exist subgroups H, K of G with K≤H such that H/K is dihedral of order and or nonabelian of order p^3 and exponent p for p > 2.
By Lemma 1.0.17, G possesses a section H/K isomorphic to either the dihedral group of order eight or, for p odd, to the nonabelian p-group of order p^3 with exponent p.
Since all proper subgroups and all proper factor groups of G are PT -groups, we must have H = G and K = 1. It is a straight forward argument to show that the dihedral group of order 8 and all nonabelian p-groups of order p^3 with exponent p do not satisfy the hypothesis.
Theorem 1.0.18:Let G be a periodic soluble group and let H be S-permutable in G. If H satisfies min-p for all primes p, then H is serial in G.
So, we can now apply Theorem 1.0.18, and deduce that G = P× Q with Q a cyclic q-group

and P $\lhd$ G where P is either an abelian p-group, p$\neq$ q, or P is the quaternion group of order 8, 2$\neq$ q.

Let A be subnormal in G with A not permutable in G. Then A is core free since proper quotients of G are PT -groups. First suppose that A is a q-group. Then A is a subnormal Sylow q-subgroup of P A. Hence P normalizes A yielding A $\lhd$ G.

So we can assume p divides $|A|$. Let A_p be a Sylow p-subgroup of A. A_p is normal in P and so there is a subgroup T permutable in G with P ∩ T = A_p. Note that A_p must be a Sylow p-subgroup of T . So T = A_p T_q with T_q a Sylow q-subgroup of T .

By choosing conjugates and renaming Q, we can assume $T_q \leq$ Q. Now, T permutable in G yields QT = TQ = A_p Q. Hence Q normalizes A_p and thus $A_p \lhd$ G. Thus G/ A_p is a PT -group so that A/ A_p is permutable in G/ A_p. We can deduce A is permutable in G and this gives a final contradiction.

Conversely, assume G is a solvable PT -group and let L be the nilpotent residual of G. By Theorem 2.1.1, L is a normal abelian Hall subgroup of G. Let C be a system normalizer of G.

Theorem 1.0.27:(Gaschiitz, Schenkman, Carter) Let G be a finite soluble group and denote by L the smallest term of the lower central series of G. If N is any system normalizer in G, then G = NL. If in addition L is abelian, then also N ∩ L = 1 and N is a complement of L .

By a result of Theorem 1.0.27 (Gaschütz, Schenkmen, and Carter), Let G be a finite soluble group and denote by L the smallest term of the lower central series of G. If N is any system normalizer in G, then G = NL. If in addition L is abelian, then also N ∩ L = 1 and N is a complement of L . Thus, G = LC and C ∩ L = 1. Note that all the complements of L in G are system normalizers of G and hence are conjugate in G.

Theorem 1.0.28:(Carter) Let G be a finite soluble group of nilpotent length at most 2.Then the system normalizers coincide with the Carter subgroups of G.

By a result of Theorem 1.0.28 (Carter), Let G be a finite soluble group of nilpotent length at most 2. Then the system normalizers coincide with the Carter subgroups of G. Thus, all the complements to L in G are necessarily Carter subgroups of G.

Also notice that C is a Hall subgroup of G. Let us show every subgroup H of G is permutable sensitive in G. Let T be a permutable subgroup of H. Since factor groups of PT -groups are again PT -groups, we can assume T is core free. Using Lemma 1.0.2, we have T L is permutable in G. Assume that T L$\neq$ G. By Theorem 2.1.1, T L is a solvable PT -group. Note

that T is permutable in H ∩T L.
By induction, there exists K permutable in T L such that H ∩ T L ∩ K = T so that H ∩ K = T. But K is permutable in T L and T L is permutable in G which implies K is permutable in G. Therefore, we may assume T L = G. T core free and T L = G imply that T is a complement to L in G. Hence T is a Carter subgroup of G. T permutable in H and T self-normalizing in G yields T = H. Therefore we have G ∩ H = T completing the proof.

Proof 1:) Suppose every subgroup of G is S-permutable sensitive in G. Using an argument similar to that used in the proof of part (2) of Theorem 2.2.2, we can argue that the hypothesis is inherited by subgroups. Now let us argue that G is a subgroup closed PST -group.
Let H be any subgroup of G and suppose N is S-permutable in K with K S-permutable in H. Then N = K∩L for some L S-permutable in H. By Lemma 1.0.1, we have K ∩ L is S-permutable in H. Thus N is S-per H and we have H is a PST -group.
Since subgroup closed PST -groups are solvable PST -groups (Corollary 5 of [3]), G is a solvable PST -group. The argument for the converse is similar to the argument used in the proof of part (2) of Theorem 1.0.7.

The assumption in Theorem 2.2.2 that every subgroup of G is S-permutable or permutable sensitive is needed to guarantee the solvability of the group G. Similarly, Theorem 2.1.5 uses the assumption of every subgroup being normal sensitive to force sol- vability. However, if we restrict the S-permutable, permutable, or normal sensitivity to the subnormal subgroups of G, then we can still deduce that G is a PST -, PT -, or T -group, respectively. In fact, for PST -groups and T -groups, we can even restrict S-permutable and normal sensitivty, respectively, to the normal subgroups. This is the content of our next theorem.

Theorem 2.2.3. Let G be a group.

1. G is a PST -group if and only if every normal subgroup of G is S-permutable sensitive in G.

2. G is a PT -group if and only if every subnormal subgroup of G is permutable sensitive in G.

3. G is a T -group if and only if every normal subgroup of G is normal sensitive in G.

Proof 1):Let G be a group.
Assume that G is a PST-group.
And let K be S-permutable in U with U a normal subgroup of G.
It is clear that U is S-permutable sensitive in G because K, being subnormal in G, is necessarily S-permutable in G.
Conversely, assume that every normal subgroup of G is S-permutable sensitive in G.
Let U and V be subgroups of G such that$U \lhd V \lhd G$.
Then there exists an S-permutable subgroup X of G such that $X \cap V = U$.
Lemma 1.0.1: Let G be a group. If H and K are S-permutable in G, then $H \cap K$ is S-permutable in G.
By Lemma 1.0.1, U is S-permutable in G.
Theorem 1.0.19: G is a T^*-group if and only if G is a PST -group.
Therefore, by Theorem 1.0.19, G is a PST -group.

Proof 2): Let G be a group.
Assume that G is a PT -group and let K be permutable in U with U a subnormal subgroup of G.
It is clear that U is permutable sensitive in G because K, being subnormal in G, is necessarily permutable in G.
Conversely, assume that every subnormal subgroup of G is permutable sensitive in G.As in the second paragraph of the proof of part (1), one can argue G is a PST - group.
If we assume that G is a p-group for some prime p, then every subgroup of G is permutable sensitive in G so that G is an Iwasawa group by part (2) of Theorem 2.2.2.
Assume that G is not a p-group.
Note that all the subnormal subgroups and all the factors groups of G satisfy the hypothesis of the theorem.

Lemma 1.0.4: Let G be a group such that every normal subgroup of G is permutable sensitive in G. Then G is an SC-group.

By Lemma 1.0.4, G is an SC-group. Let D be the solvable residual of G and assume D = 1.

Then G/D is a PT -group.

Lemma 1.0.5: Let D be the solvable residual of a group G. G is an SC-group if and only if G/D is supersolvable, D/Z(D) is a direct product of G-invariant simple groups, and Z(D) is supersolvably embedded in G (that is, there is a G-admissible series in Z(D) with cyclic factors).

By Lemma 1.0.5, D/Z(D) = U_1/Z(D) ×···× U_k/Z(D) where $U_i \lhd G$ and U_i /Z(D) is a simple (nonabelian) group. Note that $U_i = 1$ for all i.

Therefore, if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ where $1 \leq r \leq k$,

then $G/U'_{i1}U'_{i2} \ldots U'_{ir}$ is a PT -group.

And hence satisfies N_p and P_p for all primes p.

By Theorem 1.0.7, G is a PT -group.

Now let us assume G is solvable.

By Lemma 1.0.4, G is supersolvable and hence contains a normal Sylow p-subgroup P where p is the largest prime divisor of the order of G.

Now both P and G/P are PT -groups by induction. This means that P and the Sylow subgroups of G/P are Iwasawa groups.

But G is a solvable PST -group so it must be a PT -group by Theorem 2.1.1.

This completes the proof.

Proof 3):Let G be a group.

Assume G is a T-group.

Let $K \lhd H$ and $H \lhd G$.

$\implies K \lhd G$.

By theorem 2.1.5, Every subgroup of G is normal sensitive in G if and only if G is a solvable T -group.

Therefore, Every subgroup of G is normal sensitive.

$\implies$ Every normal subgroup of G is normal sensitive.

Conversely, Assume that every normal subgroup of G is normal sensitive in G.

Let U and V be subgroups such that $U \leq V \leq G$ and $V \lhd G$

Then there exists a normal subgroup X of G such that,

$X \cap V = U$

Therefore, U is a normal subgroup in G.Therefore, G is a T- group.

We now turn to some local considerations for the concepts of normal, permutable, and S-permutable sensitivity. Consider the following definitions.

Definition 2.2.4. Let G be a group and p a prime. Then

1. G is a Y_p^*-group if, for each p-subgroups K of G, each subgroups H of K is S-permutable sensitive in $N_G(K)$.
2. G is an X_p^*-group if each subgroup of a Sylow p-subgroup P of G is permutable sensitive in $N_G(P)$.
3. G is a C_p^*-group if each subgroup of a Sylow p-subgroup P of G is normal sensitive in $N_G(P)$.

We establish the following result which localizes the different sensitivity concepts in T -, PT-, and PST -groups.

Theorem 2.2.5. Let G be a group.

1. G is a solvable PST -group if and only if G is a Y_p^*-group for all primes p.
2. G is a solvable PT -group if and only if G is an X_p^*-group for all primes p.
3. G is a solvable T -group if and only if G is a C_p^*-group for all primes p.

Theorem 2.2.5 is a consequence of Theorem 2.1.3 and the following result.

Theorem 2.2.6. Let p be a prime.

1. $Y_p = Y_p^*$

2. $X_p = X_p^*$

3. $C_p = C_p^*$

Proof 1): Assume that G is a Y_p-group.
Let L, H, and K be p-subgroups of G with $L \leq H \leq K$.
Then L is S-permutable in $N = N_G$(K).
Thus H is S-permutable sensitive in N.
Thus, G is a Y_p^* -group.
Conversely, assume that G is a Y_p^* -group.
And let H and K be p-subgroups of G with $H \leq K$.
Then K is S-permutable sensitive in $N = N_G$(K).
And since H is S-permutable in K there exists an S-permutable subgroup L of N such that $H = L \cap K$.
L and K are both S-permutable subgroups of N and hence, by Lemma 1.0.1,
$H = L \cap K$ is S-permutable in N.
It follows that G is a Y_p-group.

Proof 2): Throughout, let P be a Sylow p-subgroup of G and put $N = N_G$(P).
Assume that G is an X_p^*-group.
First let us argue that P is an Iwasawa group.
It is enough to show that P is a PT -group.
Let $H \leq P$.
If C is a permutable subgroup of H, then $C = H \cap L$ with L permutable in N since G is an X_p^*-group.
Since $C = H \cap (P \cap L)$ and $P \cap L$ is permutable in P, we see that H is permutable sensitive in P. Applying part (2) of Theorem 2.2.2, we see that P is a PT -group and hence an Iwasawa group.
Let $X \leq P$.
Since P is an Iwasawa group, we have X permutable in P. Thus, there exists a permutable subgroup Y of N such that $Y \cap P = X$.

By Lemma 1.0.1, X is S-permutable in N.

Theorem 1.0.21:Let G be a group. Suppose that p is a prime number and that H is an S-permutable p-subgroup of G. If the Sylow p-subgroups of G are modular (respectively, Dedekind), then H is permutable (respectively, normal)in G.

Now we can apply theorem 1.0.21 and deduce X is permutable in N.

Thus G is an X_p-group.

Conversely, assume that G is an X_p-group.

That G is an X_p^* -group follows from the fact that P is an Iwasawa group.

Proof 3): Throughout, let P be a Sylow p-subgroup of G.

Assume that G is an C_p^*-group.

First let us argue that P is a Dedekind group.

It is enough to show that P is a T -group.

Let $H \leq P$.

If C is a normal subgroup of H, then $C = H \cap L$ with L normal in N since G is an C_p^*-group.

Since $C = H \cap (P \cap L)$ and $P \cap L$ is normal in P, we see that H is normal sensitive in P.

Applying Theorem 2.1.5, Every subgroup of G is normal sensitive in G if and only if G is a solvable T -group.

We see that P is a T -group and hence a Dedekind group.

Let $X \leq P$.

Since P is an Dedekind group, we have X normal in P.

Thus, there exists a normal subgroup Y of N such that $Y \cap P = X$.

X is normal in N.

Thus G is an C_p-group.

Conversely, assume that G is an C_p-group.

That G is an C_p^* -group follows from the fact that P is a Dedekind group.

Chapter 3

On The Direct Product Of Nearly S-Permutable Subgroups

3.1 Introduction

One of the earliest results about permutable subgroups found in [7] by Isaacs,stated that every permutable subgroup is subnormal. The term S-permutable subgroup introduced by Ore in [13] as a subgroup H is said to be S-permutable in G if it permutes with all Sylow subgroups of G. In [11], Kegel proved that S-permutable subgroups are necessarily subnormal. In [19], Al-Sharo introduced the notion nearly S-permutable subgroup, as a subgroup H of G is called nearly S-permutable in G if for every prime p such that gcd(p, $|H|$) = 1 and for every subgroup K of G containing H, the normalizer N_K(H) contains some Sylow p-subgroup of K. Then he showed that the nearly S-permutable subgroup need not be subnormal in general. Next, Ikram in [1] studied the lattices of nearly S-permutable subgroups, in particular an example was constructed to show that the set of all nearly S-permutable subgroups of a group need not distributive nor modular lattices. In this article, we study the direct product of nearly S-permutable subgroups and discuss some algebraic properties of this product.

3.2 Preliminaries

Our notions are fairly standard, all groups in this research are finite. The next definitions and results elaborate the terms: permutable, S-permutable and nearly S-permutable subgroups. These terms will be studied in the next sections.

Definition 3.2.1. [9]Let G be a group, and let H and K be two subgroups of G. We say that H permutes with K if HK = KH.

Definition 3.2.2. [9] Let G be a group and let H be a subgroup of G. Then H is said to be permutable if it satisfies the following equivalence conditions:

1. It permutes (commutes) with every subgroup of G.
2. Its product with every subgroup of G is a subgroup.
3. It permutes with every cyclic subgroup.

Definition 3.2.3. [12] A subgroup H of a group G is called p-subgroup of G, if every element in H has order a power of p.

Theorem 3.2.4. [12] Let G be a finite group and p be a prime that divides |G|. Then G has an element of order p and, consequently, a subgroup of order p.

Proof: We form the set X of all p-tuples $(g_1, g_2, \dots, g_p)$ of elements of G having the property that the product of coordinates in G is e.
That is, X={ $(g_1, g_2, \dots, g_p) \,|\, g_i \in$ G and $g_1\, g_2 \dots g_p$ = e}
We claim p divides $|X|$.
In forming p-tuples in X we may let $g_1, g_2, \dots, g_{p-1}$ be any elements of G, and g_p is then uniquely determined as $(g_1, g_2, \dots, g_{p-1})^{-1}$.
Thus $|X| = |G|^{p-1}$.
And since p divides $|G|$, we see that p divides $|X|$.
Let σ be the cycle (1,2,3, $\dots$, p) in S_p.
We let σ act on X by
$(g_1, g_2, \dots, g_p)\, \sigma$=$(g_1\, \sigma, g_2\, \sigma, \dots, g_p\, \sigma)$=$(g_2, g_3, \dots, g_{p-1}, g_1)$

Note that $(g_2, g_3, \ldots, g_{p-1}, g_1) \in$ X for g_1 $(g_2, g_3 \ldots g_p)$= e implies that

g_1 =$(g_2, g_3 \ldots g_p)^{-1}$, so $(g_2, g_3 \ldots g_p)g_1$= e also.

Thus σ acts on X, and we consider the subgroup $\langle\sigma\rangle$ of S_p to act on X by iteration in the obvious way.

Now, $|\langle\sigma\rangle| = p$.

Theorem 1.0.22: Let G be a group of order p^n and let X be a finite G-set.

Then $|X|$= $|X_G|$ (mod p)

So we may apply Theorem 1.0.22, and we know that $|X|$= $|X_{\langle\sigma\rangle}|$ (mod p).

Since p divides $|X|$, it must divide $|X_{\langle\sigma\rangle}|$ also.

Let us examine $X_{\langle\sigma\rangle}$.

Now $(g_1, g_2, \ldots, g_p)$ is left fixed by σ, and hence by $\langle\sigma\rangle$, if and only if g_1= g_2= $\ldots$ = g_p.

We know atleast one element in $X_{\langle\sigma\rangle}$, namely (e, e, ..., e).

Since p divides $|X_{\langle\sigma\rangle}|$, there must be atleast p elements in $X_{\langle\sigma\rangle}$.

Hence there exists some elements a $\in$ G, $a \neq e$, such that (a, a, ..., a) $\in$ $X_{\langle\sigma\rangle}$

a^p= e, so a has order p.

Ofcourse, $\langle a\rangle$ is a subgroup of G of order p.

Corollary 3.2.5. *[12] Let H be a finite subgroup. Then H is a p-subgroup if and only if $|H|$ is a power of p.*

Remark 3.2.6.

1. The set of all Sylow p-subgroups of a group G is denoted by Syl_p(G).
2. Every Sylow p-subgroup is p-subgroup, but the converse not necessary true.
3. Let p be a prime. If p $\nmid |G|$, then the only Sylow p-subgroup of G is the trivial subgroup $\{e\}$.
4. If $|G| = p^n, n \in N$, then G itself is the only Sylow p-subgroup of G.

Definition 3.2.7. [11]Let H be a subgroup of a group G. Then H is Sylow permutable (S-permutable) if it permutes with all Sylow p-subgroups of G for all primes p.

Remark 3.2.8. Every subgroup of an abelian group is S-permutable.

Proof: Let H be a subgroup of an abelian group G and K be a Sylow subgroup of G. Then:

$HK = \{hk | h \in H, k \in K\}$

$HK = \{kh|h \in H, k \in K\}$ (G is abelian)

$HK = KH$

Hence, H is S-permutable.

Remark 3.2.9. Every normal subgroup of a group is permutable.

Remark 3.2.10. For a group G there are at least two S-permutable subgroups, the trivial subgroup and the group itself.

Proof: Consider a group G, and let {e} be the trivial subgroup.
Then {e}H = H{e} = H and GH = GH = G for any Sylow p-subgroup H of G.

Remark 3.2.11. S-permutability does not imply permutability.

Example 3.2.12. *For any p-group G and a non permutable subgroup H, the only Sylow p-subgroup of G is G itself, which is permute with every subgroup.*
Let G = D_4 *= {e, r,* r^2*,* r^3*, s, rs,* r^2s*,* r^3s*}*
and H ={e, rs}
If K= {e, s}, and $K \leq H$*, then*
HK = {e, s, rs, r} and KH = {e, s, rs,r^3 *}*
Therefore $HK \neq KH$*, so H is not permutable.*
But H is S-permutable since HG = GH = G and G is the only Sylow p-subgroup of G.

Remark 3.2.13. [11] Every S-permutable subgroup is subnormal.

Corollory 3.2.14. *Every permutable subgroup is subnormal.*

Definition 3.2.15. [19] Let H be a subgroup of a group G. We say that H is nearly S-permutable in G if for every prime p with gcd(p, $|H|$) = 1 and for every subgroup K of G containing H, the normalizer N_K(H) contains some Sylow p-subgroup of K.

Remark 3.2.16. Nearly S-permutability does not implies S-permutability.

Example 3.2.17. *For the symmetric group* S_4*, consider the subgroup H as*
H = {( ),(1, 2),(1, 4),(2, 4),(1, 4, 2),(1, 2, 4)}.
Certainly, H is a nearly S-permutable subgroup but not permutable.

Theorem 3.2.18. Every subgroup of nilpotent group is nearly S-permutable.

Lemma 3.2.19. [19] Let H be a S-permutable subgroup of G. Then H is S-permutable subgroup of K, wherever $H \leq K \leq G$.

Lemma 3.2.20. [2]Let H be a nearly S-permutable subgroup of G. Then H is nearly S-permutable subgroup of K, wherever $H \leq K \leq G$.

Now, we state the following result.

Theorem 3.2.21. Every S-permutable subgroup of a group G is nearly S- permutable.

Proof: :Let H be a S-permutable subgroup of G.
Suppose on the contrary that H is not nearly S-permutable subgroup in G.
So, there exists a prime p such that gcd(p, $|H|$) = 1
And there exists a subgroup K of G such that $H \leq K \leq G$,
But P not contained in N_K(H), for all P $\in Syl_p$(K).
Using Lemma 3.2.19, H is S-permutable subgroup of K,
that is HP = PH, for all P $\in Syl_p$(K).
So, P is a subgroup of N_K(H), which is a contradiction.
Hence, H is S-permutable subgroup in G.

Remark 3.2.22. [2] Let us denote by π(n) the set of all prime divisors of n. If H is subgroup of G such that $\pi(|H|) = \pi(|G|)$, then H is nearly S-permutable in G.

Remark 3.2.23.
1. Permutability implies S-permutability, and S-permutability implies nearly S-permutability.
2. The trivial subgroup and the improper subgroup are nearly S-permutable.

Remark 3.2.24. Every subgroup of an abelian group is nearly S-permutable.

Proof: Let H be a subgroup of an abelian group G.
Then H is normal, and so H is permutable.
This implies that H is nearly S-permutable.

Remark 3.2.25. Let G be a p-group. Then all subgroups of G are nearly S-permutable.

Remark 3.2.26. The intersection of two nearly S-permutable subgroups not necessary nearly S-permutable, as the next example shows.

Example 3.2.27. *Consider G=S_4 and and the nearly S-permutable subgroups:*
$H_1 = \langle(1,4),(1,4,3)\rangle$ *and* $H_2 = \langle(3,4),(2,4,3)\rangle$.
Now $\pi(G) = \pi(H_1) = \pi(H_2)$
Which implies that H_1 *an* H_2 *are nearly S- permutable subgroups of G.*
But $H_1 \cap H_2 = \langle(3,4)\rangle$, *which is not nearly S-permutable.*

We denote the set of all S-permutable (nearly S-permutable) subgroups of a group G by SP(G) (NSP(G)).We denote the number of S-permutable (nearly S-permutable) subgroups of a group G by #SP(G) (#NSP(G)).

Remark 3.2.28. Clearly, Z_n is an abelian group, so by Remark 24 we have that all subgroups of Z_n are nearly S-permutable subgroups.
In this case #NSP(G) is number of subgroups of G.
For Z_p, p-prime, we have #NSP(Z_p)= 2.
In general, #NSP(Z_n) = T(n), where T(n) is the number of prime divisors of the integer n.

Example 3.2.29. *The following table shows some groups and their #SP(G) and #NSP(G):*

G	#SP(G)	#NSP(G)
Z_3	2	2
Z_9	3	3
S_3	3	3
A_4	3	3
S_4	4	8
D_8	10	10
D_{10}	3	3
D_{12}	7	7
D_{18}	4	7

Remark 3.2.30. [2] Nearly S-permutable subgroups need not to be sub- normal.

Example 3.2.31. *Let G =* D_{18}, *which contains 3 subgroups of order 6, these subgroups are nearly S-permutable in G but not subnormal.*

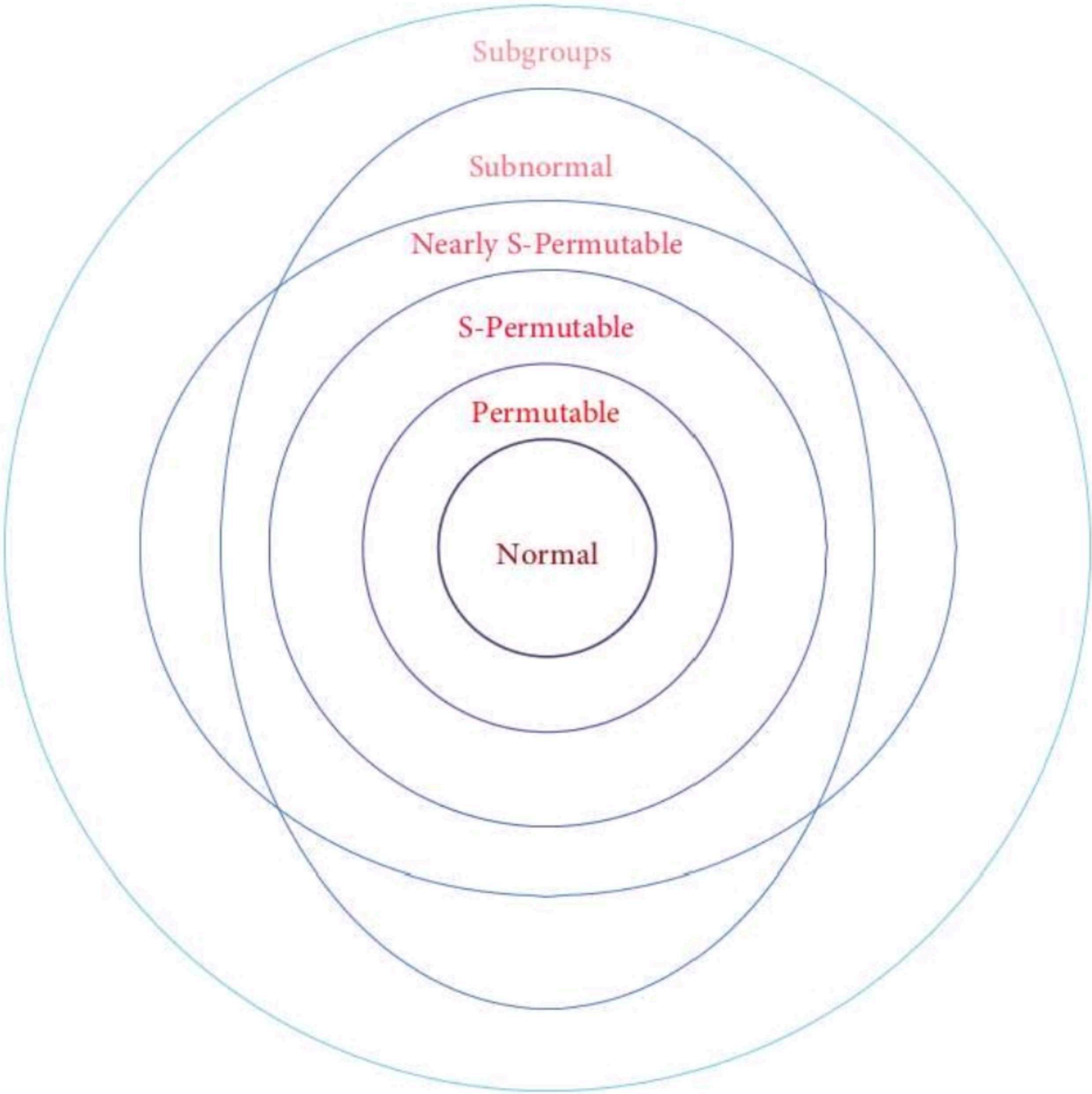

Figure 3.1: The relation between: Permutability, S-permutability, Nearly S-permutability, Normality and Sub-normality using Venn Diagram.

3.3 Direct Product of Nearly S-permutable subgroups

In this part, we study the permutability of the direct product of S-permutable subgroups. This can be shown by the next theorem.

Theorem 3.3.1. Let G_1 and G_2 be two groups with relatively prime orders. Then the direct product of nearly S-permutable subgroups of G_1 and G_2 is nearly S-permutable subgroup.

Proof: Let A_1 be a nearly S-permutable subgroup in G_1 and A_2 be a nearly S-permutable subgroup in G_2. Suppose on the contrary that A = $A_1 \times A_2$ is not nearly S-permutable subgroup in G = $G_1 \times G_2$.

Then there is a prime p such that gcd(p, $|A|$) = 1 and a subgroup K in G such that $A \leq K \leq G$ and N_K(A) does not contain P = $P_1 \times P_2$ where P is the Sylow p-subgroup of K.

Since K contains $A_1 \times A_2$, then K must factor as a product of two subgroups $K_1 \times K_2$ = K, where $A_1 \leq k_1 \leq G_1$ and $A_2 \leq K_2 \leq G_2$.

Moreover, $N_{K_1 \times K_2}(A_1 \times A_2) = N_{K_1}(A_1) \times N_{K_2}(A_2)$

and P = $P_1 \times P_2$, where $P_1 \in Syl_p(K_1)$ and $P_2 \in Syl_p(K_2)$.

Since P is not contained in N_K(A), then either P_1 is not contained in $N_{K_1}(A_1)$ or P_2 is not contained in $N_{K_2}(A_2)$.

But this implies that either A_1 is not nearly S- permutable subgroup in G_1 or A_2 is not nearly S-permutable subgroup in G_2,

Which is a contradiction. Hence the claim.

Remark 3.3.2. The direct product of non nearly S-permutable subgroups could be nearly S-permutable subgroup.

Example 3.3.3. *Consider the symmetric group S_4, and the non nearly S-permutable subgroups H_1 = $\langle(1,2)\rangle$ and H_2 = $\langle(1,2,3)\rangle$. It is clear that $H_1 \times H_2$ = $\langle(1,2),(1,2,3)\rangle$ is nearly S-permutable subgroups of $S_4 \times S_4$, since it is of order 2 · 3 and S4 × S4 of order* $2^6 \cdot 3^2$

Chapter 4

Conjecture

4.1 Conjecture

Let G be a group. Let H be subgroup of G.

Let $P(H) = \{$K is a subgroup of G/ $HK = KH$ $\}$

To find $P(H)$ for all subgroups H of various finite groups like S_3, S_4, D_4, D_p, etc
And find groups for which $|$P(H)$|$ /$|$G$|$

1. $G = S_3 = D_3 = \{e, a, ab, ab^2, b, b^2\}$

   Following are the subgroups of G

   - $H_1 = \{e\}$
   - $H_2 = \{e, a\}$
   - $H_3 = \{e, ab\}$

- $H_4 = \{e, ab^2\}$

- $H_5 = \{e, b, b^2\}$

- $H_6 = S_3 = D_3$

Following are the $P(H)$ of all H

- $P(H_1)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6\}$
- $P(H_2)$ =$\{H_1, H_2, H_5, H_6\}$
- $P(H_3)$ =$\{H_1, H_3, H_5, H_6\}$
- $P(H_4)$ =$\{H_1, H_4, H_5, H_6\}$
- $P(H_5)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6\}$
- $P(H_6)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6\}$

Here, only orders of $P(H_1)$, $P(H_5)$ and $P(H_6)$ divides order of G.

2. $G = D_4 = \{e, a, ab, ab^2, ab^3, b, b^2, b^3\}$

Following are the subgroups of G

- $H_1 = \{e\}$
- $H_2 = \{e, a\}$
- $H_3 = \{e, ab\}$
- $H_4 = \{e, ab^2\}$
- $H_5 = \{e, ab^3\}$
- $H_6 = \{e, b, b^2, b^3\}$
- $H_7 = D_4$

Following are the $P(H)$ of all H

- $P(H_1)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$
- $P(H_2)$ =$\{H_1, H_2, H_4, H_6, H_7\}$
- $P(H_3)$ =$\{H_1, H_3, H_5, H_6, H_7\}$
- $P(H_4)$ =$\{H_1, H_2, H_4, H_6, H_7\}$
- $P(H_5)$ =$\{H_1, H_3, H_5, H_6, H_7\}$
- $P(H_6)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$
- $P(H_7)$ =$\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$

Here, no orders of $P(H)$ divides order of G.

3. $G = D_p$ ={e, a, ab, ab$^2, \ldots, ab^{p-1}, b, b^2, \ldots, b^{p-1}\}$

   Following are the subgroups of G

   - $H_1 = \{e\}$
   - $H_2 = \{e, a\}$
   - $H_3 = \{e, ab\}$
   - $H_4 = \{e, ab^2\}$
   - $H_{2k-2} = \{e, ab^{p-1}\}$
   - $H_{2k-1} = \{e, b, b^2, \ldots, b^{p-1}\}$
   - $H_{2k} = D_p$

   Here, order of $P(H_1)$ divides order of G. And order of $P(H_{2k-1})$ and $P(H_{2k-2})$ divides order of G, when p=2, where k=1,2, ..., p.

4. $G = S_4 = \{I, (1234), (1243), (1324), (1342), (1423), (1432), (123), (132), (124), (142), (134), (143), (234), (243), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)\}$

   Here, all orders of $P(K)$, $\forall$ subgroup K $\in$ G, divides order of G.

We conjecture that for $G = S_n$ where n $\geq$ 4, $|$ P(H) $| \not| |$ G $|$

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