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On k-Hypergraceful labelings of Complete Graphs

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Abstract. A (p, q)-graph $G = (V, E)$, that is, $|V(G)| = p$ and $|E(G)| = q$, is said to be k-hypergraceful if there exists a decomposition of G into edge induced subgraphs $G_1, G_2, \ldots, G_k$ having sizes $m_1, m_2, \ldots, m_k$ respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$, the set of labels received by the edges of G_i is precisely $\{1, 2, \ldots, m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. The decomposition $\{G_i\}$, if it exists, is then called a k-hypergraceful decomposition of G and f is called a k-hypergraceful labeling of G. In this paper, we characterize k-hypergraceful complete graphs K_p when $p - 4 \leq k \leq p - 1$. We also prove that the cycle C_n is 3-hypergraceful if $n \equiv 1 (\text{mod } 4)$ and 2-hypergraceful if $n \equiv 2 (\text{mod } 4)$.

Keywords: k-hypergraceful labeling, k-hypergraceful decomposition, k-hypergraceful graphs.

Mathematics Subject Classification (2000): 05C 78.

1. Introduction

For standard terminology and notations in graph theory we follow West [14] and for signed graphs we follow Chartrand [5] and Zaslavsky [15, 16].

Most graph labeling methods trace their origin to the one introduced by Rosa [10].

Let G be a graph of order p and size q. A graceful labeling of G is an injection $f : V(G) \to \{0, 1, \ldots, q\}$ such that when each edge uv is assigned the label $g_f(uv) = |f(u) - f(v)|$, the resulting edge labels are all distinct. Such a function g_f is called the induced edge function and a graph which admits such a labeling is called a *graceful graph* [4, 6, 7, 10].

The notion of graceful labeling has been extended to signed graphs by Acharya and Singh [2, 3] and Singh [11].

A *signed graph* S (or simply *sigraph*) is a graph $G = (V, E)$ together with a function $s : E(G) \to \{+, -\}$ called its signing function, which assigns a sign $+$ or $-$ to each edge. The graph G is called the underlying graph of the sigraph S. The set of all positive and negative edges of S are denoted by $E^+(S)$ and $E^-(S)$ respectively so that $E^+(S) \cup E^-(S) = E(S)$ is the edge set of S. Further, if $|E^+(S)| = m$ and $|E^-(S)| = n$ so that $m + n = q$, then S is called a (p, m, n)-sigraph. An all-positive sigraph S is one in which $E^+(S) = E(S)$ and an all-negative sigraph is one in which $E^-(S) = E(S)$. A sigraph is said to be homogeneous if it is either all-positive or all-negative, and heterogeneous otherwise.

Let S be a (p, m, n)-sigraph. For any injection $f : V(S) \to \{0, 1, \ldots, q = m + n\}$, the induced edge labeling g_f is defined by $g_f(uv) = s(uv)|f(u) - f(v)|$, for all $uv \in E(S)$ where $s(uv)$ is the sign of the edge uv. The function f is said to be a *graceful labeling* of S if $g_f(E^+(S)) = \{1, 2, \ldots, m\}$ and $g_f(E^-(S)) = \{-1, -2, \ldots, -n\}$. A signed graph which admits a graceful labeling is called a *graceful signed graph*.

The notion of hypergraceful decomposition of graphs was first introduced by Acharya [1], which is a generalization of graceful graphs and graceful signed graphs [11, 12].

A (p, q)-graph $G = (V, E)$ is said to be *k-hypergraceful* if there exists a decomposition of G into edge induced subgraphs $G_1, G_2, \ldots, G_k$ having sizes $m_1, m_2, \ldots, m_k$ respectively, and an injective labeling $f : V(G) \to \{0, 1, \ldots, q\}$, such that when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$, the set of labels received by the edges of G_i is precisely $\{1, 2, \ldots, m_i\}$ for each $i \in \{1, 2, \ldots, k\}$. The decomposition $\{G_i\}$, if it exists, is then called a *k-hypergraceful decomposition* of G and f is called a *k-hypergraceful labeling* of G. Further, G is said to be *hypergraceful* if it possesses a k-hypergraceful decomposition for some k.

k-hypergraceful labeling for a graph G for $k = 1, 2, 3$ and 4 is given in Figure 1.

The friendship graph F_3 is known to be nongraceful [8]. It is also known that no signed graph on F_3 is graceful [13]. Therefore, F_3 is neither a 1-hypergraceful nor a 2-hypergraceful graph. In Figure 2 we show 3-hypergraceful and 4-hypergraceful labelings of F_3.

We need the following results.

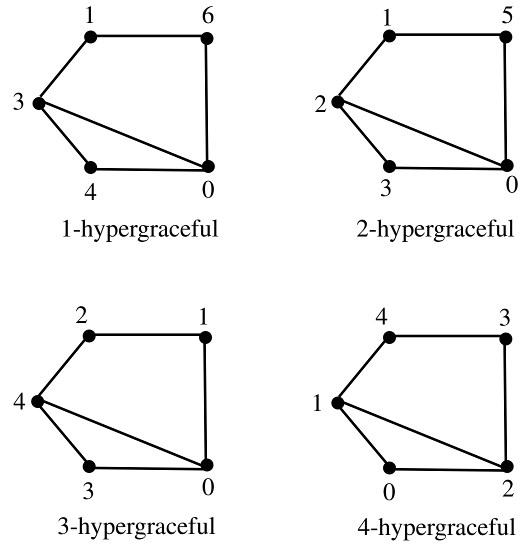

Figure 1. k-hypergraceful labelings of a graph.

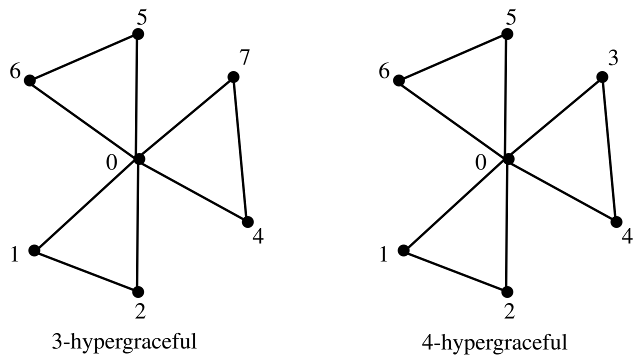

Figure 2. k-hypergraceful labeling of F_3 for $k = 3, 4$.

Theorem 1.1 ([7]). *A complete graph K_p is graceful if and only if $p \leq 4$.*

Theorem 1.2 ([9]). *A necessary condition for a (p, q)-graph $G = (V, E)$ to be k-hypergraceful with decomposition $G_1, G_2, \ldots, G_k$ is that it is possible to partition its vertex set V into two subsets V_o and V_e such that for each integer $i \in \{1, 2, \ldots, k\}$ there are exactly $\lfloor \frac{m_i+1}{2} \rfloor$ edges of G_i each of which joins a vertex of V_o with one of V_e.*

Lemma 1.3 ([9]). *If for no integer j, $0 \leq j \leq k$, $p - 2j$ is a perfect square, then K_p is not k-hypergraceful with respect to any decomposition of K_p.*

Remark 1.4. *If for some integer j, there exists a k-hypergraceful decomposition of K_p for which $p - 2j$ is a perfect square, then j represents the number of G_i's with odd size.*

By the *negation of a signed graph* S, we mean a signed graph $\eta(S)$ which is obtained from S by changing the sign of every edge to its opposite. If a signed graph S is graceful with a graceful labeling f, then the negation of the signed graph S is also graceful under the same f.

Lemma 1.5 ([9]). *If any integer p is such that none of p, $p-2$, $p-4$ is a perfect square, then no signed graph on K_p is graceful.*

Theorem 1.6 ([9]).

(i) *No signed graph on K_p, $p \geq 6$, is graceful.*
(ii) *Every signed graph on K_p, $p \leq 3$, is graceful.*
(iii) *A signed graph on K_4 is graceful if and only if the number of negative edges in it is not three.*
(iv) *A signed graph S on K_5 with n negative edges is graceful if and only if either $n = 1$ or $n = 3$ and the three negative edges in S are not incident at the same vertex or $\eta(S)$ satisfies similar conditions.*

2. Main Results

In this section, we characterize k-hypergraceful complete graphs K_p when $p - 4 \leq k \leq p - 1$. We present our results through a series of lemmas. We use the following notation.

Let $\pi_p = (a_1, a_2, \ldots, a_t)$ be a sequence of positive integers with $a_1 \leq a_2 \leq \cdots \leq a_t$ and $t = \binom{p}{2}$. If a_i occurs r_i times in the sequence, then we write the sequence as $\pi_p = (a_1^{r_1}, a_2^{r_2}, \ldots, a_s^{r_s})$.

Lemma 2.1. *The complete graph K_p is $(p-4)$-hypergraceful if $p \geq 8$ and p is even.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 3, 4, 6\}$ and $T = \{8, 9, \ldots, p+3\}$. Note that the elements of the sets S and T are from the set $\{0, 1, \ldots, q\}$ which are the labels of the vertices of K_p. It can be easily verified that $\pi_8 = (1^4, 2^4, 3^4, 4^3, 5^3, 6^3, 7^2, 8^2, 9^1, 10^1, 11^1)$ and $\pi_{10} = (1^6, 2^6, 3^6, 4^5, 5^4, 6^4, 7^3, 8^3, 9^3, 10^2, 11^1, 12^1, 13^1)$. Now let $p \geq 12$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \{1, 2, 3^2, 4, 6\}$, $L_2 = \{1^{p-5}, 2^{p-6}, 3^{p-7}, \ldots, (p-6)^2, (p-5)^1\}$ and $L_3 = \{2, 3, 4^2, 5^3, 6^3, 7^3, 8^4, 9^4, \ldots, (p-3)^4, (p-2)^3, (p-1)^3, p^2, (p+1), (p+2), (p+3)\}$.

Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \dots, (p+3)^{r_{p+3}})$ where

$$r_i = \begin{cases} p-4 & \text{if } 1 \le i \le 3; \\ p-5 & \text{if } i = 4; \\ p-6 & \text{if } i = 5, 6; \\ p-8 & \text{if } i = 7; \\ p-i & \text{if } 8 \le i \le p-4; \\ 4 & \text{if } i = p-3; \\ 3 & \text{if } i = p-2, p-1; \\ 2 & \text{if } i = p; \\ 1 & \text{if } i = p+1, p+2, p+3. \end{cases}$$

Clearly $r_i \le p-4$ for all i and $r_i \le r_j$ if $i \le j$. Hence π_p gives a $(p-4)$-hypergraceful decomposition of K_p. □

Lemma 2.2. *The complete graph K_p is $(p-4)$-hypergraceful if $p = 4t+1$, where $t \ge 2$.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 2, 4, 5\}$ and $T = \{8, 9, \dots, p+3\}$. It can be easily verified that $\pi_9 = (1^5, 2^5, 3^4, 4^4, 5^3, 6^3, 7^3, 8^3, 9^2, 10^2, 11^1, 12^1)$. Now let $p \ge 13$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \{1, 2^2, 3, 4, 5\}$, $L_2 = \{1^{p-5}, 2^{p-6}, 3^{p-7}, \dots, (p-6)^2, (p-5)^1\}$ and $L_3 = \{3, 4^2, 5^2, 6^3, 7^3, 8^4, 9^4, \dots, (p-2)^4, (p-1)^3, p^2, (p+1)^2, (p+2)^1, (p+3)^1\}$. Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \dots, (p+3)^{r_{p+3}})$, where

$$r_i = \begin{cases} p-4 & \text{if } i = 1, 2; \\ p-5 & \text{if } i = 3, 4; \\ p-6 & \text{if } i = 5; \\ p-7 & \text{if } i = 6; \\ p-8 & \text{if } i = 7; \\ p-i & \text{if } 8 \le i \le p-4; \\ 4 & \text{if } i = p-3, p-2; \\ 3 & \text{if } i = p-1; \\ 2 & \text{if } i = p, p+1; \\ 1 & \text{if } i = p+2, p+3. \end{cases}$$

Clearly $r_i \le p-4$ for all i and $r_i \le r_j$ if $i \le j$. Hence π_p gives a $(p-4)$-hypergraceful decomposition of K_p. □

Lemma 2.3. *The complete graph K_p is $(p-4)$-hypergraceful, if $p = 4t + 3$, where $t \geq 3$.*

Proof. The labeling given in Lemma 2.1 is also a $(p-4)$-hypergraceful labeling of K_p where $p = 4t + 3$ and $t \geq 3$. □

Lemma 2.4. *The complete graph K_{11} is 7-hypergraceful.*

Proof. Let f be the labeling of K_{11} with the elements of the set $S \cup T$ where $S = \{0, 4, 6, 7\}$ and $T = \{9, 10, 11, 12, 13, 14, 15\}$. It can be easily verified that $\pi_{11} = (1^7, 2^7, 3^7, 4^6, 5^5, 6^5, 7^4, 8^3, 9^3, 10^2, 11^2, 12^1, 13^1, 14^1, 15^1)$. Hence f gives a 7-hypergraceful labeling of K_{11}. □

Lemma 2.5. *The complete graph K_7 is not 3-hypergraceful.*

Proof. Suppose there exists a 3-hypergraceful labeling of K_7 with label set S with decomposition G_1, G_2, G_3 of sizes (m_1, m_2, m_3). Since $7-2j$ is a perfect square when $j = 3$, it follows from Remark 1.4 that each m_i is odd. Hence the possible cases for (m_1, m_2, m_3) are

$(1, i, 20 - i)$ where $i = 1, 3, 5, 7$ or 9,

$(3, i, 18 - i)$ where $i = 3, 5, 7$ or 9,

$(5, i, 16 - i)$ where $i = 5$ or 7 and

$(7, 7, 7)$.

We claim that f does not induce any of the above twelve decompositions.

Case 1. $(m_1, m_2, m_3) = (1, 1, 19)$.

The sequence of edge labels is $(1^3, 2^1, 3^1, \ldots, 19^1)$. Without loss of generality, we may assume that $0, 19, 1 \in S$. Now to get the label 1 for the second edge, two consecutive integers i , $i + 1$ must be in S for some $i \geq 2$. However in this case the edge label i occurs twice which is a contradiction. Therefore, f does not induce the decomposition $(1, 1, 19)$.

Case 2. (1, 3, 17).

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^1, 5^1, \ldots, 17^1)$. Without loss of generality, we may assume that $0, 17, 1 \in S$. Now to get the label 1 for the second edge, i and $i + 1$ must be in S and $i \leq 3$. Hence $2, 3 \in S$. Now $4, 5$ cannot belong to S. So to get label 3 for the second edge, 6 must be in S. Now 7 cannot be an edge label. Therefore, f does not induce the decomposition $(1, 3, 17)$.

Case 3. (1, 5, 15).

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^1, 7^1, \dots, 15^1)$. Without loss of generality, we may assume that $0, 15, 1 \in S$. To get two more edges with label 1, we have the following possibilities

(a) $2, 3 \in S$
(b) $3, 4, 5 \in S$
(c) $2, 4, 5 \in S$

In case (a) for edges to have edge labels 3 and 11, we must have $6, 11 \in S$ and the label 9 is repeated twice, which is a contradiction. In cases (b) and (c) label 5 for second edge cannot appear. Therefore, f does not induce the decomposition $(1, 5, 15)$.

Case 4. (1, 7, 13).

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1, \dots, 13^1)$. Without loss of generality, we may assume that $0, 13, 1 \in S$. Now the set $\{6, 7\}$ cannot be a subset of S. To get label 1 for three edges we have the following possibilities

(a) $2, 3 \in S$
(b) $3, 4, 7, 8 \in S$ or $4, 5, 7, 8 \in S$
(c) $2, i, i + 1 \in S$ for $i = 4, 5$ or $j, j + 1, j + 2 \in S$ for $j = 3, 4$

In case (a) label 7 for two edges cannot be obtained. In case (b) the edge label 3 repeats more than twice which gives a contradiction. In case (c) if $2, i, i + 1 \in S$ then label 10 for an edge cannot appear and if $j, j + 1, j + 2 \in S$ then label 11 for an edge cannot appear. Therefore, f does not induce the decomposition $(1, 7, 13)$.

Case 5. (1, 9, 11).

The sequence of edge labels is $(1^3, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^1, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get label 9 for two edges, 2 and 9 must be in S. So to get label 8 for second edge, 8 must be in S. Now the label 3 for the second edge cannot be obtained. Therefore, f does not induce the decomposition $(1, 9, 11)$.

Case 6. (3, 3, 15).

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^1, 5^1, \dots, 15^1)$. Without loss of generality, we may assume that $0, 15, 1 \in S$. To get label 1 for 3 edges, we have to include 2 and 3 in S and to get label 11 for an edge we must have 11 in S. Now label 7 for an edge cannot be obtained. Therefore, f does not induce the decomposition $(3, 3, 15)$.

Case 7. (3, 5, 13).

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^1, 7^1, \ldots, 13^1)$. Without loss of generality, we may assume that $0, 13, 1 \in S$. To get label 1 for two more edges, we have the following possibilities

(a) $2, 3 \in S$
(b) $3, 4, 5 \in S$
(c) $2, 4, 5 \in S$

In case (a) to get label 9 for an edge, we must have $9 \in S$. Now the label 3 for an edge cannot be obtained. In case (b) label 11 for an edge cannot appear and in case (c) label 10 for an edge cannot appear. Therefore, f does not induce the decomposition $(3, 5, 13)$.

Case 8. (3, 7, 11).

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^2, 5^2, 6^2, 7^2, 8^1, 9^1 \ldots, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get the edge label 9 we have the following two cases

(a) $9 \in S$
(b) $2 \in S$

In case (a) to get label 7 for two edges, 4 and 7 must be in S. Now the label 1 for the second edge cannot appear. In case (b) to get label 8 for an edge, either 8 or 3 must be in S. If $3 \in S$, then label 7 for two edges cannot appear. If $8 \in S$, then to get label 7 for a second edge, either 4 must be in S or 7 must be in S. In either case edge label 5 cannot appear. Therefore, f does not induce the decomposition $(3, 7, 11)$.

Case 9. (3, 9, 9).

In order to get the sequence of edge labels as $(1^3, 2^3, 3^3, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2)$, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 9\}$, which is not possible, as we cannot get label 9 for two edges.

Case 10. (5, 5, 11).

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^3, 5^3, 6^1, 7^1, \ldots, 11^1)$. Without loss of generality, we may assume that $0, 11, 1 \in S$. To get label 1 for two more edges, we have the following possibilities

(a) $2, 3 \in S$
(b) $3, 4, 5 \in S$
(c) $2, 4, 5 \in S$

In case (a) label 5 for three edges cannot appear. In cases (b) and (c) label 5 for two edges cannot appear. Therefore, f does not induce the decomposition $(5, 5, 11)$.

Case 11. (5, 7, 9).

The sequence of edge labels is $(1^3, 2^3, 3^3, 4^3, 5^3, 6^2, 7^2, 8^1, 9^1)$, Without loss of generality, we may assume that $0, 9, 1 \in S$. To obtain label 7 for two edges, the integers 2 and 7 must be in S. Now to obtain label 6 for a second edge, either $6 \in S$ or $3 \in S$. In both the cases label 5 for another edge cannot be obtained. Therefore, f does not induce the decomposition $(5, 7, 9)$.

Case 12. (7, 7, 7).

In this case, to get the sequence of edge labels $(1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3)$, we have to assign the labels to the vertices of K_7 from the set $\{0, 1, \ldots, 7\}$. One can easily see that no labeling from this set can give label 7 for three edges. Thus we see that none of the above decompositions have a 3-hypergraceful labeling of K_7. Hence K_7 is not 3-hypergraceful. □

Theorem 2.6. *The complete graph K_p is $(p-4)$-hypergraceful if and only if $p \geq 8$.*

Proof. Suppose $p \geq 8$. If $p = 2t$, $t = 4, 5, \ldots$, then by Lemma 2.1, K_p is $(p-4)$-hypergraceful. If $p = 4t+1$, $t \geq 2$, then by Lemma 2.2, K_p is $(p-4)$-hypergraceful. If $p = 4t+3$, $t \geq 3$, then by Lemma 2.3, K_p is $(p-4)$-hypergraceful. Finally, by Lemma 2.4, K_{11} is 7-hypegraceful. Therefore if $p \geq 8$ then K_p is $(p-4)$-hypergraceful.

Conversely, Suppose that K_p is $(p-4)$-hypergraceful. We need to prove that $p \geq 8$. Instead, we prove the contrapositive statement. Suppose that $p < 8$. By Theorem 1.1, K_5 is nongraceful; by Theorem 1.6, K_6 is not 2-hypergraceful and by Lemma 2.5, K_7 is not 3-hypergraceful. Therefore, if $p < 8$ then K_p is not $(p-4)$-hypergraceful. □

Lemma 2.7. *The complete graph K_p is $(p-3)$-hypergraceful if $p \geq 7$.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0, 2\}$ and $T = \{5, 6, \ldots, p+2\}$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \{2\}$, $L_2 = \{1^{p-3}, 2^{p-4}, 3^{p-5}, \ldots, (p-4)^2, (p-3)^1\}$ and $L_3 = \{3, 4, 5^2, 6^2, 7^2, \ldots, p^2, (p+1)^1, (p+2)^1\}$. Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, (p+2)^{r_{p+2}})$,

$$\text{where } r_i = \begin{cases} p-3, & i = 1, 2; \\ p-4, & i = 3; \\ p-5, & i = 4; \\ p-i, & 5 \leq i \leq p-2; \\ 2, & i = p-1, p; \\ 1, & i = p+1, p+2. \end{cases}$$

Clearly $r_i \leq p-3$ for all i and $r_i \leq r_j$ if $i \leq j$. Hence π_p gives a $(p-3)$-hypergraceful decomposition of K_p. □

Lemma 2.8. *The complete graph K_6 is 3-hypergraceful.*

Proof. Consider the labeling of K_6 with the elements of {0, 1, 3, 4, 5, 7}. One can easily verify that the corresponding sequence of induced edge labels is $(1^3, 2^3, 3^3, 4^3, 5^1, 6^1, 7^1)$. The decomposition G_1, G_2 and G_3 of K_6 have sizes (4, 4, 7). □

Theorem 2.9. *The complete graph K_p is $(p-3)$-hypergraceful for all $p \geq 4$.*

Proof. The result follows from Theorem 1.1, Theorem 1.6, Lemma 2.7 and Lemma 2.8. □

Theorem 2.10. *The complete graph K_p is $(p-2)$-hypergraceful for all $p \geq 3$.*

Proof. Let f be a labeling of K_p with the elements of the set $S \cup T$ where $S = \{0\}$ and $T = \{2, 3, \ldots, p\}$. Let $L_1 = \{g_f(uv) : u, v \in S\}$, $L_2 = \{g_f(uv) : u, v \in T\}$ and $L_3 = \{g_f(uv) : u \in S, v \in T\}$, where the integers in each L_i are in ascending order. Then $L_1 = \emptyset$, $L_2 = \{1^{p-2}, 2^{p-3}, 3^{p-4}, \ldots, (p-4)^3, (p-3)^2, (p-2)^1\}$ and $L_3 = \{2, 3, \ldots, p-1, p\}$. Hence it follows that $\pi_p = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots, p^{r_p})$,

$$\text{where } r_i = \begin{cases} p-2, & i = 1; \\ p-i, & 2 \leq i \leq p-1; \\ 1, & i = p. \end{cases}$$

Clearly $r_i \leq p-2$ for all i and $r_i \leq r_j$ if $i \leq j$. Hence π_p gives a $(p-2)$-hypergraceful decomposition of K_p. □

Theorem 2.11. *The complete graph K_p is $(p-1)$-hypergraceful for all $p \geq 2$.*

Proof. We label the vertices of K_p from the set $\{0, 1, \ldots, p-1\}$. It can be easily verified that the sequence of edge labels $\pi_p = (1^{p-1}, 2^{p-2}, 3^{p-3}, \ldots, (p-2)^2, (p-1)^1)$. Hence K_p is $(p-1)$-hypergraceful for all $p \geq 2$.. □

We now proceed to investigate the existence of k-hypergraceful labelings of cycles. It is known that if $n \equiv 1$ or $2 (\bmod\ 4)$, then the cycle C_n is nongraceful [7] and if $n \equiv 1 (\bmod\ 4)$, then C_n is also not 2-hypergraceful [2]. In the following theorems we determine the least k for which C_n, $n \equiv 1$ or $2 (\bmod\ 4)$ is k-hypergraceful.

Theorem 2.12. *If $n \equiv 1 (\bmod\ 4)$, then C_n is 3-hypergraceful.*

Proof. Let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}, a_1)$. Let $f : V(C_n) \to \{0, 1, \ldots, n\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0, & \text{for } \ i = 1; \\ i, & \text{for } 2 \leq i \leq \frac{n+1}{2}; \end{cases}$$

$$\text{and } f(b_i) = \begin{cases} 1, & \text{for } i = 1; \\ n+2-i, & \text{for } 2 \leq i \leq \frac{n-1}{4}; \\ n+1-i, & \text{for } \frac{n-1}{4} + 1 \leq i \leq \frac{n-1}{2}. \end{cases}$$

It can be easily verified that f is injective and the sequence of corresponding edge labels is $(1^3, 2^1, 3^1, \ldots, (n-2)^1\}$. Hence C_n is 3-hypergraceful. □

Theorem 2.13. *If* $n \equiv 2(\text{mod } 4)$*, then* C_n *is 2-hypergraceful.*

Proof. Let $C_n = (a_1, b_1, a_2, b_2, \ldots, a_{\frac{n}{2}}, b_{\frac{n}{2}}, a_1)$. Let $f : V(C_n) \to \{0, 1, \ldots, n\}$ be defined as follows:

$$f(a_i) = \begin{cases} 0, & \text{for } i = 1; \\ n+2-i, & \text{for } 2 \leq i \leq \frac{n+2}{4}; \\ n+1-i, & \text{for } \frac{n+2}{4} + 1 \leq i \leq \frac{n}{2}. \end{cases}$$

$$\text{and } f(b_i) = i, \quad \text{for} \quad 1 \leq i \leq \tfrac{n}{2}.$$

It can be easily verified that f is injective and the sequence of corresponding edge labels is $(1^2, 2^1, \ldots, (n-1)^1\}$. Hence C_n is 2-hypergraceful. □

3. Conclusion and Scope

In this paper we have investigated the existence of k-hypergraceful labeling of complete graphs and cycles. In this connection we propose the following conjecture.

Conjecture 3.1. For any connected graph G, there exists a positive integer k such that G is k-hypergraceful.

The hypergraceful index $h_i(G)$ is then defined to be the least positive integer k such that G is k-hypergraceful. Since $h_i(G) = 1$ if and only if G is graceful, this parameter gives another measure of gracefulness of graphs. It follows from Theorem 2.6 that $h_i(K_p) \leq p - 4$ for all $p \geq 8$. Also it follows from Theorem 2.12 and Theorem 2.13 that $h_i(C_n) = 2$ if $n \equiv 2(mod\, 4)$ and 3 if $n \equiv 1(mod\, 4)$. The most well-known conjecture on graceful labeling is Kotzig's conjecture which states that every nontrivial tree is graceful; which still remains open. We propose the following weaker conjecture.

Conjecture 3.2. Every nontrivial tree is 2-hypergraceful.

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